Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 18, No. 1/2026

THE ORDER TOPOLOGY IN PARTIALLY ORDERED SETS*

Francisco Javier García-Pacheco†

This manuscript is dedicated with great devotion to Prof. Biagio Ricceri, to whom I feel deeply thankful, among many things, for showing to me his Antiproximinal Conjecture, whose weak form was fully solved by me. The strong form remains open.

DOI 10.56082/annalsarscimath.2026.1.161

Abstract

The order topology has been historically defined only for totally ordered sets. Here, the order topology in partially ordered sets will be constructed. Several attempts have been provided in the recent literature on partially ordered groups, rings and modules. This manuscript contains a full construction that provides solid foundational ground to the previous attempts and serves to pay tribute to the topological trajectory of Prof. Ricceri.

Keywords: partially ordered set, order topology, semicontinuity.
MSC: 06F30.

1 Introduction

The historical development of General Topology is full of milestones. One of them was produced at the appearance of [11, Theorem 1.1], a principle with applications to topological mini-max theorems, which provides new proofs

*Accepted for publication on November 19, 2025
†garcia.pacheco@uca.es, Department of Mathematics, College of Engineering, Marine Science Institute, University of Cadiz, Avda. de la Universidad 10, 11519 Puerto Real, Spain

of some known results or improves a number of recent contributions in this area [12].

The order topology has been historically defined only for totally ordered sets (tosets) with at least two points [1, 2, 7–10]. The order topology is the most natural way to endow a toset with a topological structure. The order topology uses open rays (sets of all elements greater than x or less than y) and open intervals (elements between x and y) as its base. This precisely mirrors how open intervals constitute the base for the standard topology on $\mathbb{R}$. The concept was formalized as an example or class of topologies in the general development of set-theoretic topology in the 1910s and 1920s, following the work of Hausdorff and the full acceptance of abstract topological spaces.

However, many algebraic structures, such as groups, rings and modules, can not be endowed with a total ordering compatible with their operations. Therefore, the order topology cannot be considered on those algebraic structures. This is why it becomes imperative to develop a construction of the order topology in partially ordered sets (posets).

Recently, a successful attempt of construction of the order topology for posets was provided with the aim of endowing partially ordered algebraic structures with the order topology. In view of this, a sufficient condition for the order topology to be a ring topology was provided for unital ordered rings [3]. Continuing this line of research, in the recent submitted preprint [5], a sufficient condition for the order topology to be a module topology was provided for ordered modules over unital ordered rings. Finally, the very recently submitted preprint [6] contains a sufficient condition that assures that the order topology be a group topology in ordered groups.

The main goal of this manuscript is to construct the order topology in general posets (not necessarily endowed with an algebraic structure) providing a stronger foundation for the order topology in partially ordered algebraic structures given in [3, 5, 6].

2 Topological posets

Given a set X, the collection of finite intersections of a nonempty subset $\mathcal{S}$ of $\mathcal{P}(X)$, $\mathcal{B}(\mathcal{S}) := \left\{\bigcap_{T\in\mathcal{T}} T : \mathcal{T} \subseteq \mathcal{S} \text{ finite}\right\}$, is closed under finite intersections. Therefore, a necessary and sufficient condition for $\mathcal{B}(\mathcal{S})$ to be a base for a topology on X is that $\bigcup_{S\in\mathcal{S}} S = X$.

Notation 1. *If X is a poset and $x \in X$, then the closed rays are denoted by $\uparrow x := [x, \infty) := \{y \in X : x \leq y\}$ and $\downarrow x := (-\infty, x] := \{y \in X : y \leq x\}$,* *and the open rays are denoted by* $\uparrow_\times x := (x, \infty) := \{y \in X : x < y\}$ *and* $\downarrow^\times x := (-\infty, x) := \{y \in X : y < x\}$.

The classical notation for bounded intervals will also be employed. The following theorem provides ground for the definition of the order topology and extends [3, Theorem 2.15].

Theorem 1. *Let* A *be a poset. The collection* $\{\uparrow_\times a : a \in A\} \cup \{\downarrow^\times a : a \in A\} \cup \{A\}$ *forms a subbase for a topology on* A. *A necessary and sufficient condition for the collection* $\{\uparrow_\times a : a \in A\} \cup \{\downarrow^\times a : a \in a\}$ *to form a subbase for a topology on* A *is that for every* $a \in A$ *either* $\uparrow_\times a \neq \varnothing$ *or* $\downarrow^\times a \neq \varnothing$.

Proof. It is clear that $\{\uparrow_\times a : a \in A\} \cup \{\downarrow^\times a : a \in A\} \cup \{A\}$ is a subbase for a topology on A. Next, observe that $\{\uparrow_\times a : a \in A\} \cup \{\downarrow^\times a : a \in A\}$ is a subbase for a topology on A if and only if $\bigcup_{a \in A} \uparrow_\times a \cup \bigcup_{a \in A} \downarrow^\times a = A$, if and only if for every $a \in A$ there exists $b \in A$ such that either $a \in \uparrow_\times b$ or $a \in \downarrow^\times b$, if and only if for every $a \in A$ there exists $b \in A$ such that either $b \in \downarrow^\times a$ or $b \in \uparrow_\times a$. $\square$

Theorem 1 makes possible the definition of order topology given in [3], giving birth to the notion of topological poset.

Definition 1 (Order topology [3])**.** *Let* X *be a poset such that for every* $x \in X$ *either* $\uparrow_\times x \neq \varnothing$ *or* $\downarrow^\times x \neq \varnothing$. *The order topology on* X *is defined as the topology generated by the subbase* $\{\uparrow_\times x : x \in X\} \cup \{\downarrow^\times x : x \in X\}$. *And we call* X *a topological poset.*

Theorem 1 makes also possible to construct the order topology in those posets X for which there exists an element x incomparable with all the rest, that is, satisfying that $\uparrow_\times x = \downarrow^\times x = \varnothing$. The next definition contains the previous one.

Definition 2 (Order topology)**.** *Let* X *be a poset. The order topology on* X *is defined as the topology generated by the subbase* $\{\uparrow_\times x : x \in X\} \cup \{\downarrow^\times x : x \in X\} \cup \{X\}$. *And we call* X *a topological poset.*

If X is a topological space endowed with the trivial topology, then the trivial ordering on X given by $x \leq y \Leftrightarrow x = y$ satisfies that its induced order topology is precisely the trivial topology. Something similar occurs with the discrete topology.

Theorem 2. *If* X *is a nonempty set, then there exists an ordering on* X *whose induced order topology is the discrete topology.*

Proof. If $X = \{x_1, \ldots, x_n\}$ is finite, then it suffices to establish the total order $x_1 < \cdots < x_n$. Suppose that X is infinite. Let us write $X = \dot{\bigcup}_{i \in I} X_i$ as a disjoint union of infinitely countable subsets. For each $i \in I$, let us write $X_i = \{x_{ij} : j \in \mathbb{N}\}$ endowed with a total order such as $x_{i1} < \cdots < x_{in} < \cdots$. Consider the partial ordering on X given by $x \leq y \Leftrightarrow \exists i \in I$ such that $x, y \in X_i$ and $x \leq y$ in X_i. Finally, it only suffices to observe that, for every $i \in I$, $\{x_{i1}\} = \downarrow^{\times} x_{i2}$ and $\{x_{ij}\} = \uparrow_{\times} x_{ij-1} \cap \downarrow^{\times} x_{ij+1}$ for each $j > 1$. $\square$

Let us next display a counterintuitive, although trivial, example showing that the relative order topology in a subset does not always coincide with the order topology of the subset.

Example 1. *Let $X := \mathbb{R}$ endowed with its usual order topology. Let $Y := \{0\} \cup (2, 3)$. Let τ_1 denote the order topology of Y as a totally ordered set. Let τ_2 denote the relative topology of Y (inherited from the order topology of X). Note that $\tau_1 \subseteq \tau_2$. However, $\tau_1 \neq \tau_2$. Indeed, $\{0\} \in \tau_2$ because $\{0\} = Y \cap (-1, 1)$. However, $\{0\} \notin \tau_1$ since $\{(-\infty, y) : y \in (2, 3)\}$ is a base of neighborhoods of 0 in τ_1.*

The following proposition unveils a sufficient condition to guarantee that the relative order topology of a subset coincide with its own order topology. It is worth recalling that if $\mathcal{S}_1, \mathcal{S}_2$ are subbases for topologies τ_1, τ_2 in a set Y and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\mathcal{B}(\mathcal{S}_1) \subseteq \mathcal{B}(\mathcal{S}_2)$, hence $\tau_1 \subseteq \tau_2$.

Proposition 1 (Relative order topology)**.** *Let X be a topological poset. Let Y be a subset of X. Then:*

1. *The order topology of Y is coarser than its inherited order topology from X.*

2. *If, for all $y \in Y$, $\uparrow_{\times} y \cap Y$ is coinitial in $\uparrow_{\times} y$ and $\downarrow^{\times} y \cap Y$ is cofinal in $\downarrow^{\times} y$, then the order topology of Y coincides with its inherited order topology from X.*

Proof.

1. The subbase of open rays within Y together with Y, which generates its order topology, is clearly contained in

$$\{\uparrow_{\times} x \cap Y : x \in X\} \cup \left\{\downarrow^{\times} x \cap Y : x \in X\right\} \cup \{X \cap Y\},$$

which is the subbase of the inherited order topology from X. As a consequence, the order topology of Y is coarser than its inherited order topology from X.

2. Let U be an open subset in the order topology of X such that $U \cap Y \neq \varnothing$ and fix an arbitrary $y \in U \cap Y$. Without loss of generality, there can be found $x_1, \ldots, x_p, x_{p+1}, \ldots, x_q \in X$ such that

$$y \in \left(\uparrow_\times x_1 \cap \cdots \cap \uparrow_\times x_p \cap \downarrow^\times x_{p+1} \cap \cdots \cap \downarrow^\times x_q\right) \cap Y \subseteq U \cap Y.$$

Then $x_i < y < x_j$ for each $i \in \{1, \ldots, p\}$ and each $j \in \{p+1, \ldots, q\}$. By hypothesis, $\uparrow_\times y \cap Y$ and $\downarrow^\times y \cap Y$ are coinitial and cofinal in $\uparrow_\times y$ and $\downarrow^\times y$, respectively. Thus, for every $i \in \{1, \ldots, p\}$ and every $j \in \{p+1, \ldots, q\}$, there can be found $y_i, y_j \in Y$ such that $y < y_j \leq x_j$ and $x_i \leq y_i < y$. As a consequence,

$$\begin{aligned} y &\in (\uparrow_\times y_1 \cap Y) \cap \cdots \cap (\uparrow_\times y_p \cap Y) \cap \left(\downarrow^\times y_{p+1} \cap Y\right) \cap \cdots \cap \left(\downarrow^\times y_q \cap Y\right) \\ &\subseteq \left(\uparrow_\times x_1 \cap \cdots \cap \uparrow_\times x_p \cap \downarrow^\times x_{p+1} \cap \cdots \cap \downarrow^\times x_q\right) \cap Y \\ &\subseteq U \cap Y. \end{aligned}$$

This means that $U \cap Y$ is open in the order topology of Y.

□

A direct consequence of Proposition 1(1) is the fact that if the order topology of Y is the discrete topology or its inherited order topology from X is the trivial topology, then the order topology of Y coincides with its inherited order topology from X. Next corollary shows that dense subsets satisfy the hypothesis of Proposition 1(2) in topological posets free of holes.

Corollary 1. *Let X be a topological poset. If X is holefree and Y is a dense subset of X, then the order topology of Y coincides with its inherited order topology from X.*

Proof. Fix an arbitrary $y \in Y$. We will show that $\uparrow_\times y \cap Y$ is coinitial in $\uparrow_\times y$. Indeed, take any $x \in \uparrow_\times y$. Then $y < x$ and by hypothesis $(y, x) \neq \varnothing$ because X is free of holes. Thus, (y, x) is a nonempty open subset of X which must intersect Y due to its density. As a consequence, there exists $y' \in (y, x) \cap Y$. Finally, notice that $y' < x$ and $y' \in \uparrow_\times y \cap Y$. This shows that $\uparrow_\times y \cap Y$ is coinitial in $\uparrow_\times y$. In a dual way, it can be shown that $\downarrow^\times y \cap Y$ is cofinal in $\downarrow^\times y$. □

The following technical lemma improves [4, Lemma 3.3] as well as mirrors how to construct a base of neighborhoods of a point for the order topology on the real line.

Lemma 1. *Let X be a topological poset. Let $x_0 \in X$. Then:*

1. *If* $\uparrow_{\times} x_0 \neq \varnothing$ *is downward directed and* $\downarrow^{\times} x_0 \neq \varnothing$ *is upward directed, then* $\{(x, y) : x < x_0 < y\}$ *is a base of neighborhoods of* x_0.
2. *If* $\uparrow_{\times} x_0 \neq \varnothing$ *is downward directed and* $\downarrow^{\times} x_0 = \varnothing$, *then* $\{(-\infty, y) : x_0 < y\}$ *is a base of neighborhoods of* x_0.
3. *If* $\uparrow_{\times} x_0 = \varnothing$ *and* $\downarrow^{\times} x_0 \neq \varnothing$ *is upward directed, then* $\{(x, \infty) : x < x_0\}$ *is a base of neighborhoods of* x_0.
4. *If* $\uparrow_{\times} x_0 = \varnothing$ *and* $\downarrow^{\times} x_0 = \varnothing$, *then* $\{X\}$ *is a base of neighborhoods of* x_0.

Proof. Let $W \subseteq X$ be an x_0-neighborhood for the order topology. We will distinguish between the four cases above:

1. There can be found $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $x_0 \in \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$. Observe that $a_i < x_0$ and $b_j > x_0$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Since $\downarrow^{\times} x_0$ is upward directed, there exists $a_0 \in \downarrow^{\times} x_0$ such that $a_0 \geq a_i$ for all $i \in \{1, \ldots, n\}$. Similarly, since $\uparrow_{\times} x_0$ is downward directed, there exists $b_0 \in \uparrow_{\times} x_0$ such that $b_0 \leq b_j$ for all $j \in \{1, \ldots, m\}$. Finally, notice that $x_0 \in (a_0, b_0) \subseteq \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$.
2. There can be found $b_1, \ldots, b_m \in X$ satisfying that $x_0 \in \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$. Observe that $b_j > x_0$ for all $j \in \{1, \ldots, m\}$. Since $\uparrow_{\times} x_0$ is downward directed, there exists $b_0 \in \uparrow_{\times} x_0$ such that $b_0 \leq b_j$ for all $j \in \{1, \ldots, m\}$. Finally, notice that $x_0 \in (-\infty, b_0) \subseteq \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$.
3. There can be found $a_1, \ldots, a_n \in X$ satisfying that $x_0 \in \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \subseteq W$. Observe that $a_i < x_0$ for all $i \in \{1, \ldots, n\}$. Since $\downarrow^{\times} x_0$ is upward directed, there exists $a_0 \in \downarrow^{\times} x_0$ such that $a_0 \geq a_i$ for all $i \in \{1, \ldots, n\}$. Finally, notice that $x_0 \in (a_0, \infty) \subseteq \uparrow_{\times} a_1 \cap \cdots \cap \uparrow_{\times} a_n \subseteq W$.
4. In this situation, x is incomparable with all the rest elements of X, hence X is the only open set containing x.

□

Observe that open rays are open in the order topology, and, consequently, open bounded intervals are open as well in the order topology. However, closed rays are not necessarily closed in the order topology. If the order is total, then closed rays are closed in the order topology because their complementary is an open ray, and, consequently, closed bounded intervals are closed as well in the order topology. Another sufficient condition for closed rays to be closed in the order topology was unveiled in [4, Lemma 3.2], result that is included (and refined) next for the sake of completion. Recall that in a poset X, the subset of incomparable elements to a certain $x \in X$ is denoted by θ_x. Recall as well that a poset X is said to be cofinal in itself provided that $X = \bigcup_{x \in X} \downarrow^{\times} x$. Dually, coinitial in itself can be defined.

Lemma 2. *Let X be a topological poset. Let $x \in X$:*

1. *If θ_x is open or cofinal in itself, then $\uparrow x$ is closed.*
2. *If θ_x is open or coinitial in itself, then $\downarrow x$ is closed.*

Proof. Only the first item will be proved and we will only assume that θ_x is cofinal in itself. Notice that $X \setminus \uparrow x = \downarrow^{\times} x \cup \theta_x$. We will show that $\downarrow^{\times} x \cup \theta_x$ is open by showing that it is a neighborhood of each of its points. Indeed, fix an arbitrary $y \in \downarrow^{\times} x \cup \theta_x$. If $y \in \downarrow^{\times} x$, then $y \in \downarrow^{\times} x \subseteq \downarrow^{\times} x \cup \theta_x$, so $\downarrow^{\times} x \cup \theta_x$ is a neighborhood of y by definition of order topology. Next, assume that $y \in \theta_x$. By hypothesis, θ_x is cofinal in itself, meaning that there exists $z \in \theta_x$ such that $y < z$. Let us prove that $y \in \downarrow^{\times} z \subseteq \downarrow^{\times} x \cup \theta_x$. Take any $w \in \downarrow^{\times} z$. If w is not comparable to x, then $w \in \theta_x$. If w is comparable to x, then we have two options. One is that $x \leq w$, which implies that $x \leq w < z$ contradicting the fact z is not comparable to x. This only leaves the other option, that is, $w < x$, meaning that $w \in \downarrow^{\times} x$. As a consequence, $y \in \downarrow^{\times} z \subseteq \downarrow^{\times} x \cup \theta_x$. This shows that $X \setminus \uparrow x = \downarrow^{\times} x \cup \theta_x$ is open because it is a neighborhood of each of its points. □

Next technical lemma refines and compiles [4, Lemma 3.4] and [4, Lemma 3.5] together. Recall that a hole in a poset is an empty nontrivial open interval such as $(x, y) = \varnothing$ for some $x < y$.

Lemma 3. *Let X be a topological poset. Let $a, b \in X$. Then:*

1. *If $\uparrow_{\times} b \neq \varnothing$ is downward directed and $\uparrow b$ is hole free, then $b \notin \operatorname{int}(\downarrow b)$. In particular, if $a < b$, then $b \notin \operatorname{int}([a, b])$.*
2. *If $\downarrow^{\times} a \neq \varnothing$ is upward directed and $\downarrow a$ is hole free, then $a \notin \operatorname{int}(\uparrow a)$. In particular, if $a < b$, then $a \notin \operatorname{int}([a, b])$.*
3. *Suppose $a < b$. If $\downarrow^{\times} b$ is upward directed and $\downarrow b$ is hole free, then $b \in \operatorname{cl}((a, b)) \subseteq \operatorname{cl}(\downarrow^{\times} b)$.*

4. Suppose $a < b$. If $\uparrow_\times a$ is downward directed and $\uparrow a$ is hole free, then $a \in \mathrm{cl}((a,b)) \subseteq \mathrm{cl}(\uparrow_\times a)$.

Proof. Only the first and third items will be proved.

1. Assume on the contrary that $b \in \mathrm{int}(\downarrow b)$. There exists $W \subseteq X$ a b-neighborhood for the order topology contained in $\downarrow b$. There are three possibilities:

   - There are $a_1, \ldots, a_n \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W$. Then $\uparrow_\times b \subseteq \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W \subseteq \downarrow b$, which is a contradiction.
   - There are $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. By hypothesis, $\uparrow_\times b \neq \varnothing$ and is downward directed, thus there exists $b_0 \in \uparrow_\times b$ such that $b_0 \leq b_j$ for each $j = 1, \ldots, m$. Again, by hypothesis, $\uparrow b$ is hole free, meaning that there exists $c \in \uparrow b$ with $b < c < b_0$. Then we reach the contradiction that $c \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W \subseteq \downarrow b$.
   - There are $b_1, \ldots, b_m \in X$ satisfying that $b \in \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. By hypothesis, $\uparrow_\times b \neq \varnothing$ and is downward directed, thus there exists $b_0 \in \uparrow_\times b$ such that $b_0 \leq b_j$ for each $j = 1, \ldots, m$. Again, by hypothesis, $\uparrow b$ is hole free, meaning that there exists $c \in \uparrow b$ with $b < c < b_0$. Thus, the contradiction that $c \in \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W \subseteq \downarrow b$ is again reached.

3. Take any b-neighborhood $W \subseteq X$ for the order topology. There are three possibilities:

   - There are $a_1, \ldots, a_n \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \subseteq W$. By hypothesis, $\downarrow^\times b$ is upward directed, meaning that there exists $a_0 \in \downarrow^\times b$ such that $a \leq a_0$ and $a_i \leq a_0$ for each $i = 1, \ldots, n$. Again, by hypothesis, $\downarrow b$ is hole free, therefore there exists $c \in \downarrow b$ with $a_0 < c < b$. Finally, $c \in (a,b) \cap \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n$.
   - There are $a_1, \ldots, a_n, b_1, \ldots, b_m \in X$ satisfying that $b \in \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m \subseteq W$. By hypothesis, $\downarrow^\times b$ is upward directed, meaning that there exists $a_0 \in \downarrow^\times b$ such that $a \leq a_0$ and $a_i \leq a_0$ for each $i = 1, \ldots, n$. Again, by hypothesis, $\downarrow b$ is hole free, therefore there exists $c \in \downarrow b$ with $a_0 < c < b$. Finally, $c \in (a,b) \cap \uparrow_\times a_1 \cap \cdots \cap \uparrow_\times a_n \cap \downarrow^\times b_1 \cap \cdots \cap \downarrow^\times b_m$.

- There are $b_1, \ldots, b_m \in X$ satisfying that $b \in \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m \subseteq W$. Since $\downarrow b$ is hole free, we can find $c \in \downarrow b$ with $a < c < b$. Then $c \in (a,b) \cap \downarrow^{\times} b_1 \cap \cdots \cap \downarrow^{\times} b_m$.

□

Lemma 1 and Lemma 3 motivate the following definition.

Definition 3 (Locally directed poset)**.** *A poset X is said to be:*

- *locally directed whenever that, for every $x \in X$, if $\uparrow_{\times} x \neq \varnothing$, then it is downward directed, and if $\downarrow^{\times} x \neq \varnothing$, then it is upward directed.*
- *strongly locally directed whenever that, for every $x \in X$, $\uparrow_{\times} x \neq \varnothing$ and is downward directed, and $\downarrow^{\times} x \neq \varnothing$ and is upward directed.*

By relying on the previous definition and by combining together Lemma 2 and Lemma 3, we obtain the following result.

Lemma 4. *Let X be a topological poset. Suppose that X is holefree and locally directed. Let $x < y$ in X be such that either θ_x and θ_y are open in X or θ_x is cofinal in itself and θ_y is coinitial in itself. Then:*

1. $\operatorname{cl}(\uparrow_{\times} x) = \uparrow x$, $\operatorname{cl}(\downarrow^{\times} y) = \downarrow y$, *and* $\operatorname{cl}((x,y)) = \operatorname{cl}((x,y]) = \operatorname{cl}([x,y)) = [x,y]$.
2. *Assume* $\downarrow^{\times} x \neq \varnothing$ *and* $\uparrow_{\times} y \neq \varnothing$. *Then* $\operatorname{int}(\uparrow x) = \uparrow_{\times} x$, $\operatorname{int}(\downarrow y) = \downarrow^{\times} y$, *and* $\operatorname{int}([x,y]) = \operatorname{int}((x,y]) = \operatorname{int}([x,y)) = (x,y)$.

Proof. According to Lemma 2, $\uparrow x, \downarrow y$ are closed, therefore, $[x,y]$ is closed as well. Then $(x,y) \subseteq \operatorname{cl}((x,y)) \subseteq [x,y]$. In virtue of Lemma 3(3,4), $\operatorname{cl}(\uparrow_{\times} x) = \uparrow x$, $\operatorname{cl}(\downarrow^{\times} y) = \downarrow y$ and $\operatorname{cl}((x,y)) = \operatorname{cl}((x,y]) = \operatorname{cl}([x,y)) = [x,y]$. Next, $\uparrow_{\times} x, \downarrow^{\times} y$ are open, hence (x,y) is open as well, so $(x,y) \subseteq \operatorname{int}([x,y]) \subseteq [x,y]$. Assume $\downarrow^{\times} x \neq \varnothing$ and $\uparrow_{\times} y \neq \varnothing$. In view of Lemma 3(1,2), $\operatorname{int}(\uparrow x) = \uparrow_{\times} x$, $\operatorname{int}(\downarrow y) = \downarrow^{\times} y$ and $\operatorname{int}([x,y]) = \operatorname{int}((x,y]) = \operatorname{int}([x,y)) = (x,y)$. □

3 Separation properties of topological posets

Let us continue our study of the order topology by unveiling its most basic separation properties. Recall that the set of isolated points of a topological space X is commonly denoted by $\operatorname{iso}(X)$.

Proposition 2. *Let X be a topological poset. Let $x \neq y$ in X. Then:*

1. *If $x < y$, then x, y can be separated by disjoint open sets. In particular, if X is totally ordered, then it is Hausdorff.*

2. *If X is locally directed and hole free, then* $\mathrm{iso}(X) = \varnothing$.

3. *If $X \setminus \{x, y\}, \downarrow^{\times} x, \downarrow^{\times} y$ are upward directed and $\uparrow_{\times} x = \uparrow_{\times} y = \varnothing$, then x, y cannot be separated by disjoint open sets. Hence X is not Hausdorff.*

Proof.

1. If $(x, y) = \varnothing$, then $y \in (x, \infty)$, $x \in (-\infty, y)$, and $(-\infty, y) \cap (x, \infty) = (x, y) = \varnothing$. If there exists $z \in X$ with $x < z < y$, then $x \in (-\infty, z)$, $y \in (z, \infty)$, and $(-\infty, z) \cap (z, \infty) = \varnothing$.

2. Assume that $x \in \mathrm{iso}(X)$. Then $\{x\}$ is open. According to Lemma 1, $\{x\}$ is of the form (a, b) or $(-\infty, a)$ or $(b, +\infty)$. In any of the three prior cases, we reach the contradiction that X has a hole.

3. First off, note that the fact that $\uparrow_{\times} x = \uparrow_{\times} y = \varnothing$ implies that x, y are incomparable. Fix arbitrary neighborhoods U_x, U_y of x, y, respectively, in the order topology. In accordance with Lemma 1(3), there exist $x_0 \in \downarrow^{\times} x$ and $y_0 \in \downarrow^{\times} y$ such that $x \in (x_0, \infty) \subseteq U_x$ and $y \in (y_0, \infty) \subseteq U_y$. Since $X \setminus \{x, y\}$ is upward directed, there exists $z \in X$ with $z \geq x_0$ and $z \geq y_0$. At this point, let us distinguish between the following cases:

   - $z = x_0$ and $z \neq y_0$. Then $x_0 \geq y_0$ so $x \in (x_0, \infty) \subseteq (x_0, \infty) \cap (y_0, \infty) \subseteq U_x \cap U_y$.
   - $z \neq x_0$ and $z = y_0$. Then $y_0 \geq x_0$ so $y \in (y_0, \infty) \subseteq (x_0, \infty) \cap (y_0, \infty) \subseteq U_x \cap U_y$.
   - $z \neq x_0, y_0$. Then $z \in (x_0, \infty) \cap (y_0, \infty) \subseteq U_x \cap U_y$.

□

Next theorem characterizes separability of the order topology in locally directed holefree posets. This result is known in the classic literature for totally ordered sets.

Theorem 3. *Let X be a locally directed holefree topological poset. Then X is separable if and only if X is second countable.*

Proof. Every second countable topological space is separable. As a consequence, let us suppose that X is separable. Let D be a countable and dense subset of X. We claim that $\mathcal{B}_D := \{(-\infty, d) : d \in D\} \cup \{(c, d) : c < d \text{ in } D\} \cup \{(c, +\infty) : c \in D\} \cup \{X\}$ is a countable base of the order topology of X. Notice that $\mathcal{B}_D$ is countable. Take an arbitrary nonempty open subset V of X and an arbitrary $v \in V$. According to Lemma 1, there exists an interval of the form $(a, b), (-\infty, b), (a, +\infty), (-\infty, \infty)$ containing v and contained in V. We will distinguish between the following four cases:

- $v \in (a, b) \subseteq V$. Since X has no holes, $(a, v), (v, b) \neq \varnothing$, hence the density of D allows $c \in (a, v), d \in (v, b)$. Then $v \in (c, d) \subseteq (a, b) \subseteq V$.
- $v \in (-\infty, b) \subseteq V$. Since X has no holes, $(v, b) \neq \varnothing$, hence the density of D allows $d \in (v, b)$. Then $x \in (-\infty, d) \subseteq (-\infty, b) \subseteq V$.
- $v \in (a, +\infty) \subseteq V$. Since X has no holes, $(a, v) \neq \varnothing$, hence the density of D allows $c \in (a, v)$. Then $v \in (c, +\infty) \subseteq (a, +\infty) \subseteq V$.
- $v \in (-\infty, \infty) \subseteq V$. Then $v \in X = (-\infty, \infty) \subseteq V$.

□

Recall that a poset is said to have the sup (inf) property if every bounded above (below) subset has a supremum (infimum). At this stage, it is worth recalling that a poset enjoys the sup property if and only if it satisfies the inf property. Our next step is to introduce a topological version of this property. But first, a technical lemma is needed.

Lemma 5. *Let X be a topological poset. Let $A \subseteq X$. If x is an upper bound of A, $y \in \mathrm{cl}(A)$, and x and y are comparable, then $y \leq x$. In particular, if X is totally ordered, then $\mathrm{cl}(A)$ is bounded above if so is A, and $\sup(A)$ exists if and only if $\sup(\mathrm{cl}(A))$ exists, in which case $\sup(A) = \sup(\mathrm{cl}(A)) \in \mathrm{cl}(A)$.*

Proof. Suppose on the contrary that $x < y$. Then $\uparrow_\times x$ is an open neighborhood of y, meaning that $\uparrow_\times x \cap A \neq \varnothing$. As a consequence, there exists $a \in A$ with $x < a$, which contradicts the fact that x is an upper bound of A. □

Lemma 5 has its dual version for lower bounds and infimums. It also motivates the following definition.

Definition 4 (Strong sup (inf) property)**.** *A topological poset X is said to have the strong sup (inf) property provided that it verifies the sup (inf) property and* $\sup(A) \in \mathrm{cl}(A)$ $(\inf(A) \in \mathrm{cl}(A))$ *for every bounded above (below) subset $A \subseteq X$.*

An immediate consequence of Lemma 5 is that every topological toset with the sup (inf) property enjoys the strong sup (inf) property.

Theorem 4. *Let X be a topological poset. If X is upward directed and satisfies the strong sup property, then X is totally ordered.*

Proof. Fix arbitrary elements $x_1 \neq x_2$ in X. We will show that they are comparable. Since X is upward directed, there exists $x_3 \in X$ such that $x_i \leq x_3$ for $i = 1, 2$. Then $\{x_1, x_2\}$ is bounded above, hence there exists $x_1 \vee x_2$ and $x_1 \vee x_2 \in \mathrm{cl}(\{x_1, x_2\})$. We distinguish now between two cases:

- $x_1 \vee x_2 = x_1$. In this case, $x_2 \leq x_1 \vee x_2 = x_1$.
- $x_1 \vee x_2 \neq x_1$. In this case, $x_1 \vee x_2 \in \uparrow_\times x_1$. Then $\uparrow_\times x_1$ is a neighborhood of $x_1 \vee x_2$ in the order topology, so $\uparrow_\times x_1 \cap \{x_1, x_2\} \neq \varnothing$, meaning that $x_1 < x_2$.

□

As expected, the previous theorem has its dual version for downward directed topological posets enjoying the strong inf property. The following result characterizes connectedness of the order topology in totally ordered sets. This result is well known in the classic literature, but we decided to include it here for the sake of completeness.

Theorem 5. *A necessary and sufficient condition for a topological toset X to be connected is that X have no holes and possess the sup property.*

Proof. Let us suppose first that X is connected. Let us prove that X has no holes. Assume on the contrary that we can find $x, y \in X$ with $x < y$ such that $(x, y) = \varnothing$. Observe that $\{(-\infty, y), (x, \infty)\}$ is a partition of X in nonempty open sets, which contradicts the fact that X is connected. Next, we will prove that X enjoys the sup property. Again, suppose on the contrary that there can be found $Y \subseteq X$ bounded above in such a way that $\sup(Y)$ does not exist. Define $U := \bigcup_{y \in Y} (-\infty, y)$ and $V := \bigcup \{(z, +\infty) : z \text{ is an upper bound of } Y\}$. Observe that U, V are open as well as disjoint. We will show now that $U, V \neq \varnothing$ and $U \cup V = X$. If $U = \varnothing$, then Y is a singleton, hence $\sup(Y)$ exists. Therefore, $U \neq \varnothing$. In case $V = \varnothing$, then there exists only one upper bound of Y, hence $\sup(Y)$ exists. Thus, $V \neq \varnothing$. Finally, let us show that $X = U \cup V$. Take an arbitrary $x_0 \in X$. If $x_0 \notin U$, then x_0 is an upper bound for Y. If, in addition, $x_0 \notin V$, then $x_0 \leq z$ for every upper bound z of Y, meaning that $x_0 = \sup(Y)$, which is impossible by assumption. Therefore, $x_0 \in U \cup V$. The arbitrariness of $x_0 \in X$ guarantees that $X = U \cup V$. The contradiction that X is not connected has been reached. Conversely, suppose that X does not have any hole and possesses the sup property. Let $\{U, V\}$ be a partition of X in nonempty open sets. Fix $u_0 \in U, v_0 \in V$. Let us assume, without any loss of generality, that $u_0 < v_0$. Define $U_0 := \{u \in U : u \leq v_0\} = U \cap (-\infty, v_0]$. Notice that U_0 is closed, $u_0 \in U_0$ and v_0 is an upper bound for U_0. By hypothesis, $\sup(U_0)$ exists, and $\sup(U_0) \leq v_0$. In virtue of Lemma 5, $\sup(U_0) \in \mathrm{cl}(U_0) = U_0$. Since $U \cap V = \varnothing$ and $\sup(U_0) \in U$, we have that $\sup(U_0) \notin V$, hence $\sup(U_0) < v_0$. Since X is free of holes, $(\sup(U_0), v_0) \neq \varnothing$. We will prove next that $(\sup(U_0), v_0) \subseteq V$. Take any $x \in (\sup(U_0), v_0)$. Since $X = U \cup V$, if $x \notin V$, then $x \in U$. Since $x < v_0$, we conclude that $x \in U_0$, which is not possible because $x > \sup(U_0)$. As a consequence, $(\sup(U_0), v_0) \subseteq V$. Observe that V is closed, meaning, by bearing in mind Lemma 4, that $\sup(U_0) \in [\sup(U_0), v_0] = \mathrm{cl}\left((\sup(U_0), v_0)\right) \subseteq \mathrm{cl}(V) = V$, which is impossible. □

4 Side topological posets

Our next step is to introduced the side order topologies, which allow the conception of lower and upper semicontinuity.

Definition 5 (Side order topologies)**.** *Let X be a poset. The right order topology on X is defined as the topology generated by the subbase of upper open rays $\{\uparrow_\times x : x \in X\} \cup \{X\}$ and we call X a right topological poset. Dually, the left order topology on X is defined as the topology generated by the subbase of lower open rays $\{\downarrow^\times x : x \in X\} \cup \{X\}$ and we call X called a left topological poset.*

Notice that both the left and right order topologies are coarser than the order topology.

Proposition 3. *Let X be a left topological toset. If $K \subseteq X$ is compact, then there exists* $\max(K)$.

Proof. Consider the family $\{\uparrow k \cap K : k \in K\}$ of closed subsets of K. Let us prove that the previous family satisfies the finite intersection property. Fix arbitrary elements $k_1, \ldots, k_p$ in K. Since K is upward directed, there exists $k_0 \in K$ such that $k_0 \geq k_j$ for all $j = 1, \ldots, p$. Then $k_0 \in \bigcap_{j=1}^{p} (\uparrow k_j \cap K) \neq \varnothing$. The compacity of K allows the existence of $k' \in \bigcap_{k \in K} (\uparrow k \cap K)$. Notice that $k' = \max(K)$. □

The following corollary is a direct consequence of Proposition 3, which has its dual version for the right order topology.

Corollary 2. *If X is a topological toset and $K \subseteq X$ is compact, then there exist* $\max(K)$ *and* $\min(K)$.

Proof. On the one hand, K is closed because the order topology is Hausdorff in totally ordered sets. On the other hand, K is compact as well for the left and right order topologies, since the side order topologies are coarser than the order topology. Therefore, by Proposition 3 and its dual version, there exist $\max(K)$ and $\min(K)$. □

The notion of semicontinutiy can now be transported to posets by means of the side order topologies.

Definition 6 (Semicontinuous map)**.** *Let X be a topological space. Let Y be a poset. A function $f : X \to Y$ is called upper (lower) semicontinuous at $x \in X$ whenever f is continuous at x when Y is endowed with the left (right) order topology. The function f is said to be upper (lower) semicontinuous if so is f at each $x \in X$.*

The following remark shows that maximal and minimal elements are always points of upper semicontinuity and lower semicontinuity, respectively.

Remark 1. *Let X be a topological space. Let Y be a poset. Let $f : X \to Y$ be a function. Let $x_0 \in X$. If $f(x_0)$ is maximal, then f is upper semicontinuous at x_0. Indeed, since $\uparrow_\times f(x_0) = \varnothing$, the only open set of the left order topology containing $f(x_0)$ is Y. If U is any neighborhood of x_0, then $f(U) \subseteq Y$. Dually, if $f(x_0)$ is minimal, then f is lower semicontinuous at x_0.*

As expected, semicontinuity is preserved by taking infs and sups of families of functions.

Lemma 6. *Let X be a topological space. Let Y be a topological poset satisfying the strong sup property. Let $\{f_i\}_{i\in I}$ be a family of functions from X to Y. Then:*

1. *If $\{f_i\}_{i\in I}$ is pointwise bounded below and f_i is upper semicontinuous for all $i \in I$, then $\inf_{i\in I} f_i$ is upper semicontinuous.*
2. *If $\{f_i\}_{i\in I}$ is pointwise bounded above and f_i is lower semicontinuous for all $i \in I$, then $\sup_{i\in I} f_i$ is lower semicontinuous.*

Proof. We will prove only the second item. Denote $f := \sup_{i\in I} f_i$. Let V be a neighborhood of $f(x_0)$ in Y for the right order topology. We will find a neighborhood U of $x_0 \in X$ such that $f(U) \subseteq V$. By construction of the right order topology, if $\downarrow^{\times} f(x_0) = \varnothing$, then $V = Y$, hence we can take $U := X$. If $\downarrow^{\times} f(x_0) \neq \varnothing$, then there are $y_1, \dots, y_n \in Y$ satisfying that $f(x_0) \in \uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n \subseteq V$. By hypothesis, $f(x_0) \in \mathrm{cl}(\{f_i(x_0) : i \in I\})$ in the order topology, meaning that $(\uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n) \cap \{f_i(x_0) : i \in I\} \neq \varnothing$, that is, there exists $j \in I$ such that $y_k < f_j(x_0)$ for all $k = 1, \dots, n$. Since f_j is lower semicontinuous at x_0, there exists a neighborhood U of x_0 such that $f_j(U) \subseteq \uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n$. Finally, $y_k < f_j(u) \leq f(u)$ for all $u \in U$, meaning that $f(U) \subseteq \uparrow_{\times} y_1 \cap \cdots \cap \uparrow_{\times} y_n \subseteq V$. □

What comes up next is a technical remark that allows to compute global sups. And, obviously, it has its dual version for infs.

Remark 2. *Let X be a set. Let Y be a poset. Let $\{f_i\}_{i\in I}$ be a pointwise bounded above family of functions from X to Y such that $\sup_{i\in I} f_i(x)$ exists for all $x \in X$. Let $f := \sup_{i\in I} f_i$. If $\sup f_i(X)$ exists for all $i \in I$ and $\sup\{\sup f_i(X) : i \in I\}$ exists as well, then $\sup f(X)$ exists and $\sup f(X) = \sup\{\sup f_i(X) : i \in I\}$. Indeed, denote $y := \sup\{\sup f_i(X) : i \in I\}$. For an arbitrarily fixed $x \in X$, $y \geq \sup f_i(X) \geq f_i(x)$ for all $i \in I$, thus y is an upper bound for $\{f_i(x) : i \in I\}$, hence $y \geq f(x)$. Then y is an upper bound for $f(X)$. Let $z \in Y$ be another upper bound for $f(X)$. Fix an arbitrary $i \in I$. For every $x \in X$, $z \geq f(x) \geq f_i(x)$. This means that z is an upper bound of $f_i(X)$, meaning that $\sup f_i(X) \leq z$. Therefore, z is an upper bound of $\{\sup f_i(X) : i \in I\}$, resulting in $y \leq z$. This shows that $y = \sup f(X)$.*

Notice that, in a topological toset X, $\sup(A) = \sup(\mathrm{cl}(A))$ for each subset $A \subseteq X$ whenever those sups exist (Lemma 5). Also, it is worth mentioning that every continuous map $f : X \to Y$ between topological spaces X, Y satisfies that $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ for each subset $A \subseteq X$.

Lemma 7. *Let X be a topological space. Let Y be a topological toset. Let $A \subseteq X$. Let $f : X \to Y$ be a lower semicontinuous function. Then $\sup f(A)$ exists if and only if $\sup f(\mathrm{cl}(A))$ exists and, in this situation, $\sup f(A) = \sup f(\mathrm{cl}(A))$.*

Proof. Suppose first that $\sup f(A)$ exists. If y is an upper bound for $f(\mathrm{cl}(A))$, then y is also an upper bound for $f(A)$, meaning that $\sup f(A) \leq y$. Let us show that $\sup f(A)$ is a upper bound for $f(\mathrm{cl}(A))$. If not, there exists $b \in \mathrm{cl}(A)$ such that $\sup f(A) < f(b)$. Then $\uparrow_{\times} \sup f(A)$ is an open neighborhood of $f(b)$ in the right order topology. Since f is lower semicontinuous, there exists a neighborhood U of b in X such that $f(U) \subseteq \uparrow_{\times} \sup f(A)$. Since $U \cap A \neq \varnothing$, by considering any $a \in U \cap A$, we reach the contradiction that $\sup f(A) < f(a)$. This shows that $\sup f(A) = \sup f(\mathrm{cl}(A))$. Conversely, suppose that $\sup f(\mathrm{cl}(A))$ exists. Observe that $\sup f(\mathrm{cl}(A))$ is an upper bound for $f(A)$. Let $y \in Y$ be an upper bound for $f(A)$. Let us show that y is an upper bound for $f(\mathrm{cl}(A))$. If not, there exists $b \in cl(A)$ such that $y < f(b)$. Then $\uparrow_{\times} y$ is an open neighborhood of $f(b)$ in the right order topology. Since f is lower semicontinuous, there exists a neighborhood U of b in X such that $f(U) \subseteq \uparrow_{\times} y$. Since $U \cap A \neq \varnothing$, by considering any $a \in U \cap A$, we reach the contradiction that $y < f(a)$. Finally, since y is an upper bound for $f(\mathrm{cl}(A))$, we obtain that $\sup f(\mathrm{cl}(A)) \leq y$. This shows that $\sup f(\mathrm{cl}(A)) = \sup f(A)$. □

In a dual way, the previous lemma has its version for lower semicontinuity. The final theorem in this manuscript constitutes a topological generalization of the famous and classical Extreme Value Theorem.

Theorem 6 (Extreme Value Theorem)**.** *If $f : X \to Y$ is an upper (lower) semicontinuous function from a compact topological space X to a toset Y, then f is bounded above (below) and attains its maximum (minimum).*

Proof. By hypothesis, f is continuous if Y is endowed with the left order topology. As a consequence, $f(X)$ is compact in Y for the left order topology. By applying Proposition 3, there exists $\max f(X)$. □

Acknowledgment. The author was supported by Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía under grants ProyExcel00780 (Operator Theory: An interdisciplinary approach) and ProyExcel01036 (Multifísica y optimización multiobjetivo de estimulación magnética transcraneal).

References

[1] S. Dolecki, *A Royal Road to Topology-Convergence of Filters*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2024.

[2] V. Kakkar, *General Topology-Metric Spaces, Dedekind Cut, Isometries, and Hilbert Cube*, De Gruyter Graduate, De Gruyter, Berlin, 2025.

[3] F.J. García-Pacheco, M.A. Moreno-Frías and M. Murillo-Arcila, Topological ordered rings and measures, *Res. Math.* 78 (2023), 16.

[4] F.J. García-Pacheco, Unit neighborhoods of zero in topological ordered rings, *Carpathian J. Math.* 41 (2025), 951-957.

[5] F.J. García-Pacheco, M.A. Moreno-Frías, M. Murillo-Arcila and S. Rahbariyan, Topological ordered modules, *Lobachevskii J. Math.* (2025).

[6] F.J. García-Pacheco, M.A. Moreno-Frías and S. Rahbariyan, Generalized topological ordered groups, *submitted* (2025).

[7] G. Laures and M. Szymik, *A Basic Course in Topology*, Compact Textbooks in Mathematics, Birkhäuser/Springer, Berlin, 2025.

[8] R. López, *Point-Set Topology - A Working Textbook*, Springer Undergraduate Mathematics Series, Springer, Cham, 2024.

[9] J.R. Munkres, *Topology: A First Course*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

[10] J.R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000.

[11] B. Ricceri, Some topological mini-max theorems via an alternative principle for multifunctions, *Arch. Math. (Basel)* 60 (1993), 367-377.

[12] B. Ricceri, On a topological minimax theorem and its applications. In *Minimax theory and applications (Erice, 1996)*, 191-216, Nonconvex Optim. Appl., 26, Kluwer Acad. Publ., Dordrecht.