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SOME SEQUENCES DERIVED FROM THE CLASSICAL PROOF OF e'S IRRATIONALITY*

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Dedicated to Prof. Biagio Ricceri on the occasion of his 70th birthday

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Abstract

We introduce some sequences approximating the constant e, related to the sequence defining the constant e and its irrationality. The main tool for constructing those sequences is a result of Cesàro-Stolz type. Some related inequalities are given.

Keywords: factorial function, constant e, rate of convergence, inequalities.

MSC: 41A21, 26D05, 26D15, 33B10.

1 Introduction

The mathematical constant e (equal to 2.71828...), is one of the most important numbers in mathematics. It arises naturally in various contexts, especially in calculus, number theory, and complex analysis.

One of the definitions of e is through the infinite series of inverse factorials

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}\right).$$

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The constant e first appeared in the work of the Swiss mathematician Jacob Bernoulli (1655-1705) in the context of compound interest around 1683. However, it was the Swiss mathematician Leonhard Euler (1707-1783) who formally introduced the symbol e and explored its properties extensively in the 18th century. Euler demonstrated many remarkable identities involving e, including its irrationality by defining the sequence

$$e_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{\theta_n}{n!n}, \quad 0 < \theta_n < 1. \tag{1}$$

Euler's work laid the foundation for the modern understanding of exponential functions and logarithms, making e a cornerstone of mathematical analysis.

2 Main results

We consider a family of sequences x_n approximating the constant e, of the form

$$x_n = 1 + \frac{1}{1!} + ... + \frac{1}{(n-1)!} + \frac{\tau_n}{n!} - e, \tag{2}$$

where the sequence τ_n depends on some real parameters. We define and discuss a sequence x_n obtained for particular values of the parameters involved, such that x_n has the highest possible rate of convergence.

Our method is inspired by a lemma of Cesàro-Stolz type presented in [1]. Precisely, if u_n is a sequence convergent to zero such that

$$\lim_{n\to\infty} n^k (u_n - u_{n+1}) = l,$$

for some $k > 1$ and $l \neq 0$, then

$$\lim_{n\to\infty} n^{k-1} u_n = \frac{l}{k-1}.$$

This is a useful tool for calculating the rate of convergence of some sequences, or to accelerate some convergencies. Consequently, many authors used this lemma and obtained new results in recent years (see, for instance, [2]- [4]).

We give here a similar result which we will use in this paper.

Lemma 1. *Let x_n be a sequence convergent to zero such that*

$$\lim_{n\to\infty} n^p (n+k)! (x_n - x_{n+1}) = l,$$

for some $p > 0$, $k \in \mathbb{Z}$ and $l \neq 0$. Then

$$\lim_{n\to\infty} n^p (n+k)! x_n = l.$$

Proof. By using the classical Cesàro-Stolz lemma, case 0/0, we have

$$\lim_{n\to\infty} n^p\,(n+k)!x_n = \lim_{n\to\infty} \frac{x_n}{\frac{1}{n^p(n+k)!}} = \lim_{n\to\infty} \frac{x_n - x_{n+1}}{\frac{1}{n^p(n+k)!} - \frac{1}{(n+1)^p(n+1+k)!}}$$

$$= \lim_{n\to\infty} \frac{x_n - x_{n+1}}{\frac{1}{(n+k)!}\left(\frac{1}{n^p} - \frac{1}{(n+1)^{p+1}(n+1+k)}\right)}$$

$$= \lim_{n\to\infty} \frac{x_n - x_{n+1}}{\frac{1}{n^p(n+k)!}\left\{n^p\left(\frac{1}{n^p} - \frac{1}{(n+1)^{p+1}(n+1+k)}\right)\right\}}$$

$$= \lim_{n\to\infty} \frac{x_n - x_{n+1}}{\frac{1}{n^p(n+k)!}} = l.$$

Note that for the last equality we have used

$$\lim_{n\to\infty}\left\{n^p\left(\frac{1}{n^p} - \frac{1}{(n+1)^{p+1}(n+1+k)}\right)\right\}$$

$$= \lim_{n\to\infty}\left(1 - \frac{n^p}{(n+1)^{p+1}(n+1+k)}\right) = 1.$$

□

In fact, this lemma says that if $x_n - x_{n+1}$ converges to zero like

$$(n^p\,(n+k)!)^{-1},$$

then x_n converges to zero like $(n^p\,(n+k)!)^{-1}$, too. In other words, the faster sequence $x_n - x_{n+1}$ is, the faster sequence x_n is obtained.

As a first example, let us define the family of sequences

$$x_n\,(a) = 1 + \frac{1}{1!} + ... + \frac{1}{(n-1)!} + \frac{1}{(n+a)\,(n-1)!} - e.$$

We have

$$\begin{aligned} x_n\,(a) - x_{n+1}\,(a) &= \left(\frac{1}{(n+a)\,(n-1)!}\right) - \left(\frac{1}{n!} + \frac{1}{(n+1+a)\,n!}\right) \\ &= \frac{1}{n!}\left(\frac{n}{(n+a)} - \left(1 + \frac{1}{(n+1+a)}\right)\right), \end{aligned}$$

so

$$x_n\,(a) - x_{n+1}\,(a) = -\frac{1}{n!}\frac{(a+1)\,n + 2a + a^2}{(n+a)\,(n+a+1)}. \tag{3}$$

We use (3) and Lemma 1 to deduce the following:

Theorem 1. *For the sequence $x_n(a)$, we have:*
a) if $a \neq -1$, then

$$\lim_{n\to\infty} n \cdot n!\,(x_n(a) - x_{n+1}(a)) = -a-1 \quad \text{and} \quad \lim_{n\to\infty} n \cdot n! \cdot x_n(a) = -a-1.$$

ii) if $a = -1$, then

$$\lim_{n\to\infty} n^2 n!\,(x_n(-1) - x_{n+1}(-1)) = 1 \quad \text{and} \quad \lim_{n\to\infty} n^2 n! \cdot x_n(-1) = 1.$$

Corollary 1. *The sequence*

$$x_n(-1) = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{(n-1)!} + \frac{1}{(n-1)!\,(n-1)} - e$$

converges to zero as $\left(n^2 n!\right)^{-1}$, while all sequences $x_n(a)$, with $a \neq -1$, converge to zero as $(n \cdot n!)^{-1}$.

Note that the sequence

$$x_{n+1}(-1) = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{1}{n!n} - e$$

is related to the sequence (1) used in proving the irrationality of e.

Now let us consider the family of sequences

$$\gamma_n(p,q,r) = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{1 - \frac{p}{n^2} + \frac{q}{n^3} - \frac{r}{n^4}}{n!n} - e,$$

depending on real parameters p, q, r. We have

$$\begin{aligned} \gamma_n(p,q,r) - \gamma_{n+1}(p,q,r) &= \frac{1 - \frac{p}{n^2} + \frac{q}{n^3} - \frac{r}{n^4}}{n!n} \\ &\quad - \frac{1}{(n+1)!} - \frac{1 - \frac{p}{(n+1)^2} + \frac{q}{(n+1)^3} - \frac{r}{(n+1)^4}}{(n+1)!\,(n+1)} \\ &= \frac{1}{n!} \cdot \frac{S(n)}{n^5 (n+1)^6}, \end{aligned}$$

so

$$\gamma_n(p,q,r) - \gamma_{n+1}(p,q,r) = \frac{1}{n!} \cdot \frac{S(n)}{n^5 (n+1)^6}, \tag{4}$$

where

$$\begin{aligned} S(n) &= (1-p)\,n^8 + (q - 5p + 4)\,n^7 + (5q - 13p - r + 6)\,n^6 \\ &\quad + (14q - 19p - 5r + 4)\,n^5 + (20q - 15p - 15r + 1)\,n^4 \\ &\quad + (15q - 6p - 20r)\,n^3 + (6q - p - 15r)\,n^2 + (q - 6r)\,n - r. \end{aligned}$$

It follows that the limit

$$\lim_{n\to\infty} n^d n! \left(\gamma_n\left(p,q,r\right) - \gamma_{n+1}\left(p,q,r\right)\right)$$

is finite, nonzero, where

$$d = 11 - \deg S\left(n\right).$$

The highest value for d is obtained when $\deg S\left(n\right)$ is as small as possible. This is attained when the first three coefficients of $S\left(n\right)$ vanish

$$\begin{cases} 1 - p = 0 \\ q - 5p + 4 = 0 \\ 5q - 13p - r + 6 = 0 \end{cases},$$

that is

$$p = 1, \quad q = 1, \quad r = -2.$$

By using (4) and Lemma 1, we can give the following:

Theorem 2. *For the sequence $\gamma_n\left(p,q,r\right)$, we have:*
i) if $p \neq 1$, then $\gamma_n\left(p,q,r\right)$ converges to zero as $\left(n^3 n!\right)^{-1}$, since

$$\lim_{n\to\infty} n^3 n! \left(\gamma_n\left(p,q\right) - \gamma_{n+1}\left(p,q\right)\right) = 1 - p \quad \textit{and} \quad \lim_{n\to\infty} n^3 n! \gamma_n\left(p,q\right) = 1 - p.$$

ii) if $p = 1$ and $q \neq 1$, then $\gamma_n\left(1,q,r\right)$ converges to zero as $\left(n^4 n!\right)^{-1}$, since

$$\lim_{n\to\infty} n^4 n! \left(\gamma_n\left(1,q,r\right) - \gamma_{n+1}\left(1,q,r\right)\right) = q-1 \quad \textit{and} \quad \lim_{n\to\infty} n^4 n! \gamma_n\left(1,q,r\right) = q-1.$$

iii) if $p = 1$, $q = 1$, and $r \neq -2$, then $\gamma_n\left(1,1,r\right)$ converges to zero as $\left(n^5 n!\right)^{-1}$, since

$$\lim_{n\to\infty} n^5 n! \left(\gamma_n\left(1,1\right) - \gamma_{n+1}\left(1,1\right)\right) = -r-2 \quad \textit{and} \quad \lim_{n\to\infty} n^5 n! \gamma_n\left(1,1\right) = -r-2.$$

iv) if $p = 1$, $q = 1$, and $r = -2$, then $\gamma_n\left(1,1,-2\right)$ converges to zero as $\left(n^6 n!\right)^{-1}$, since

$$\lim_{n\to\infty} n^6 n! \left(\gamma_n\left(1,1,-2\right) - \gamma_{n+1}\left(1,1,-2\right)\right) = 9 \quad \textit{and} \quad \lim_{n\to\infty} n^6 n! \gamma_n\left(1,1,-2\right) = 9.$$

Corollary 2. *The sequence*

$$\gamma_n\left(1,1,-2\right) = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4}}{n! n} - e$$

converges to zero as $\left(n^6 n!\right)^{-1}$; it has the highest rate of convergence, through all sequences $\gamma_n\left(p,q,r\right)$, where p,q,r are real parameters.

We can consider the approximation formula

$$\theta_n \approx 1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4}, \quad n \to \infty,$$

as we can prove the next:

Theorem 3. *The following inequalities hold true, for all integers $n \geq 3$:*

$$1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{\alpha_n}{n!n} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{\beta_n}{n!n},$$

where

$$\alpha_n = 1 - \frac{1}{n^2} + \frac{1}{n^3}, \quad \beta_n = 1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4}.$$

Proof. We have

$$1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{\alpha_n}{n!n} - e = \gamma_n (1, 1, 0)$$

and

$$1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} + \frac{\beta_n}{n!n} - e = \gamma_n (1, 1, -2) .$$

By using (4), we deduce that for all integers $n \geq 3$:

$$\gamma_n (1, 1, 0) - \gamma_{n+1} (1, 1, 0) = -\frac{1}{n!} \cdot \frac{(2n + 1) \left(n^3 - 2n - n^2 - 1\right)}{n^4 (n + 1)^5} < 0$$

and

$$\gamma_n (1, 1, -2) - \gamma_{n+1} (1, 1, -2) = \frac{1}{n!} \cdot \frac{13n + 35n^2 + 49n^3 + 36n^4 + 9n^5 + 2}{n^5 (n + 1)^6} > 0.$$

Consequently, the sequence $\gamma_n (1, 1, 0)$ is strictly increasing, while the sequence $\gamma_n (1, 1, -2)$ is strictly decreasing. They converge to zero, so

$$\gamma_n (1, 1, 0) < 0 \quad \text{and} \quad \gamma_n (1, 1, -2) > 0.$$

The proof is complete. □

3 Final remarks

Our method permits to calculate more terms in the asymptotic series, as $n \to \infty$:

$$\begin{aligned} \theta_n &= 1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4} - \frac{9}{n^5} + \frac{9}{n^6} \\ &\quad + \frac{50}{n^7} - \frac{267}{n^8} + \frac{413}{n^9} + \frac{2180}{n^{10}} - \frac{17731}{n^{11}} + O\left(\frac{1}{n^{12}}\right), \end{aligned}$$

and we are convinced that it can be a starting point for obtaining new results in the approximation theory.

Moreover, by using the same method presented above, for the sequence τ_n defined by (2), we have, as $n \to \infty$

$$\begin{aligned} \tau_n &= 1 + \frac{1}{n} - \frac{1}{n^3} + \frac{1}{n^4} + \frac{2}{n^5} \\ &\quad - \frac{9}{n^6} + \frac{9}{n^7} + \frac{50}{n^8} - \frac{267}{n^9} + O\left(\frac{1}{n^{10}}\right). \end{aligned}$$

Theorem 4. *The following inequalities hold true, for all integers $n \geq 1$:*

$$1 + \frac{1}{1!} + ... + \frac{1}{(n-1)!} + \frac{\rho_n}{n!} < e < 1 + \frac{1}{1!} + ... + \frac{1}{(n-1)!} + \frac{\sigma_n}{n!},$$

where

$$\rho_n = 1 + \frac{1}{n} - \frac{1}{n^3} \quad \text{and} \quad \sigma_n = 1 + \frac{1}{n}.$$

Proof. Let us denote

$$\begin{aligned} r_n &= 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{\rho_n}{n!} - e \\ &= 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1 + \frac{1}{n} - \frac{1}{n^3}}{n!} - e \end{aligned}$$

and

$$\begin{aligned} s_n &= 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{\sigma_n}{n!} - e \\ &= 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1 + \frac{1}{n}}{n!} - e. \end{aligned}$$

We have

$$\begin{aligned} r_{n+1} - r_n &= \frac{1}{n!} + \frac{1 + \frac{1}{n+1} - \frac{1}{(n+1)^3}}{(n+1)!} - \frac{1 + \frac{1}{n} - \frac{1}{n^3}}{n!} \\ &= \frac{1}{n!} \cdot \frac{4n + 5n^2 + n^3 + 1}{n^3 (n+1)^4} > 0 \end{aligned}$$

and

$$\begin{aligned} s_{n+1}-s_n &= \frac{1}{n!}+\frac{1+\frac{1}{n+1}}{(n+1)!}-\frac{1+\frac{1}{n}}{n!} \\ &= -\frac{1}{n!}\cdot\frac{1}{n\,(n+1)^2}<0. \end{aligned}$$

Consequently, the sequence r_n is strictly increasing, while the sequence s_n is strictly decreasing. They converge to zero, so

$$r_n < 0 \quad \text{and} \quad s_n > 0.$$

The proof is now complete. □

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