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EKELAND VARIATIONAL PRINCIPLE OVER PREORDERED MONOIDS*

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Dedicated to Prof. Biagio Ricceri on the occasion of his 70th birthday

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Abstract

We establish a generalization of Ekeland's variational principle for submonotone maps defined on a preordered pseudometric space and with values in a preordered monoid. The proof relies on an ordering principle for more general preordered sets.

Keywords: preordered pseudometric space, preordered monoid, ordering principle, submonotone map, variational principle.
MSC: 58E30, 06A75, 49J52.

1 Introduction

The classical Ekeland principle can be stated as follows.

Ekeland's variational principle. *Let (M, d) be a complete metric space and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, proper (i.e., $f \not\equiv +\infty$), and bounded below. Then, for every $x \in \operatorname{dom}(f) := f^{-1}(\mathbb{R})$, there is $y \in \operatorname{dom}(f)$ such that*

- $d(x, y) \leq f(x) - f(y)$;

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- *$d(y,z) > f(y) - f(z)$ for all $z \in M$, $z \neq y$.*

As corollaries, one has the existence of "almost minimum" point of arbitrarily small strong slope, or the existence of minimizing Palais–Smale sequence. Thus Ekeland's variational principle is especially fruitful for studying minimization problems for lower semicontinuous maps on a metric space.

In [1], Ekeland's principle is generalized to the context of a submonotone map $f : P \to \mathbb{R} \cup \{+\infty\}$ defined on a set P endowed with a pseudometric d (instead of a metric) and a preorder $\leq$, i.e., a reflexive and transitive binary relation. The classical situation is retrieved if d is a metric and $\leq$ is the trivial preorder, i.e., such that $x \leq y$ for all x, y.

In this paper, our aim is to consider maps $f : P \to \mathbb{K}$, still defined on a preordered set P, but with values in a commutative monoid $(\mathbb{K}, +)$ endowed with a preorder (instead of $\mathbb{R} \cup \{+\infty\}$, which is itself a monoid endowed with a total order).

In the present situation, the problem of finding "almost minimum" points x for f (i.e., whose value $f(x)$ is "almost the smallest" value of f) is replaced by the problem of finding points x whose value $f(x)$ is "almost minimal" with respect to the preorder on $\mathbb{K}$. A first step towards this purpose is to generalize Ekeland's principle over preordered monoids. This is the object of this paper.

The paper is organized as follows. In Section 2, we show an ordering principle (Proposition 1) which asserts the existence of certain d-maximal points in a preordered set P, with respect to a map d more general than a pseudometric. This extends [2, Theorem 2]. In Section 3, we derive our extension of Ekeland's principle (Theorem 1). The precise setting of this result is given in the axioms (K1)–(K5), (P1)–(P2) which we introduce in the due places and illustrate with various remarks and examples.

2 Basic setting and a general ordering principle

The setting of this section is quite rudimentary: we consider two nonempty sets $\mathbb{K}$ and P, with the following structure.

(K1) $\mathbb{K}$ is a nonempty set, with a distinguished element $0 \in \mathbb{K}$, and a countable collection of subsets $\mathcal{V}$ such that

$$(\forall V, V' \in \mathcal{V}, \quad V \cap V' \in \mathcal{V}) \quad \text{and} \quad \bigcap_{V \in \mathcal{V}} V = \{0\}$$

(thus in particular $0 \in V$ for all $V \in \mathcal{V}$).

(P1) P is a nonempty set equipped with a preorder $\leq$ (i.e., a binary relation which is reflexive and transitive, but not necessarily antisymmetric) and a map

$$d : P^2_{\leq} := \{(x, y) \in P \times P : x \leq y\} \to \mathbb{K}$$

such that $d(x, x) = 0$ for all $x \in P$.

Based on these assumptions, we will use the following terminology:

- We say that a sequence $(\alpha_n) \subset \mathbb{K}$ *converges to* 0, and we write $\alpha_n \to 0$ if, for every $V \in \mathcal{V}$ there is $n_0 \in \mathbb{N}$ such that, whenever $n \geq n_0$ there holds $\alpha_n \in V$.
- A sequence $(x_n) \subset P$ is said to be $\leq$-*ascending* if $x_n \leq x_m$ whenever $n \leq m$. Then we say that:
  - $x \in P$ is an *upper bound* of (x_n) if $x_n \leq x$ for all n.
  - (x_n) $\leq$-*converges to* $x \in P$ if x is an upper bound of (x_n) and $d(x_n, x) \to 0$; we write $x_n \overset{\leq}{\to} x$ in this case.
  - (x_n) is $\leq$-*convergent* if there is $x \in P$ such that $x_n \overset{\leq}{\to} x$.
  - (x_n) is a *Cauchy sequence* (respectively, a *semi-Cauchy sequence*) if for every $V \in \mathcal{V}$ there is a rank n_0 such that $d(x_n, x_m) \in V$ whenever $n_0 \leq n \leq m$ (respectively, $d(x_{n_0}, x_n) \in V$ whenever $n_0 \leq n$).
- A sequence is called a $\leq$-*Cauchy* (resp., $\leq$-*semi-Cauchy*) *sequence* if it is both $\leq$-ascending and a Cauchy (resp., semi-Cauchy) sequence. (In the same way, whenever we will speak of a $\leq$-convergent sequence, it will be assumed $\leq$-ascending.)
- We say that P is $\leq$-*complete* if any $\leq$-Cauchy sequence is $\leq$-convergent.
- An element $x \in P$ is called *d-maximal* if

  $$\forall u, v \in P, \quad x \leq u \leq v \quad \Rightarrow \quad d(u, v) = 0.$$

  We will say that x is *d-semi-maximal* if the following weaker condition holds:

  $$\forall u \in P, \quad x \leq u \quad \Rightarrow \quad d(x, u) = 0.$$

- In the same way, we say that $x \in P$ is *d-minimal* (resp., *d-semi-minimal*) if

$$\forall u, v \in P, \quad u \leq v \leq x \;\Rightarrow\; d(u,v) = 0$$
$$(\text{resp.}, \;\; \forall u \in P, \quad u \leq x \;\Rightarrow\; d(u,x) = 0).$$

The goal of this section is to prove a general criterion of existence of d-maximal elements in the preordered set P: see Proposition 1 below. Before that, we give comments on assumptions (K1)–(P1) and the above definitions, and show preliminary facts.

Example 1. A straightforward model for (K1) and (P1) is the following:

- $\mathbb{K} = \mathbb{R}$ (or $\mathbb{R} \cup \{+\infty\}$), equipped with the system $\mathcal{V}$ of neighborhoods of 0 consisting of intervals $I \subset \mathbb{R}$ with rational bounds;
- P is endowed with a metric $d : P \times P \to [0, +\infty) \subset \mathbb{R}$ and the trivial preorder $\leq$, i.e., such that $x \leq y$ for all $x, y \in P$.

Then we retrieve the standard notions of convergence, Cauchy sequence, and completeness. This is the setting of the classical Ekeland's principle (see Section 1).

The assumptions (K1)–(P1), and the complementary assumptions that we will make in the next section, aim to incorporate the standard setting above in a considerably wider generality.

The reason why (K1) involves a countable collection $\mathcal{V}$ is for having the following lemma.

Lemma 1. *Under (K1), there is a sequence* $(V_k)_{k \geq 1} \subset \mathcal{V}$ *such that*

(i) $V_k \supset V_{k+1}$ *for all* $k \geq 1$*;*

(ii) $\bigcap_{k \geq 1} V_k = \{0\}$*;*

(iii) *for all* $V \in \mathcal{V}$*, there is* $k \geq 1$ *such that* $V \supset V_k$*.*

Proof. Since $\mathcal{V}$ is countable, we can write $\mathcal{V} = \{U_k\}_{k \geq 1}$. Then by setting $V_k := U_1 \cap \ldots \cap U_k$ for all $k \geq 1$, we obtain a sequence $(V_k)_{k \geq 1}$ still contained in $\mathcal{V}$ (since $\mathcal{V}$ is stable under finite intersections) and satisfying the required conditions (for condition (iii), we have that every $V \in \mathcal{V}$ is of the form $V = U_k$ for some $k \geq 1$, thus by construction $V \supset V_k$). □

Remark 1. If $\mathcal{V}$ is not assumed to be countable in (K1), then Lemma 1 is not valid. Take for instance $\mathbb{K} = \mathbb{R}$ equipped with the collection $\mathcal{V}$ of all cofinite subsets $V \subset \mathbb{R}$ containing 0. Then $\mathcal{V}$ is stable under finite intersection and $\bigcap_{V \in \mathcal{V}} V = \{0\}$. However, for every sequence $(V_k)_{k \geq 1} \subset \mathcal{V}$, the subset $\mathbb{R} \setminus \bigcap_{k \geq 1} V_k = \bigcup_{k \geq 1} (\mathbb{R} \setminus V_k)$ is at most countable, hence $\bigcap_{k \geq 1} V_k \neq \{0\}$.

Remark 2. In (K1), changing $\mathcal{V}$ to another countable collection of subsets $\mathcal{V}'$ (satisfying the conditions of (K1)) may of course modify the property that a sequence converges to 0. For the purpose of discussion in this remark, we write $\alpha_n \overset{\mathcal{V}}{\to} 0$ (resp., $\alpha_n \overset{\mathcal{V}'}{\to} 0$) for convergence with respect to $\mathcal{V}$ (resp., $\mathcal{V}'$).

(a) Say that $\mathcal{V}$ *refines* $\mathcal{V}'$ if, for every $V' \in \mathcal{V}'$, there is $V \in \mathcal{V}$ with $V' \supset V$. Then:

$$\mathcal{V} \text{ refines } \mathcal{V}' \quad \Leftrightarrow \quad \big(\forall (\alpha_n) \subset \mathbb{K}, \quad \alpha_n \overset{\mathcal{V}}{\to} 0 \quad \Rightarrow \quad \alpha_n \overset{\mathcal{V}'}{\to} 0\big).$$

Indeed, the proof of $\Rightarrow$ is straightforward. For showing $\Leftarrow$, we use the sequence $(V_k)_{k \geq 1} \subset \mathcal{V}$ given by Lemma 1. If $\mathcal{V}$ does not refine $\mathcal{V}'$, then there is $V' \in \mathcal{V}'$ such that $V' \not\supset V$ for all $V \in \mathcal{V}$. For every $k \geq 1$ this yields an element $\alpha_k \in V_k$ such that $\alpha_k \notin V'$. We have $\alpha_k \overset{\mathcal{V}}{\to} 0$ by construction of the sequence (and the properties stated in Lemma 1), but (α_k) does not converge to 0 with respect to $\mathcal{V}'$ (since V' does not contain any term of the sequence).

(b) Say that $\mathcal{V}$ and $\mathcal{V}'$ are *equivalent* if $\mathcal{V}$ refines $\mathcal{V}'$ and $\mathcal{V}'$ refines $\mathcal{V}$. By part (a) above:

$$\mathcal{V} \text{ is equivalent to } \mathcal{V}' \quad \Leftrightarrow \quad \big(\forall (\alpha_n) \subset \mathbb{K}, \quad \alpha_n \overset{\mathcal{V}}{\to} 0 \quad \Leftrightarrow \quad \alpha_n \overset{\mathcal{V}'}{\to} 0\big).$$

For example, in $\mathbb{K} = \mathbb{R}^2$, the collection $\mathcal{V} = \{I_q\}_{q \in \mathbb{Q}_+^*}$ with $I_q := \{(x, 0) : |x| < q\}$ is not equivalent to $\mathcal{V}' = \{I'_q\}_{q \in \mathbb{Q}_+^*}$ with $I'_q := \{(0, y) : |y| < q\}$. We have $(\frac{1}{n}, 0) \overset{\mathcal{V}}{\to} 0$ but $(\frac{1}{n}, 0) \overset{\mathcal{V}'}{\not\to} 0$.

(c) In what follows the collection $\mathcal{V}$ of (K1) is fixed, but it would be harmless to replace it by an equivalent collection of subsets also fulfilling the conditions of (K1). As an example, the collection of subsets $\mathcal{V}$ of (K1) is equivalent to the collection $\{V_k\}_{k \geq 1}$ produced in Lemma 1; thus we could freely assume that $\mathcal{V} = \{V_k\}_{k \geq 1}$.

Remark 3. We may declare that a subset $A \subset P$ is $\leq$-*closed* if

$$\forall x \in P, \quad (\exists (x_n) \subset A, \quad x_n \overset{\leq}{\rightarrow} x) \quad \Rightarrow \quad x \in A$$

(where the notation $x_n \overset{\leq}{\rightarrow} x$ means that the sequence (x_n) is $\leq$-ascending and $\leq$-converges to x). Then a subset U is declared $\leq$-*open* if its complement

$P \setminus U$ is $\leq$-closed. It is straightforward to check that this defines a topology on P.

Remark 4. In the definitions above, we are focusing on $\leq$-ascending sequences and the notion of $\leq$-convergence because the map d is only defined over pairs $(x, y) \in P^2_{\leq}$, so that $d(x_n, x)$ has no meaning if we do not require a priori that $x_n \leq x$.

Assume d defined on the whole set $P \times P$ instead of just the subset $P^2_{\leq}$; this is the setting of [1] and [2] (with $\mathbb{K} = \mathbb{R}$). Then more general definitions can be given, which are considered in the aforementioned references:

- We say that a (not necessarily $\leq$-ascending) sequence $(x_n) \subset P$ *converges to* $x \in P$ if $d(x_n, x) \to 0$.
- We say that (x_n) is *convergent* if it converges to some $x \in P$.
- The preorder $\leq$ is called *self-closed* if whenever (x_n) is a $\leq$-ascending sequence which converges to $x \in P$, we have $x_n \leq x$ for all n.

In other words, $\leq$ is self-closed if and only if, for every $\leq$-ascending sequence $(x_n) \subset P$ and every element $x \in P$, we have:

$$(x_n) \text{ converges to } x \quad \Leftrightarrow \quad (x_n) \leq\text{-converges to } x$$

(the implication $\Leftarrow$ is always true).

The condition that P is $\leq$-complete is then implied by the condition:

$$\begin{array}{l}\text{every } \leq\text{-Cauchy sequence } (x_n) \subset P \text{ is convergent,} \\ \text{and the preorder } \leq \text{ is self-closed.}\end{array} \tag{1}$$

We stress that $\leq$-completeness of P is in general a weaker condition than (1). For instance, let $\mathbb{K} = \mathbb{R}$ and $\mathcal{V}$ be as in Example 1, and let $P = [0, 1] \cup \{-1\}$ be endowed with the standard total order $\leq$ and the map $d : P \times P \to [0, +\infty)$, $(x, y) \mapsto ||x| - |y||$. In this case, P is $\leq$-complete: if (x_n) is a $\leq$-Cauchy sequence, then it is either constant equal to -1 (in which case it $\leq$-converges to -1) or it is a nondecreasing sequence contained in $[0, 1]$ for n large enough (in which case it $\leq$-converges to some $\ell \in [0, 1]$); in both cases (x_n) is $\leq$-convergent. However, $\leq$ is not self-closed, because the sequence $(1 - \frac{1}{n})$ converges to -1 although $-1 < 1 - \frac{1}{n}$ for all n.

Remark 5. (a) A $\leq$-convergent sequence is not always a Cauchy sequence: take for instance $\mathbb{K} = \mathbb{R}$ and $\mathcal{V}$ as in Example 1 and let $P = [0, 1]$ be equipped with its standard total order $\leq$ and the map $d : P^2_{\leq} \to \mathbb{R}$ given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x < y < 1, \\ 1-x & \text{if } x < y = 1. \end{cases}$$

The sequence $(x_n) \subset P$ defined by $x_n = 1 - \frac{1}{n}$ for all $n \geq 1$ is $\leq$-convergent (to 1) but it is not a Cauchy sequence. Here a $\leq$-ascending sequence $(y_n) \subset P$ is a Cauchy sequence if and only if it is stationary (which implies $\leq$-convergent). This fact shows that P is $\leq$-complete.

(b) If d satisfies the property

$$u \leq v \leq w \quad \Rightarrow \quad (\forall V \in \mathcal{V}, \;\; d(u,w), d(v,w) \in V \quad \Rightarrow \quad d(u,v) \in V) \tag{2}$$

for all $u, v, w \in P$, then it holds that every $\leq$-convergent sequence is also a $\leq$-Cauchy sequence. Indeed, assume that $(x_n) \subset P$ is $\leq$-convergent to some x. Then for every $V \in \mathcal{V}$ there is a rank n_0 with $d(x_n, x) \in V$ whenever $n \geq n_0$. If $m \geq n \geq n_0$, we then have $x_n \leq x_m \leq x$ and $d(x_n, x), d(x_m, x) \in V$, which (by virtue of (2)) yields $d(x_n, x_m) \in V$.

(c) Assume that d satisfies the next property, which is a "right counterpart" of (2):

$$u \leq v \leq w \quad \Rightarrow \quad (\forall V \in \mathcal{V}, \quad d(u,w) \in V \quad \Rightarrow \quad d(v,w) \in V) \tag{3}$$

for all $u, v, w \in P$. If (x_n) is a $\leq$-ascending sequence which is $\leq$-convergent, then every limit has the following property:

$$x_n \overset{\leq}{\Rightarrow} x \quad \Rightarrow \quad \begin{cases} x \text{ is an upper bound of } (x_n) \text{ and} \\ x \text{ is } d\text{-semi-minimal among upper bounds of } (x_n). \end{cases} \tag{4}$$

Indeed, since $x_n \overset{\leq}{\Rightarrow} x$, we have by definition $x_n \leq x$ for all n. Now let y be another upper bound of (x_n) such that $y \leq x$. For every $V \in \mathcal{V}$, we have $d(x_n, x) \in V$ for n large enough; since $x_n \leq y \leq x$ we derive $d(y, x) \in V$ by virtue of (3). Whence $d(y,x) \in \bigcap_{V \in \mathcal{V}} V = \{0\}$ (by (K1)), so that $d(y, x) = 0$.

If we assume both (2) and (3), we obtain the following more precise property of the limits:

$$x_n \overset{\leq}{\Rightarrow} x \quad \Rightarrow \quad \begin{cases} x \text{ is an upper bound of } (x_n) \text{ and} \\ x \text{ is } d\text{-minimal among upper bounds of } (x_n). \end{cases} \tag{5}$$

Indeed, it remains to show the d-minimality of x, so let $y, z \in P$ be such that $x_n \leq y \leq z \leq x$ for all n. For every $V \in \mathcal{V}$, we have $d(x_n, x) \in V$

whenever n is large enough. Then (3) implies that $d(y, x), d(z, x) \in V$ and (2) yields in turn $d(y, z) \in V$. Since the latter inclusion holds for every $V \in \mathcal{V}$, condition (K1) ensures that $d(y, z) = 0$.

However, the converse of (5) is not true (and a fortiori the converse of (4) is not true neither). Take $\mathbb{K} = \mathbb{R}$ and $\mathcal{V}$ as in Example 1, and let $P = [0, 1) \cup \{2\}$ be equipped with its standard total order $\leq$ and standard metric d. The sequence (x_n) defined by $x_n = 1 - \frac{1}{n}$ for all $n \geq 1$ is $\leq$-ascending, and admits 2 as a d-minimal upper bound (in fact 2 is the unique upper bound of (x_n)) but (x_n) does not $\leq$-converge to 2.

(d) Finally, we emphasize that a $\leq$-convergent sequence $(x_n) \subset P$ may have several limits – even when (2) and (3) hold (so that the characterization of the limit of part (c) is valid), and even if the map d is nondegenerate (i.e., $\forall (x, y) \in P^2_{\leq},\ d(x, y) = 0 \Leftrightarrow x = y$). Take for instance $\mathbb{K} = \mathbb{R}$ and $\mathcal{V}$ as in Example 1, let $P = [0, 1] \cup \{2\}$ be endowed with the partial order $\leq$ (resp., the map d) whose restriction to $[0, 1]$ is the standard order (resp., the standard metric) and such that $x \leq 2$ for all $x \in [0, 1)$ and $1, 2$ are not comparable (resp., $d(x, 2) = 1 - x$ for all $x \in [0, 1)$). Then the sequence $(1 - \frac{1}{n})$ is $\leq$-ascending and $\leq$-converges to both 1 and 2.

Remark 6. (a) In addition to the notions of d-maximal and d-semi-maximal elements in P, there is of course the notion of *maximal* element, with respect to the preorder $\leq$: $x \in P$ is maximal if for all $u \in P$ the relation $x \leq u$ implies $x = u$ (the latter notion only concerns the preorder and does not involve the map d).

In general, we have clearly the implications

$$x \text{ is maximal} \quad \Rightarrow \quad x \text{ is } d\text{-maximal} \quad \Rightarrow \quad x \text{ is } d\text{-semi-maximal} \tag{6}$$

but none of the reversed implications is valid. For instance, if $P = \mathbb{R}$ is equipped with the standard order $\leq$ and the map d such that $d(x, y) = 0$ for all $x, y \in P^2_{\leq}$, then every element of P is both d-maximal and d-semi-maximal, but P contains no maximal element. Now if $P = [0, 1]$ is endowed with the standard order $\leq$ and the map $d : P^2_{\leq} \to \mathbb{R}$ such that $d(x, y) = y - x$ for $0 < x \leq y \leq 1$ and $d(0, y) = 0$ for all $y \in [0, 1]$, then 0 is d-semi-maximal, but not d-maximal.

(b) The three conditions of (6) become equivalent if d is nondegenerate (i.e., such that $\forall (x, y) \in P^2_{\leq},\ d(x, y) = 0 \ \Leftrightarrow \ x = y$). Indeed, in this case, it is clear that d-semi-maximality implies maximality.

(c) If it holds that

$$\forall u, v, w \in P, \quad u \leq v \leq w \quad \Rightarrow \quad (d(u, v) = d(u, w) = 0 \ \Rightarrow \ d(v, w) = 0), \tag{7}$$

then every d-semi-maximal element becomes also d-maximal. Note that condition (7) is weaker than condition (3) stated in Remark 5 (c), because $\bigcap_{V \in \mathcal{V}} V = \{0\}$. Also condition (7) is weaker than the condition that d is nondegenerate, involved in part (b) above (indeed, (7) automatically holds if d is nondegenerate, but it also holds if $d \equiv 0$).

The following result is the announced criterion of existence of d-maximal elements; it extends [2, Theorem 2].

Proposition 1. *Assume that (K1) and (P1) hold; assume in addition that*

(i) *every $\leq$-ascending sequence in P has an upper bound*

and that one of the following two conditions holds:

(ii) *every $\leq$-ascending sequence in P is a Cauchy sequence;*

(ii)′ *every $\leq$-ascending sequence $(x_n) \subset P$ is a semi-Cauchy sequence and satisfies*

$$\forall u, v \in P, \quad (u \leq v \quad \text{and} \quad x_n \overset{\leq}{\to} u, v) \quad \Rightarrow \quad d(u, v) = 0.$$

Then, for every $x \in P$, there exists a d-maximal element x' with $x \leq x'$.

Proof. We first show the following claim:

Claim. (a) *If* (ii) *holds then for every $x \in P$ and $V \in \mathcal{V}$, there is $y \in P$ with $x \leq y$ and such that for all $u, v \in P$ with $y \leq u \leq v$, we have $d(u, v) \in V$.*

(b) *If* (ii)′ *holds then for every $x \in P$ and $V \in \mathcal{V}$, there is $y \in P$ with $x \leq y$ and such that for all $u \in P$ with $y \leq u$, we have $d(y, u) \in V$.*

For showing part (a) of the Claim, assume to the contrary that there are $x \in P$ and $V \in \mathcal{V}$ such that for every $y \in P$ with $x \leq y$, we can find $u, v \in P$ with $y \leq u \leq v$ and $d(u, v) \notin V$. We construct a $\leq$-ascending sequence $(y_n) \subset P$ by induction:

- Set $y_0 = x$.
- Assuming that $y_0 \leq \ldots \leq y_{2n}$ have been constructed, by invoking the above property, we find $y_{2n+1}, y_{2n+2} \in P$ such that $y_{2n} \leq y_{2n+1} \leq y_{2n+2}$ and $d(y_{2n+1}, y_{2n+2}) \notin V$.

Altogether we get a $\leq$-ascending sequence (y_n) such that $d(y_{2n+1}, y_{2n+2}) \notin V$ for all n. The latter fact implies that (y_n) cannot be a Cauchy sequence, a contradiction with (ii).

Next we show part (b) of the Claim. Arguing again by contradiction, assume that there are $x \in P$ and $V \in \mathcal{V}$ such that for every $y \in P$ with $x \leq y$ there is some $u \in P$ with $y \leq u$ and $d(y, u) \notin V$. Here we construct a sequence (y_n) as follows: set $y_0 = x$ and once $y_0 \leq \ldots \leq y_n$ are constructed choose $y_{n+1} \geq y_n$ such that $d(y_n, y_{n+1}) \notin V$. The so-obtained sequence $(y_n) \subset P$ is $\leq$-ascending and such that $d(y_n, y_{n+1}) \notin V$ for all n. The latter fact prevents it from being semi-Cauchy, a contradiction with (ii)$'$. The Claim is established.

Fix $x \in P$ and let us produce a d-maximal element $x' \in P$ with $x \leq x'$. To do this, we rely on the construction of a $\leq$-ascending sequence $(x_n) \subset P$. For this construction, we use the Claim and the sequence $(V_k)_{k\geq 1} \subset \mathcal{V}$ provided by Lemma 1.

Set first $x_0 = x$. Once $x_0 \leq \ldots \leq x_{n-1}$ are constructed, by virtue of the Claim, we find $x_n \geq x_{n-1}$ such that

$$x_n \leq u \leq v \quad \Rightarrow \quad d(u, v) \in V_n \qquad \text{if (ii) holds,} \tag{8}$$

$$\text{respectively,} \quad x_n \leq u \quad \Rightarrow \quad d(x_n, u) \in V_n \qquad \text{if (ii)}' \text{ holds.} \tag{9}$$

We obtain a $\leq$-ascending sequence (x_n) which satisfies (8) (respectively, (9)) for all $n \geq 1$.

By assumption (i), the sequence (x_n) has a upper bound x'. In particular $x = x_0 \leq x'$. It remains to verify that x' is d-maximal. To this end, let $u, v \in P$ be such that $x' \leq u \leq v$. For all $k \geq 1$ we thus have $x_k \leq x' \leq u \leq v$. If (ii) holds, then (8) implies that $d(u, v) \in V_k$ for all $k \geq 1$, which yields $d(u, v) = 0$ because $\bigcap_{k\geq 1} V_k = \{0\}$.

Now assume that (ii)$'$ holds. Given $k \geq 1$, for every $n \geq k$, condition (9) implies that $d(x_n, u) \in V_n \subset V_k$. This shows that $d(x_n, u) \to 0$, whence $x_n \overset{\leq}{\Rightarrow} u$. Similarly, (9) implies that $d(x_n, v) \in V_k$ for all $n \geq k$, and we also obtain that $x_n \overset{\leq}{\Rightarrow} v$. Due to (ii)$'$, we conclude that $d(u, v) = 0$. In all the cases, we obtain that x' is d-maximal, and the proof of the proposition is complete. $\square$

Example 2. Let $\mathbb{K} = \mathbb{R}$ and $\mathcal{V}$ be as in Example 1. Let P be the closed unit ball of $\mathbb{R}^2$ endowed with its standard metric d and the partial order $\leq$ given by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. Then, every $\leq$-ascending sequence $(z_n) = ((x_n, y_n)) \subset P$ consists of two nondecreasing sequences $(x_n), (y_n) \subset [-1, 1]$, which easily implies that (z_n) is $\leq$-convergent to some point $z = (x, y) \in P$, and $z_n \leq z$ for all n. Thus (i) and (ii) are satisfied and Proposition 1 can be applied. Note that the d-maximal elements of P are exactly the points $(\cos\theta, \sin\theta)$ with $\theta \in [0, \frac{\pi}{2}]$. Thus even in this simple example, the applicability of the proposition does not mean that P has a single d-maximal element, or a finite number of d-maximal elements.

3 Extension of Ekeland's principle

We now introduce more assumptions on our two sets $\mathbb{K}$ and P.

(K2) $\mathbb{K}$ is endowed with an operation $+$ and a preorder $\leq$ such that:

(a) $(\mathbb{K}, +)$ is a commutative monoid with neutral element 0;

(b) letting $\mathbb{K}' \subset \mathbb{K}$ denote the subset of invertible elements, we have

$$\forall \alpha, \beta \in \mathbb{K}, \quad (\alpha \leq \beta \quad \text{and} \quad \beta \in \mathbb{K}') \quad \Rightarrow \quad \alpha \in \mathbb{K}';$$

(c) the preorder $\leq$ is compatible with $+$ in the sense that

$$\forall \alpha, \beta, \gamma \in \mathbb{K}, \quad \alpha \leq \beta \quad \Rightarrow \quad \alpha + \gamma \leq \beta + \gamma,$$

so that in particular

$$\forall \alpha \in \mathbb{K}',\ \forall \beta \in \mathbb{K}, \quad \alpha \leq \beta \quad \Leftrightarrow \quad \beta - \alpha \geq 0.$$

We set $\mathbb{K}_+ := \{\alpha \in \mathbb{K} : \alpha \geq 0\}$, $\mathbb{K}'_+ := \mathbb{K}' \cap \mathbb{K}_+$, and $\mathbb{K}_- := \{\alpha \in \mathbb{K} : \alpha \leq 0\} \subset \mathbb{K}'$.

(K3) For all $V \in \mathcal{V}$, we have

$$\forall \alpha, \beta \in \mathbb{K}', \quad (0 \leq \alpha \leq \beta \quad \text{and} \quad \beta \in V) \quad \Rightarrow \quad \alpha \in V.$$

(K4) (Generalized Archimedean condition) For every $\alpha \in \mathbb{K}'$ and $V \in \mathcal{V}$, there is $n \geq 1$ such that

$$\forall \beta_1, \ldots, \beta_n \in \mathbb{K}'_+ \setminus V, \quad \beta_1 + \ldots + \beta_n \not\leq \alpha.$$

(K5) (Passing to the limit condition) For all $\alpha \in \mathbb{K}$,

$$(\exists (\beta_n) \subset \mathbb{K}_+ \quad \text{with} \quad \beta_n \to 0 \quad \text{and} \quad \alpha \leq \beta_n\ \forall n) \quad \Rightarrow \quad \alpha \leq 0.$$

(P2) We have $d(x, y) \in \mathbb{K}_+$ for all $(x, y) \in P^2_{\leq}$, and d is $\leq$-triangular in the sense that

$$\forall x, y, z \in P, \quad x \leq y \leq z \quad \Rightarrow \quad d(x, z) \leq d(x, y) + d(y, z).$$

(For simplifying the presentation, we use the same symbol $\leq$ to denote the preorder on $\mathbb{K}$ and the preorder on P, but this will generate no ambiguity.)

Example 3. (a) As in Example 1, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{R} \cup \{+\infty\}$, and let $\mathcal{V}$ be the system of neighborhoods of 0 formed by real intervals with rational bounds. Let in addition $\mathbb{K}$ be equipped with its standard operation $+$ and the standard total order $\leq$. Then all the assumptions (K1)–(K5) are fulfilled. Note that if $\mathbb{K} = \mathbb{R}$ then $(\mathbb{K}, +)$ is a group (every element is invertible). If $\mathbb{K} = \mathbb{R} \cup \{+\infty\}$, then $(\mathbb{K}, +)$ is just a monoid with $\mathbb{K}' = \mathbb{R}$.

(b) Let $\mathbb{K} = \mathbb{C}$ be equipped with the usual addition $+$, let $\mathcal{V}$ be the collection of open discs $V_q := \{z \in \mathbb{C} : |z| < q\}$ of rational radius $q \in \mathbb{Q}$, $q > 0$. We consider the partial order $\leq$ on $\mathbb{C}$ defined by letting $z \leq z'$ if $|\Im(z' - z)| \leq \Re(z' - z)$ (where $\Re$ and $\Im$ stand for real and imaginary parts). It is easy to see that conditions (K1)–(K5) are fulfilled. The subset $\mathbb{C}_+ = \{z \in \mathbb{C} : z \geq 0\}$ is the closed cone $\{re^{i\theta} : r \geq 0,\ |\theta| \leq \frac{\pi}{4}\}$, and it contains in particular the real interval $[0, +\infty)$.

Remark 7. All examples and counterexamples produced for illustrating the various remarks in Section 2 actually fit conditions (K1)–(K5), (P1)–(P2). Therefore, these remarks also apply to the setting of the present section.

Remark 8. Conditions (K1), (K2), (K4) actually imply that the restriction of the preorder $\leq$ to $\mathbb{K}'$ has to be antisymmetric, thus a partial order.

Indeed, arguing by contradiction, assume that there are $\alpha, \beta \in \mathbb{K}'$ distinct such that $\alpha \leq \beta$ and $\beta \leq \alpha$. Then $\gamma := \beta - \alpha$ and $-\gamma$ both belong to $\mathbb{K}_+$ and are nonzero. Due to (K1), there is $V \in \mathcal{V}$ such that $\gamma, -\gamma \in \mathbb{K}_+ \setminus V$. Setting $\beta_i = \gamma$ for odd i and $\beta_i = -\gamma$ for even i, we obtain a sequence $(\beta_i)_{i \geq 1} \subset \mathbb{K}_+ \setminus V$ such that every partial sum $\beta_1 + \ldots + \beta_n$ is either 0 or γ, thus always $\leq \gamma$. This contradicts (K4).

Remark 9. (a) For a pair of subsets $A \subset X \subset \mathbb{K}$, define the $\leq$-saturation of A in X as the set

$$\mathrm{sat}_X(A) = \{\alpha \in X : \exists \beta \in A,\ \alpha \leq \beta\}.$$

Say that A is $\leq$-saturated in X if $A = \mathrm{sat}_X(A)$. Some of the above assumptions can be rephrased with this terminology:

- Condition (K2) (b) means that the subset $\mathbb{K}'$ of invertible elements is $\leq$-saturated in $\mathbb{K}$.
- (K3) means that $V \cap \mathbb{K}'_+$ is $\leq$-saturated in $\mathbb{K}'_+$ for all $V \in \mathcal{V}$.
- Finally, under (K1)–(K2), condition (K5) becomes equivalent to

$$\bigcap_{V \in \mathcal{V}} \mathrm{sat}_{\mathbb{K}}(V \cap \mathbb{K}_+) = \mathbb{K}_-. \tag{10}$$

The latter assertion requires a justification, which can be done as follows.

Assume that (10) holds and let us show condition (K5). Let $\alpha \in \mathbb{K}$ and a sequence $(\beta_n) \subset \mathbb{K}_+$ be such that $\alpha \leq \beta_n$ for all n and $\beta_n \to 0$. For every $V \in \mathcal{V}$, we have $\beta_n \in V \cap \mathbb{K}_+$ for n large enough, which yields $\alpha \in \mathrm{sat}_{\mathbb{K}}(V \cap \mathbb{K}_+)$; then (10) implies that $\alpha \in \mathbb{K}_-$, as desired in (K5).

Now suppose that (K5) holds and let us show (10). The inclusion $\supset$ in (10) comes from the fact that every $\alpha \in \mathbb{K}_-$ satisfies $\alpha \leq 0$, and $0 \in V \cap \mathbb{K}_+$ for every $V \in \mathcal{V}$, whence $\alpha \in \mathrm{sat}_{\mathbb{K}}(V \cap \mathbb{K}_+)$ by definition of the $\leq$-saturation. Now let $\alpha \in \mathbb{K}$ be an element that belongs to the left hand side of (10). Let $\{V_k\}_{k \geq 1} \subset \mathcal{V}$ be as in Lemma 1. For every $k \geq 1$, we have $\alpha \in \mathrm{sat}_{\mathbb{K}}(V_k \cap \mathbb{K}_+)$, hence there is $\beta_k \in V_k \cap \mathbb{K}_+$ such that $\alpha \leq \beta_k$. By construction (and the properties stated in Lemma 1), the sequence (β_k) is contained in $\mathbb{K}_+$ and satisfies $\beta_k \to 0$. Whence $\alpha \in \mathbb{K}_-$, by virtue of (K5). This completes the argument.

(b) Observe that (K1)–(K3) imply the following property: for all $\alpha \in \mathbb{K}$,

$$(\exists (\beta_n) \subset \mathbb{K}'_+ \quad \text{with} \quad \beta_n \to 0 \quad \text{and} \quad \alpha \leq \beta_n \ \forall n) \quad \Rightarrow \quad \alpha \not> 0.$$

Thus (K5) automatically holds if every sequence $(\beta_n) \subset \mathbb{K}_+$ with $\beta_n \to 0$ satisfies $\beta_n \in \mathbb{K}'$ for large enough n (this holds for instance if there is $V \in \mathcal{V}$ such that $V \cap \mathbb{K}_+ \subset \mathbb{K}'$) and if $\leq$ is a total order on $\mathbb{K}'$.

Let us justify the above observation. Suppose to the contrary that there are $\alpha \in \mathbb{K}$ and a sequence $(\beta_n) \subset \mathbb{K}'_+$ such that $0 < \alpha \leq \beta_n$ for all n and $\beta_n \to 0$. For every $V \in \mathcal{V}$, we have $\beta_n \in V$ for n large enough, whence $\alpha \in V$ by virtue of (K2) (b) and (K3) and, therefore, $\alpha \in \bigcap_{V \in \mathcal{V}} V = \{0\}$ (see (K1)), a contradiction.

We consider functions $f : P \to \mathbb{K}$.

- Given such a function f, its *domain* is the subset $\mathrm{dom}(f) = f^{-1}(\mathbb{K}') = \{x \in P : f(x) \in \mathbb{K}'\} \subset P$.
- We say that f is *proper* if $\mathrm{dom}(f) \neq \emptyset$.

- We say that f is $\leq$-*submonotone* if, for every $\leq$-ascending sequence $(x_n) \subset \mathrm{dom}(f)$ and every element $x \in P$,

$$(f(x_n) \geq f(x_m) \ \forall n \leq m \quad \text{and} \quad x_n \overset{\leq}{\Rightarrow} x) \quad \Rightarrow \quad f(x_n) \geq f(x) \ \forall n.$$

- Finally, we will say that f is *locally finitely bounded from below* if for every $x \in \mathrm{dom}(f)$ there is a finite subset $M = M(x) \subset \mathbb{K}'$ such that

$$\forall y \in P, \quad (x \leq y \quad \text{and} \quad f(y) \leq f(x)) \quad \Rightarrow \quad \exists \mu \in M,\ f(y) \geq \mu.$$

Remark 10. The condition that f is $\leq$-submonotone is weaker than the following condition, which is a generalization of lower semicontinuity to the present context:

$$\forall \alpha \in \mathbb{K}', \quad \{x \in P : f(x) \leq \alpha\} \ \text{ is } \leq\text{-closed} \tag{11}$$

(see Remark 3). Indeed, it is easy to see that f will be $\leq$-submonotone whenever (11) holds. Now, letting $\mathbb{K} = \mathbb{R}$ and $\mathcal{V}$ as in Example 1, and considering $P = [0, 1]$ endowed with its standard metric and total order, the function $f : P \to \mathbb{R}$ such that $f(x) = x$ for all $x \in [0, 1)$ and $f(1) = 2$ is $\leq$-submonotone but does not satisfy (11).

Note also that, for arbitrary $\mathbb{K}$ and P, a function $f : P \to \mathbb{K}$ is $\leq$-submonotone if it is nonincreasing, i.e., for all $x, y \in P$ such that $x \leq y$, we have $f(x) \geq f(y)$. (Indeed, the $\leq$-convergence $x_n \overset{\leq}{\Rightarrow} x$ incorporates that $x_n \leq x$ for all n, which directly implies $f(x_n) \geq f(x)$.)

As an opposite special situation, f is also automatically $\leq$-submonotone when its restriction to $\mathrm{dom}(f)$ is increasing in the sense that

$$\forall x, y \in \mathrm{dom}(f), \quad (x \leq y \quad \text{and} \quad x \neq y) \quad \Rightarrow \quad f(x) < f(y),$$

and provided that d is nondegenerate. (Indeed, a $\leq$-ascending sequence $(x_n) \subset \mathrm{dom}(f)$ such that $f(x_n) \geq f(x_m)$ whenever $n \leq m$ must then be constant. If $x_n \overset{\leq}{\Rightarrow} x$, then the fact that d is nondegenerate yields $x_n = x$ for all n and so $f(x) = f(x_n)$.)

Remark 11. The condition that f is locally finitely bounded from below is much weaker than requiring that $f(P)$ has a lower bound. For instance, if f is nondecreasing in the sense that the following holds:

$$\forall x, y \in P, \quad x \leq y \quad \Rightarrow \quad f(x) \leq f(y),$$

then f is automatically locally finitely bounded from below.

As another example, let $\mathbb{K} = \mathbb{C}$, $\mathcal{V}$, and the partial order $\leq$ on $\mathbb{C}$ be as in Example 3 (b). Then it is easy to see that every subset $A \subset \mathbb{C}$ which is bounded (in the usual sense, i.e., contained in a ball) has a lower bound (i.e., there is $\mu \in \mathbb{C}$ with $\mu \leq \alpha$ for all $\alpha \in A$). This observation implies that whenever $f : P \to \mathbb{C}$ is a map satisfying the condition

$$\forall \alpha \in \mathbb{C}, \quad f(P) \cap \{\beta \in \mathbb{C} : \beta \leq \alpha\} \text{ is bounded,} \tag{12}$$

then f will be locally finitely bounded from below – even if the image $f(P)$ is unbounded below. Take for example $P = \mathbb{R}$ endowed with the trivial preorder $\leq$ and the standard metric $d : \mathbb{R} \times \mathbb{R} \to [0, +\infty) \subset \mathbb{C}_+$ and consider the function $f : \mathbb{R} \to \mathbb{C}$, $x \mapsto |x|^\eta \sin(x) + ix$ with $\eta \in (0, 1)$. Then it is straightforward to check that (12) holds, so that f is locally finitely bounded from below. Also in this example the fact that f is continuous ensures that it is $\leq$-submonotone.

However, if the preorder on P is trivial and the restriction of the preorder on $\mathbb{K}$ is a total order on $\mathbb{K}'$, then the condition of being locally finitely bounded from below is equivalent to the condition that $f(P) \cap \mathbb{K}'$ has a lower bound.

We now give our generalization of Ekeland's principle.

Theorem 1. *Assume that (K1)–(K5), (P1)–(P2) hold. Assume that P is $\leq$-complete. Let $f : P \to \mathbb{K}$ be $\leq$-submonotone and locally finitely bounded from below.*

Then, for every $x \in \operatorname{dom}(f)$, there exists $y \in \operatorname{dom}(f)$ satisfying the following conditions:

(a) $x \leq y$;

(b) $d(x, y) \leq f(x) - f(y)$;

(c) *for all $(z, z') \in \operatorname{dom}(f) \times \operatorname{dom}(f)$ such that $y \leq z \leq z'$, we have*

$$(d(y, z) \leq f(y) - f(z) \ \ \text{and} \ \ d(z, z') \leq f(z) - f(z')) \quad \Rightarrow \quad d(z, z') = 0.$$

Proof. We consider the set

$$P' = \{u \in P : x \leq u \text{ and } f(u) \leq f(x)\} \subset \operatorname{dom}(f),$$

where the last inclusion is shown as follows: for every $u \in P'$, we have $f(u) \leq f(x) \in \mathbb{K}'$ whence $f(u) \in \mathbb{K}'$, i.e., $u \in \operatorname{dom}(f)$, by virtue of (K2) (b).

By exploiting the fact that f is locally finitely bounded from below, we find a finite set $M = M(x) \subset \mathbb{K}'$ such that

$$\forall u \in P', \quad \exists \mu \in M, \quad \mu \leq f(u). \tag{13}$$

We define a binary relation $\preceq$ on P' by setting

$$u \preceq v \quad \text{if} \quad (u \leq v \quad \text{and} \quad d(u,v) \leq f(u) - f(v))$$

and check that $\preceq$ is a preorder:

- The reflexivity of $\preceq$ comes from the fact that the relations considered on $\mathbb{K}$ and P are reflexive, and $d(u,u) = 0$ for all u.
- Let $u, v, w \in P'$ be such that $u \preceq v$ and $v \preceq w$. Then, $u \leq v$, $v \leq w$, $d(u,v) \leq f(u) - f(v)$, and $d(v,w) \leq f(v) - f(w)$.
  - The first two inequalities yield $u \leq w$ by transitivity of the relation $\leq$ on P.
  - The last two inequalities first tell us that $d(u,v)$ and $d(v,w)$ belong to $\mathbb{K}'$ by virtue of (K2) (b), since $f(u), f(v), f(w) \in \mathbb{K}'$. Then we can invoke (P2) and (K2) (c) which yield

$$\begin{aligned} d(u,w) &\leq d(u,v) + d(v,w) \\ &\leq f(u) - f(v) + f(v) - f(w) = f(u) - f(w). \end{aligned}$$

  Whence $u \preceq v$, and the transitivity is established, which completes the proof that $\preceq$ is a preorder on P'.

Clearly $(P')^2_{\preceq} := \{(u,v) \in P' \times P' : \ u \preceq v\} \subset P^2_{\leq}$ and thus the restriction $d' := d|_{(P')^2_{\preceq}} : (P')^2_{\preceq} \to \mathbb{K}$ is well defined; the triple $(P', \preceq, d')$ satisfies (P1).

Let us show that $(P', \preceq, d')$ also satisfies the hypotheses (i) and (ii) of Proposition 1. Let $(u_n)_{n \geq 0} \subset P'$ be an arbitrary $\preceq$-ascending sequence, which (by definition of $\preceq$) means that

$$(u_n) \text{ is } \leq\text{-ascending} \tag{14}$$

and

$$(0 \leq)\ d(u_n, u_m) \leq f(u_n) - f(u_m) \quad \text{whenever } n \leq m. \tag{15}$$

We claim that

$$\forall V \in \mathcal{V}, \quad \exists n_0, \quad n_0 \leq n \leq m \quad \Rightarrow \quad f(u_n) - f(u_m) \in V. \tag{16}$$

Assume to the contrary that there is $V \in \mathcal{V}$ such that for any n_0 we can find $m \geq n \geq n_0$ with $f(u_n) - f(u_m) \notin V$. Then we construct a nondecreasing sequence of indices $(\alpha_k) \subset \mathbb{N}$ as follows:

- Set $\alpha_0 = 0$.
- Once $\alpha_0 \leq \ldots \leq \alpha_{2k}$ are constructed, choose $\alpha_{2k+1}, \alpha_{2k+2}$ with $\alpha_{2k} \leq \alpha_{2k+1} \leq \alpha_{2k+2}$ such that $f(u_{\alpha_{2k+1}}) - f(u_{\alpha_{2k+2}}) \notin V$.

The so-obtained sequence (α_k) being nondecreasing, relation (15) implies that $f(u_{\alpha_k}) - f(u_{\alpha_{k+1}}) \geq 0$ for all $k \geq 0$, whence

$$f(u_{\alpha_{2k-2}}) - f(u_{\alpha_{2k}}) \geq f(u_{\alpha_{2k-1}}) - f(u_{\alpha_{2k}}) =: \beta_k$$

for all $k \geq 1$ by virtue of (K2) (c). Note also that, by construction, we have

$$\forall k \geq 1, \quad \beta_k \in \mathbb{K}'_+ \setminus V.$$

On the one hand, by invoking again (K2) (c) and using (13), we have

$$\begin{aligned} \forall n \geq 1, \quad \exists \mu \in M, \quad f(u_0) - \mu \quad &\geq \quad f(u_0) - f(u_{\alpha_{2n}}) \\ &= \quad \sum_{k=1}^{n} (f(u_{\alpha_{2k-2}}) - f(u_{\alpha_{2k}})) \\ &\geq \quad \beta_1 + \ldots + \beta_n. \end{aligned} \tag{17}$$

On the other hand, by virtue of (K4), for every $\mu \in M$ there is an integer $n_\mu \geq 1$ with $\beta_1 + \ldots + \beta_{n_\mu} \not\leq f(u_0) - \mu$, thus a fortiori $\beta_1 + \ldots + \beta_n \not\leq f(u_0) - \mu$ for all $n \geq n_\mu$ (since $\beta_k \geq 0$ for all k and by (K2) (c)). Since M is finite, we can set $n := \max\{n_\mu : \mu \in M\}$ and infer that

$$\forall \mu \in M, \quad \beta_1 + \ldots + \beta_n \not\leq f(u_0) - \mu.$$

This is contradictory with (17). Therefore, (16) is established.

By combining (15), (16), and (K3), we obtain that

$$(u_n) \text{ is a Cauchy sequence.} \tag{18}$$

Since (u_n) was an arbitrary $\preceq$-ascending sequence in P', we have therefore checked condition (ii) of Proposition 1.

In order to apply the proposition, it remains to complete the verification of (i), that is, to show that (u_n) has an upper bound in the preordered set $(P', \preceq)$. Note that (14) and (18) imply that (u_n) is also a $\leq$-Cauchy sequence in P. The assumption that P is $\leq$-complete yields an element $u \in P$ such that $u_n \overset{\leq}{\Rightarrow} u$. Let us show that this element u is the desired upper bound.

By definition of the $\leq$-convergence $u_n \overset{\leq}{\Rightarrow} u$, we have in particular

$$u_n \leq u \quad \text{for all } n \tag{19}$$

thus $x \leq u$ (since $u_n \in P'$). From (15) and (K2) (c), we see that the sequence $(f(u_n)) \subset \mathbb{K}'$ is such that $f(u_n) \geq f(u_m)$ whenever $n \leq m$. Using that f is $\leq$-submonotone, we get

$$f(u) \leq f(u_n) \quad \text{for all } n \tag{20}$$

thus $f(u) \leq f(x)$ (since $u_n \in P'$). This already shows that $u \in P'$.

Relation (19) also implies that, for all n and m such that $n \leq m$, we have $u_n \leq u_m \leq u$. By invoking (P2) and using (15), we obtain that

$$\begin{aligned} d(u_n, u) &\leq d(u_n, u_m) + d(u_m, u) \\ &\leq f(u_n) - f(u_m) + d(u_m, u) \\ &= f(u_n) - f(u) + f(u) - f(u_m) + d(u_m, u). \end{aligned}$$

Knowing that $f(u_n), f(u_m), f(u) \in \mathbb{K}'$ (since $u_n, u_m, u \in P'$) and $f(u) - f(u_m) \leq 0$ (by (20)), by (K2) (c) we deduce that

$$d(u_n, u) - f(u_n) + f(u) \leq d(u_m, u) \quad \text{whenever } n \leq m.$$

Since $d(u_m, u) \to 0$ (because $u_m \overset{\leq}{\to} u$), by virtue of (K5) we infer that $d(u_n, u) - f(u_n) + f(u) \leq 0$, whence

$$d(u_n, u) \leq f(u_n) - f(u) \quad \text{for all } n$$

(since $f(u_n), f(u) \in \mathbb{K}'$). This fact combined with (19) shows that

$$u_n \preceq u \quad \text{for all } n. \tag{21}$$

Therefore, u is an upper bound of (u_n) in the preordered set $(P', \preceq)$, which shows that condition (i) of Proposition 1 is valid, and finally allows us to apply this proposition to P' endowed with the preorder $\preceq$ and the restriction d' of d.

The application of Proposition 1 provides an element $y \in P'$ (so $y \in \operatorname{dom}(f)$) such that $x \preceq y$ and y is d'-maximal in P'. The fact that $x \preceq y$ exactly means that y satisfies parts (a) and (b) of the theorem. For showing (c), let $z, z' \in \operatorname{dom}(f)$ be such that

$$y \leq z \leq z', \quad d(y, z) \leq f(y) - f(z), \quad \text{and} \quad d(z, z') \leq f(z) - f(z') \tag{22}$$

and let us show that this forces $d(z, z') = 0$. The first part of (22) yields $x \leq y \leq z \leq z'$. Since z, z' belong to $\operatorname{dom}(f)$, we get $f(z), f(z') \in \mathbb{K}'$. Then, by (K2) (c) and the fact that d takes values in $\mathbb{K}_+$ (see (P2)), the last two inequalities in (22) imply $f(z') \leq f(z) \leq f(y) \leq f(x)$. Whence $z, z' \in P'$. Now (22) means that $y \preceq z \preceq z'$, and the property that y is d'-maximal in P' yields finally $d(z, z') = d'(z, z') = 0$. $\square$

Example 4. As an illustration of the result, we point out a property of "existence of almost minimal point" in a general context.

Let $(\mathbb{K},+,\leq)$ be a preordered monoid satisfying conditions (K1)–(K5). Let P be an arbitrary nonempty set and $f : P \to \mathbb{K}$ a function which is locally finitely bounded from below and such that $\mathrm{dom}(f) \neq \emptyset$. Then, for every $\alpha \in \mathbb{K}'_+$, $\alpha \neq 0$, there is $y_\alpha \in \mathrm{dom}(f)$ with

$$\forall z \in P, \quad f(z) \not\leq f(y_\alpha) - \alpha.$$

Indeed, let $\preceq$ be the trivial preorder on P. Define $d_\alpha : P^2_\preceq = P \times P \to \mathbb{K}$ by letting

$$d_\alpha(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \alpha & \text{if } x \neq y. \end{cases}$$

It is clear that (P1)–(P2) hold. Moreover, every sequence $(x_n) \subset P$ is $\preceq$-ascending (since the preorder on P is trivial), and (x_n) is a Cauchy sequence if and only if it is stationary, which is in fact also equivalent to having that (x_n) is $\preceq$-convergent. Thus, endowed with d_α, the set P is $\preceq$-complete.

The fact that every $\preceq$-convergent sequence in P is stationary (and thus has a unique limit since d_α is nondegenerate) also easily implies that f is $\preceq$-submonotone. We can apply Theorem 1: fixing $x \in \mathrm{dom}(f)$, there is $y_\alpha \in \mathrm{dom}(f)$ such that

$$\forall z \in \mathrm{dom}(f), \quad d_\alpha(y_\alpha, z) \leq f(y_\alpha) - f(z) \quad \Rightarrow \quad d_\alpha(y_\alpha, z) = 0. \tag{23}$$

Now, arguing by contradiction, assume that there is $z \in P$ with $f(z) \leq f(y_\alpha) - \alpha$. Then $f(z) \in \mathbb{K}'$, i.e., $z \in \mathrm{dom}(f)$. Moreover, since $d_\alpha(y_\alpha, z) \leq \alpha$, the previous inequality implies $d_\alpha(y_\alpha, z) \leq f(y_\alpha) - f(z)$, which yields $d_\alpha(y_\alpha, z) = 0$ by virtue of (23). However, we then have $y_\alpha = z$ by definition of d_α, so that the relation $f(z) \leq f(y_\alpha) - \alpha$ becomes impossible as it implies $\alpha \in \mathbb{K}_- \cap \mathbb{K}'_+ \setminus \{0\}$ whereas $\mathbb{K}_- \cap \mathbb{K}'_+ = \{0\}$ (since $\leq$ is a partial order on $\mathbb{K}'$; see Remark 8). This completes the argument.

References

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