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WAVE SOLUTIONS FOR HYPERBOLIC SYSTEMS*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

In this paper we propose a reduction procedure for determining generalized traveling waves for first order quasilinear hyperbolic non-homogeneous systems. The basic idea is to look for solutions of the governing model that satisfy a further set of differential constraints. Some applications are given for a barotropic fluid with a source term.

Keywords: differential constraints, traveling waves, ideal fluid.
MSC: 35C07, 35C05, 35L40.

1 Introduction

Determining exact solutions of partial differential equations (PDEs) is of great interest not only from a theoretical point of view but also for possible applications. To this end, over the years many mathematical approaches have been proposed, most of them based on group analysis (i.e., classical and nonclassical symmetries [1,2], weak symmetries [18], conditional symmetries [10] (see also [14])).

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Within such a theoretical framework, in 1964 J.J. Yanenko proposed the Method of Differential Equations [24] and he applied it to the fluid-dynamics equations. In order to explain better the basic idea of such an approach and also for further convenience, we give the following simple example.

Let us consider the PDE

$$u_t + a(u)u_x = f(u). \tag{1}$$

A particular class of exact solutions admitted by (1) are the famous traveling waves, where

$$u = U(\sigma), \qquad \sigma = x - s\,t, \tag{2}$$

with s constant. In order to calculate the traveling waves of (1) we have to substitute the ansatz (2) into (1) and solve the resulting ordinary differential equation. The function (2) also satisfies the linear PDE

$$u_t + su_x = 0. \tag{3}$$

Therefore, if we cannot determine the traveling waves of (1), we can look for the particular solutions of (1) which also satisfy (3). In such a case, since the equation (3) selects the class of exact solutions of (1) we are looking for, they play the role of differential constraints. Of course, since the unknown $u(x,t)$ must satisfy (1) along with (3) an overdetermined system is obtained and some compatibility conditions must be required. In this simple case, it is easy to verify that equations (1) and (3) are always compatible.

More in general, given a system of PDEs

$$F^i(x, t, \mathbf{U}, \mathbf{U}_x, \mathbf{U}_t, \mathbf{U}_{xx}, \mathbf{U}_{xt}, ...) = 0; \qquad i = 1, .., N \tag{4}$$

Yanenko proposed to append to it a further set of differential constraints

$$G^k(x, t, \mathbf{U}, \mathbf{U}_x, \mathbf{U}_t, \mathbf{U}_{xx}, \mathbf{U}_{xt}, ...) = 0; \qquad k = 1, .., M \tag{5}$$

and to look for the exact solutions of (4) and (5). Of course, the compatibility of the overdetermined system (4) and (5) must be required. The method is general and, in fact, it includes many of the known approaches for determining exact solutions of PDEs. Unfortunately, such a generality leads to an enormous complicated algorithm such that, without any further hypotheses, the method is not always useful for studying problems of interest for the applications. To overcome such a difficulty, in [7] – [19] it was required that (4) and (5) are in involution (i.e. no new differential relations can be obtained from them by differentiation). The involutivness requirement simplifies the algorithms of the method, in particular for hyperbolic systems of equations. In fact, many results concerning wave problems described by hyperbolic systems are obtained [15] – [13]. Moreover, an interesting application of the method to a parabolic model was given in [20].

Within such a framework, here we develop a reduction procedure which permits to determine generalized traveling wave solutions for first order quasilinear nonhomogeneous hyperbolic systems.

The paper is organized as follows. In section 2 we recall briefly the algorithm of the method of differential constraints applied to hyperbolic systems. In section 3 we illustrate how the use of the $k-$Riemann invariants may simplify such a procedure. In section 4 an approach for characterizing generalized traveling waves is developed. Finally, some conclusions and final remarks are given in section 5.

2 General procedure

In this section we illustrate the procedure related to the Method of Differential Constraints for a first order quasilinear strictly hyperbolic system. Let us consider the quasilinear system

$$\mathbf{U}_t + A\left(\mathbf{U}\right)\mathbf{U}_x = \mathbf{B}\left(\mathbf{U}\right), \tag{6}$$

where $\mathbf{U} \in \mathbf{R}^N$ is the field vector, A the $N \times N$ matrix coefficients, $\mathbf{B} \in \mathbf{R}^N$ the source vector, while t and x denote, respectively, time and space coordinates. We assume the hyperbolicity (in the $t-$direction) of (6) and denote by $\lambda^i\left(\mathbf{U}\right)$ the eigenvalues of A (characteristic speeds) while the corresponding right and left eigenvectors are indicated, respectively, by $\mathbf{d}^i\left(\mathbf{U}\right)$ and $\mathbf{l}^i\left(\mathbf{U}\right)$. Moreover, we assume $\lambda^i \neq \lambda^j$, $\forall i \neq j$ (that is, the system (6) is strictly hyperbolic). We choose $\mathbf{d}^i$ and $\mathbf{l}^i$ so that the orthonormal condition is satisfied ($\mathbf{d}^i \cdot \mathbf{l}^j = \delta^{ij}$). We add to (6) the set of differential constraints

$$\mathbf{C}^i(x,t,\mathbf{U}) \cdot \mathbf{U}_x = p^i(x,t,\mathbf{U}) \qquad i = 1, ..., M \leq N, \tag{7}$$

where the functions C^i and p^i are still not specified. For requiring the compatibility between (6) and (7) we differentiate them with respect to t and x, so that we find

$$\left(\frac{\partial \mathbf{C}^i}{\partial t} + \frac{\partial \mathbf{C}^i}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial t}\right) \cdot \mathbf{U}_x + \mathbf{C}^i \cdot \left(\frac{dB}{dx} - \frac{d(AU_x)}{dx}\right) = \frac{\partial p^i}{\partial t} + \frac{\partial p^i}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial t} \tag{8}$$

$$\left(\frac{\partial \mathbf{C}^i}{\partial x} + \frac{\partial \mathbf{C}^i}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial x}\right) \cdot \mathbf{U}_x + \mathbf{C}^i \cdot \mathbf{U}_{xx} = \frac{\partial p^i}{\partial x} + \frac{\partial p^i}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial x} \tag{9}$$

where $\frac{d}{dx}$ means the total derivative with respect to x. In order to eliminate $\mathbf{U}_{xx}$ between (8) and (9) it soon follows that the vectors $\mathbf{C}_i$ must belong to the subspace of the left eigenvectors of A so that, owing to the strictly hyperbolicity of (6), without loss of generality we can choose $\mathbf{C}_i = \mathbf{l}^i$ and therefore we prove that the most general first-order differential constraints admitted by (6) have the form

$$\mathbf{l}^i \cdot \mathbf{U}_x = p^i(x, t, \mathbf{U}) \qquad i = 1, ..., M \leq N. \tag{10}$$

The case of great interest for the nonlinear wave problem is when the number of constraints is $M = N - 1$. In this case, from (6) and (10) we have

$$\mathbf{U}_t = \mathbf{B} - \sum_{i=1}^{N-1} p^i \lambda^i \mathbf{d}^i - \pi \lambda^N \mathbf{d}^N \tag{11}$$

$$\mathbf{U}_x = \sum_{i=1}^{N-1} p^i \mathbf{d}^i + \pi \mathbf{d}^N, \tag{12}$$

where $\pi(x, t)$ is arbitrary. If we wont that the overdetermined system (6), (10) is in involution, from (11), (12) we have to require $\mathbf{U}_{tx} = \mathbf{U}_{xt}\ \forall \pi$, so that the following compatibility conditions are obtained

$$\begin{aligned} & p^i_t + \lambda^i p^i_x + \nabla p^i \left(\mathbf{B} - \sum_{j=1}^{N-1} p^j \left(\lambda^j - \lambda^i\right) \mathbf{d}^j \right) \\ & + \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} p^j p^k \left(\lambda^j - \lambda^k\right) \mathbf{l}^i \nabla \mathbf{d}^j \mathbf{d}^k \\ & + \sum_{k=1}^{N-1} p^k \left(\mathbf{l}^i \left(\nabla \mathbf{d}^k \mathbf{B} - \nabla \mathbf{B}\ \mathbf{d}^k \right) + p^i \nabla \lambda^i \mathbf{d}^k \right) = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} & \left(\lambda^i - \lambda^N\right) \nabla p^i \mathbf{d}^N + \sum_{k=1}^{N-1} p^k \left(\lambda^k - \lambda^N\right) \mathbf{l}^i \left(\nabla \mathbf{d}^k \mathbf{d}^N - \nabla \mathbf{d}^N \mathbf{d}^k \right) \\ & + \mathbf{l}^i \left(\nabla \mathbf{d}^N \mathbf{B} - \nabla \mathbf{B}\ \mathbf{d}^N \right) + p^i \nabla \lambda^i \mathbf{d}^N = 0, \end{aligned} \tag{14}$$

where $\nabla = \frac{\partial}{\partial \mathbf{U}}$ and $i = 1...(N - 1)$.

Furthermore, from (11) and (12) we obtain

$$\mathbf{U}_t + \lambda^N \mathbf{U}_x = \mathbf{B} + \sum_{i=1}^{N-1} p^i \left(\lambda^N - \lambda^i\right) \mathbf{d}^i. \tag{15}$$

Since the left-hand side of system (15) involves the derivative of $\mathbf{U}$ along the characteristics associated to $\lambda^{(N)}$, equations (15) can be integrated by using the standard method of characteristics. By substituting the resulting solutions into the constraints (10) we get (see [14] for more details)

$$\mathbf{l}^i(\mathbf{U}_0(x)) \cdot \frac{d\mathbf{U}_0(x)}{dx} = p^i\,(x, 0, \mathbf{U}_0) \qquad i = 1 \ldots N-1, \tag{16}$$

where $\mathbf{U}_0(x) = \mathbf{U}(x,0)$. Since the N initial conditions $\mathbf{U}_0(x)$ must satisfy the $N-1$ constraints (16) the solutions which can be obtained by integration of (15) are determined in terms of one arbitrary functions.

It is of some interest to notice that in the case where $\mathbf{B} = 0$ and $p^i = 0$, the compatibility conditions (13) and (14) are identically satisfied and the above-illustrated procedure permits to determine the classical simple wave solutions.

3 An alternative approach

The crucial point of the method of differential constraints is to study and possibly to solve the compatibility conditions (13), (14). Unfortunately, it is a very hard task not only to find the general solution of (13), (14) but also to determine particular solutions of such an overdetermined system. In order to simplify the analysis of (13), (14), quite recently in [9] an alternative approach based on the use of the Riemann invariants was proposed.

The basic idea is the following. We fix one of the characteristic speeds of (6) (for instance, we can choose, without loss of generality, λ^N) and we compute its Riemann invariants defined by

$$\nabla R^\alpha \cdot \mathbf{d}^N = 0, \quad \alpha = 1, ..., N-1. \tag{17}$$

It is well known that associated to λ^N there exist $N-1$ Riemann invariants whose gradients are linearly independent (see, for instance, [23]). Therefore, owing to (17), we can write

$$\nabla R^\alpha = \sigma^\alpha_\beta \, \mathbf{l}^\beta, \quad \alpha, \beta = 1, ..., N-1, \tag{18}$$

where σ^α_β are the components of ∇R^α with respect to the basis of the left eigenvectors. Moreover, here and what follows, the greek indices vary from 1 to $N-1$. Taking (18) into account, the constraints (10) assume the form

$$\frac{\partial R^\alpha}{\partial x} = \sigma^\alpha_\beta \, q^\beta \tag{19}$$

and, in turn, equations (15) give

$$\frac{\partial R^\alpha}{\partial t} + \lambda^N \frac{\partial R^\alpha}{\partial x} = \sigma^\alpha_\beta\, \mathbf{l}^\beta \cdot \mathbf{B} + \left(\lambda^N - \lambda^\beta\right) \sigma^\alpha_\beta\, q^\beta. \tag{20}$$

We choose one of the field variables of $\mathbf{U}$ (say, for instance, $v = u_j$) and we add to the $N-1$ equations (20) the $j-$th equation arising from (15)

$$\frac{\partial v}{\partial t} + \lambda^N \frac{\partial v}{\partial x} = B_j + \left(\lambda^N - \lambda^\beta\right) q^\beta d^\beta_j, \tag{21}$$

where B_j and d^α_j denote, respectively, the $j-$th component of $\mathbf{B}$ and $\mathbf{d}^\alpha$. Of course, we can always choose v in such a way the variable transformation

$$R^\alpha = R^\alpha(\mathbf{U}), \quad v = u_j \tag{22}$$

is not singular. Therefore, the equations (15) transform to (20) and (21) while the constraints (10) take the form (19). Integration of (20), (21) along with (19) gives, through the change of variables (22), exact solutions of (6), (10). Furthermore, by requiring the the involutiveness of the overdetermined system (19)-(21), the following compatibility conditions are obtained:

$$\left(\lambda^\beta - \lambda^N\right) \sigma^\alpha_\beta \frac{\partial q^\beta}{\partial v} = \left(\left(\lambda^N - \lambda^\beta\right) \frac{\partial \sigma^\alpha_\beta}{\partial v} - \sigma^\alpha_\beta \frac{\partial \lambda^\beta}{\partial v}\right) q^\beta + \frac{\partial}{\partial v}\left(\sigma^\alpha_\beta\, \mathbf{l}^\beta \cdot \mathbf{B}\right) \tag{23}$$

$$\frac{\partial w^\alpha}{\partial R^\gamma} z^\gamma - \frac{\partial z^\alpha}{\partial R^\gamma} w^\gamma + \frac{\partial w^\alpha}{\partial v}\left(B_j + \left(\lambda^N - \lambda^\gamma\right) q^\gamma d^\gamma_j\right) = 0 \tag{24}$$

where, for simplicity, we set

$$w^\alpha = \sigma^\alpha_\beta\, q^\beta, \qquad z^\alpha = \sigma^\alpha_\beta\, \mathbf{l}^\beta \cdot \mathbf{B} - \lambda^\beta \sigma^\alpha_\beta\, q^\beta. \tag{25}$$

The analysis of the equations (23), (24) is not so hard as that of (13), (14). In fact, we notice that the $N-1$ equations (23) characterize a linear ODE-like system in the unknown q^α which, due the strictly hyperbolicity of (6), can be written in normal form. Once the functions q^α are determined from (23), substituting them in (24), we find a set of $N-1$ structural conditions that the coefficients of the system (6) must satisfy to guarantee the compatibility among (6) and (10).

Two significant cases where (23) and (24) are solved under suitable structural conditions are the following (see [9] for a more general analysis).

i) We assume $q^\alpha = 0$ so that from (23), (24) we find

$$\sigma^\alpha_\beta \, \mathbf{l}^\beta \cdot \mathbf{B} = F^\alpha(R^\gamma), \tag{26}$$

where F^α are not specified functions. If the structural condition (26) is satisfied, then, from (19) we find $R^\alpha = R^\alpha(t)$ and taking into account (20), (21) exact solutions of (6) are obtained by solving the system

$$\frac{dR^\alpha}{dt} = F^\alpha(R^\gamma), \tag{27}$$

$$\frac{\partial v}{\partial t} + \lambda^N \left(v, R^\alpha(t)\right) \frac{\partial v}{\partial x} = B_j \left(v, R^\alpha(t)\right). \tag{28}$$

In passing we notice that the equations (27) are decoupled from (28). In fact, once $R^\alpha(t)$ are determined from (27), exact solutions of the governing system can be obtained by solving the quasilinear non-autonomous PDE (28). Furthermore, when $\mathbf{B} = 0$ also $F^\alpha = 0$, the compatibility conditions (26) are identically satisfied and the equations (27), (28) characterize the simple waves

$$R^\alpha\left(\mathbf{U}\right) = k^\alpha$$
$$u_j = v = v_0(\xi), \qquad x = \lambda^N\left(v_0\left(\xi\right), k^\alpha\right) t + \xi,$$

where k^α are arbitrary constants and $v_0(x) = v(x, 0)$.

ii) We require

$$\sigma^\alpha_\beta \left(\mathbf{l}^\beta \cdot \mathbf{B} - \lambda^\beta q^\beta\right) = F^\alpha(R^\gamma), \qquad \sigma^\alpha_\beta \, q^\beta = G^\alpha(R^\gamma) \tag{29}$$

so that condition (23) is identically satisfied, while from (24) we find

$$\frac{dG^\alpha}{dR^\beta} F^\beta - \frac{dF^\alpha}{dR^\beta} G^\beta = 0 \tag{30}$$

where F^α and G^α are not specified. The functions q^α can be calculated by solving the linear algebraic system $(29)_2$ while from $(29)_1$ supplemented by (30) a set of $N-1$ structural conditions which must be satisfied by the coefficients of (6) in order that such a procedure holds, are obtained.

Furthermore, equations (20) assume the form

$$\frac{\partial R^\alpha}{\partial t} + \lambda^N \frac{\partial R^\alpha}{\partial x} = F^\alpha + \lambda^N G^\alpha \tag{31}$$

while the constraints (21) specialize to

$$\frac{\partial R^\alpha}{\partial x} = G^\alpha. \tag{32}$$

It is of interest to notice that if $\lambda^N(R^\gamma)$, then equations (31) are decoupled from (21). Thus, once $R^\alpha(x,t)$ are determined from (31), exact solutions of (6) can be obtained by integrating the PDE (21) by means of the method of characteristics. Of course, the corresponding initial data must obey the constraints (32).

4 Generalized traveling waves

Within the framework of the method of differential constraints, the main aim of this section is to develop a reduction procedure which permit to determine a class of exact solutions which generalize the classical traveling waves.

To this end we append to system (6) the following constraints

$$\mathbf{U}_t + s\mathbf{U}_x = \mathbf{F}\left(\mathbf{U}\right), \tag{33}$$

where s is a constant and $\mathbf{F}\left(\mathbf{U}\right)$ is unspecified. We decompose the vector $\mathbf{U}_x$ along the basis of the right eingenvectors

$$\mathbf{U}_x = \pi_j \mathbf{d}^j \tag{34}$$

so that, from (6) and (33) we have

$$\mathbf{U}_t = \mathbf{F} - s\pi_j \mathbf{d}^j, \qquad \mathbf{U}_x = \pi_j \mathbf{d}^j, \tag{35}$$

where

$$\pi_i = \frac{1}{\lambda^i - s}\,\mathbf{l}^i \cdot \left(\mathbf{B} - \mathbf{F}\right). \tag{36}$$

It is simple to verify that the compatibility between relations (35) leads to

$$F_i \frac{\partial \pi_s}{\partial u_i} = l^s_k \left(\frac{\partial F_k}{\partial u_i} d^j_i - \frac{\partial d^j_k}{\partial u_i} F_i \right) \pi_j. \tag{37}$$

Once π_i are determined by solving the linear PDEs system (37), from (36) we find $\mathbf{F}$ and by integration of (35) a class of exact solutions of (6) are found. Such a solutions generalize the classical traveling waves because when $\mathbf{F} = 0$, the compatibility conditions (37) are identically satisfied and from (35) traveling wave solutions are found.

Remark 1. Owing to (35) along with (36), if there exists a value $\mathbf{U}^\star$ of the field $\mathbf{U}$ such that $\lambda^i\left(\mathbf{U}^\star\right) = s$, a singularity in the traveling wave solution may appear depending if $\left(\mathbf{l}^i \cdot (\mathbf{B}-\mathbf{F})\right)_{U=U^\star}$ vanishes or not. In the last case a sub-shock appears [21].

As an example, here we apply our procedure the Euler system describing a barotropic fluid

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{38}$$

$$u_t + uu_x + \frac{c^2}{\rho}\rho_x = f(\rho, u), \tag{39}$$

where ρ is the mass density, u the velocity, $c = \sqrt{p_\rho}$ the sound velocity with $p(\rho)$ the pressure while $f(\rho, u)$ denotes a force term. The characteristics velocities of (38), (39) are

$$\lambda^1 = u - c, \qquad \lambda^2 = u + c \tag{40}$$

while the corresponding left and right eigenvectors are

$$\mathbf{l}^1 = \frac{1}{2}\left(1, -\frac{\rho}{c}\right), \qquad \mathbf{l}^2 = \frac{1}{2}\left(1, \frac{\rho}{c}\right) \tag{41}$$

$$\mathbf{d}^1 = \left(1, -\frac{c}{\rho}\right)^T, \qquad \mathbf{d}^2 = \left(1, \frac{c}{\rho}\right)^T, \tag{42}$$

where T means for transposition.

Next, we consider the compatibility conditions (37) where, for simplicity, we assume $F_1 = 0$. In such a case equations (37) assume the form

$$F_2 \frac{\partial}{\partial u}(\pi_1 + \pi_2) = 0$$

$$F_2 \frac{\partial}{\partial u}(\pi_2 - \pi_1) = \frac{\rho}{c}\left(\frac{\partial F_2}{\partial \rho}(\pi_1 + \pi_2) + \frac{c}{\rho}\frac{\partial F_2}{\partial u}(\pi_2 - \pi_1)\right)$$

whose integration gives

$$\pi_2 - \pi_1 = \frac{1}{\Phi_u}\left(-\frac{\rho\alpha(\rho)}{c(\rho)}\Phi_\rho + \beta(\rho)\right), \qquad \pi_2 + \pi_1 = \alpha(\rho), \tag{43}$$

where $\alpha(\rho)$ and $\beta(\rho)$ are not specified, while

$$\Phi_u = \frac{1}{F_2(\rho, u)}.$$

Owing to (43), from (36) we determine F_2 as well as the source term $f(\rho, u)$. For instance, if $F_2(u)$ we soon find

$$F_2 = k_1(u - s), \qquad f = k_1(u - s) + k_1 \frac{c\beta}{\rho}\left((u - s)^2 - c^2\right), \tag{44}$$

where k_1 is a constant. Furthermore, the function $\alpha(\rho)$ must assume the form

$$\alpha = -k_1 c\beta.$$

In such a case, integration of (35) leads to

$$\rho = R(\sigma), \qquad u = s + \frac{a_0}{R(\sigma)} e^{k_1 t}, \tag{45}$$

where $\sigma = x - st$, a_0 is a constant, while $R(\sigma)$ is given by solving the ODE

$$\frac{dR}{d\sigma} = -k_1 c(R)\beta(R). \tag{46}$$

We notice that the function $\beta(\rho)$ that is involved in the force term $(44)_2$ is still not specified. For instance, if we choose

$$\beta = \frac{\rho}{c}$$

from (45), (46) the following generalized traveling wave solution is obtained

$$\rho = \rho_0 e^{-k_1(x - st)}, \qquad u = s + \frac{a_0}{\rho_0} e^{k_1(x - 2st)} \tag{47}$$

where ρ_0 is a constant. Finally, we notice that the solution (45), (46) is obtained $\forall p(\rho)$.

5 Conclusions and final remarks

In this paper we developed a reduction procedure for determining a special class of exact solutions admitted by hyperbolic first order systems. The idea is to append to the original systems a further set of particular nonhomogeneous differential constraints which, in the homogeneous case, characterize traveling waves. Therefore, the solutions we obtained generalize the known traveling wave solutions.

To accomplish such a procedure, we are led to integrate the linear PDEs system (37) and, in turn, from (36) to calculate the vector $\mathbf{F}(\mathbf{U})$ involved in the constraints (33). Once π_i or, equivalently, $\mathbf{F}$ are determined, generalized traveling waves of (6) can be obtained by solving the ODEs (35). We applied such a procedure to the hyperbolic system describing a barotropic fluid. The exact solutions we obtained for such equations are determined for any pressure law.

We conclude by noticing that the traveling waves are, in general, non admitted by non-autonomous systems. Our procedure can be applied also to systems like (6) where the matrix coefficients A and/or the source vector $\mathbf{B}$ depend on the field $\mathbf{U}$ as well as on the variables x and t. In fact, in such a case we have to require that the source vector $\mathbf{F}$ involved in the constraints (33) depends on $(\mathbf{U}, x, t)$. As a consequence, the compatibility conditions (37) must be modified by adding the derivatives of π_i with respect to x and t and from (35) generalized traveling waves are obtained also in the non-autonomous case.

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