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A PLANAR MACHINE FOR THE COMPLEX EXPONENTIAL*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

Mechanical devices for tracing curves offer a tangible approach to geometry, complementing symbolic abstraction with physical construction. Starting from the historical machine for the real exponential, we introduce a planar mechanism that solves $f = f'$ in the complex domain, with a natural extension of the real case. To enhance accessibility and visualization, the behavior of the machine is illustrated using Dynamic Geometry simulations.

Keywords: complex exponential, mathematical machines, history of scientific instruments, dynamic geometry system, mathematics education.

MSC: 51M15, 97I80, 01A50.

1 Introduction

Throughout the history of mathematics, mechanical devices for tracing curves have served as more than just visualization tools—they embody a distinct way of understanding geometry, rooted not in symbolic abstraction but in real-world construction.

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While algebraic curves admit mechanical construction through classical linkage systems—culminating in the 1876 universality theorem by Alfred Bray Kempe (1849-1922) in [3], which proves that any algebraic curve can be drawn using suitable linkages—this framework reaches its limits when applied to transcendental curves, such as the exponential. To transcend these boundaries, one must move beyond pure positional linkages and incorporate directional constraints: in particular, the introduction of a wheel to guide the tangent marks a shift from positional mechanics to differential machines. We formalized this conceptual extension in [5], where machines that combine links and wheels are used to impose differential conditions.

In this spirit, the real exponential function has inspired the design of a historical machine that realizes the differential condition $f' = f$ by imposing a geometric constraint: the constant subtangent. At first glance, the complex exponential function seems to resist such mechanical embodiment and its behavior can appear incompatible with a purely planar construction. However, in this paper, we introduce a planar machine that realizes the complex exponential by generalizing the solution of $f = f'$ from the real to the complex case in a quite natural way. We also performed a dynamic simulation in GeoGebra, the link to which is provided in the conclusions.

2 Historical background

To address the problem of tracing curves by tangent conditions, we require some historical context, as pointed out with D. Crippa in [1], where more references are available. In 1638, Florimond de Beaune (1601–1652) posed a subtle yet groundbreaking question to René Descartes (1596–1650): Find the curve whose subtangent remains constant. In a Cartesian plane, the subtangent of a curve at a point (x, y) is the horizontal distance between the projection of the point onto the abscissa, namely $(x, 0)$, and the point where the tangent line at (x, y) intersects the x axis. Descartes solved the problem identifying the exponential curve and confirming its unique properties (cf. the left of Fig. 1). However, his solution remained purely theoretical. He offered no proposal for a device to draw the curve mechanically, and more broadly, no insight into how inverse tangent problems could be realized physically. For Descartes, such transcendental curves lived only in the realm of symbols, not instruments.

This left a vacuum in the mechanical visualization of such curves. Nearly four decades later, Claude Perrault's 1676 tractrix construction changed the narrative. Unlike de Beaune's problem, the tractrix emerged from a real physical situation. Imagine dragging an object (such as a chain watch) across a flat surface by means of a taut string, while the dragging point moves along a straight path. The string remained tangent to the path at every moment, embodying the very kind of condition de Beaune had asked about, except now the tangent was imposed through friction and constraint, not only theoretically.

In this sense, the tractrix marks the first true mechanical solution to an inverse tangent problem. Its geometry was not just imagined; it was performed. Although rudimentary, Perrault's design set the stage for more sophisticated curve-tracing machines developed in the first half of the 18th century. Among these, Giovanni Poleni (1683–1761) developed some devices that proved particularly noteworthy. He described them with an engineer's precision in a letter to Jacob Hermann published in [7]. By incorporating guiding wheels and bars to actively steer the tangent, he extended the tractrix principle to more abstract curves, inventing and constructing a machine for the exponential, as visible on the right of Fig. 1.

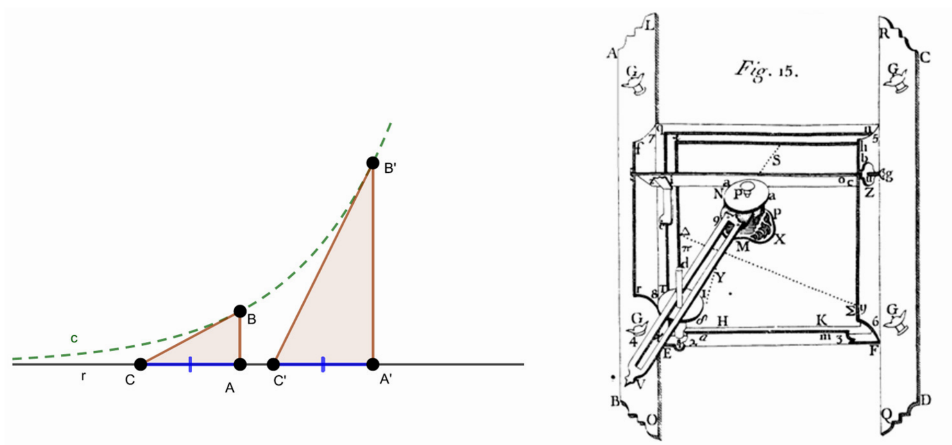

Figure 1: Left: the constancy of the subtangent for the exponential. Right: Poleni's machine for the exponential curve (reproduced from [7]).

To get an idea of the behavior of the machine, you can explore the dynamic simulation https://www.geogebra.org/m/sbqy4pdk. The point $A = (x, 0)$ is free to slide along the abscissa, and A is our input; the point B must remain on a vertical rod attached to A. The point $C = (x - 1, 0)$ moves in function to A-for example, by a rod. If we put in B a wheel perpendicular to the plane (represented as a thick segment centered in B) oriented as BC, the wheel poses the condition that the tangent in B has to point to C. That means that the slope of BC is the ordinate of B (we consider $AC = 1$ as unit-length subtangent). If we add the initial condition $B = (0, 1)$, the point B is constrained to stay on $(x, \exp(x))$.

3 From real to complex case

We have seen that, by extending linkages with wheels, it is possible to introduce a machine for the exponential in the real case. With the Argand-Gauss plane, we can consider any point of the plane as a complex number. For example, Emch in [2] introduces linkages performing complex algebraic operations between points considered as dynamical complex values. Hence, it is quite natural to ask whether it is possible to use this class of planar machines to construct the complex exponential by solving $f = f'$ on complex values. Quoting Needham [6] (p. 194):

> "In the ordinary real calculus we have a potent means of visualizing the derivative f' of a function f from $\mathbb{R}$ to $\mathbb{R}$, namely, as the slope of the graph $y = f(x)$. See Fig. 2(left). Unfortunately, due to our lack of four-dimensional imagination, we can't draw the graph of a complex function, and hence we cannot generalize this particular conception of the derivative in any obvious way.
>
> As a first step towards a successful generalization, we simply split the axes apart, so that Fig. 2(left) becomes Fig. 2(right)."

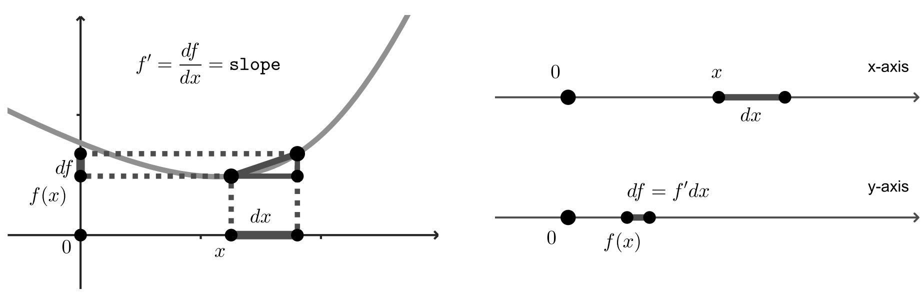

Figure 2: (Left) The usual representation of the real derivative as the slope of the graph $y = f(x)$. (Right) The representation of the same function f splitting the axes, evincing how the variation of the input dx implies a variation in the output df. Figure redrawn following [6] (p. 195).

According to Needham's purpose of visualizing the complex derivative as the two-dimensional case of the real one, respecting Cauchy-Riemann equations, the first suggestion is to represent the motion of input and output of the function on two distinct graphs. Thus, for the real case, we have two one-dimensional graphs, and for the complex case two two-dimensional graphs, one for the input and the other for the output.

We, too, aim to cope with the complex derivative by avoiding the introduction of a sensitively unimaginable four-dimensional space, but in a different way. Indeed, instead of splitting the input and the output in different planes, we keep staying on a single two-dimensional plane so that, for example, we can generalize the visualization of the derivative as the slope of the real function. But staying on a two-dimensional plane instead of a four-dimensional space will cause to lose the bijective relation between the graph and the function. This means that we will no longer have the property that a point in the plane defines a unique input/output couple (so the function will no longer be defined by its planar graph), but in the meanwhile if we consider a certain point as input, we can dynamically find the position of the related output (i.e., we lose the static representation but obtain a dynamic one). In particular, we try to reach the complex derivative extending in a "natural way" the real one.

By adopting complex coordinates, the usual representation of the graph of a real function is the graph of the output point $B = x + if(x)$, where the input point is $A = x \in \mathbb{R}$. Let's introduce the *pivot point* $P = A - \frac{f(A)}{f'(A)}$, the intersection of the tangent to $x + if(x)$ and the real axis (it is used in Newton's method of numerical approximation). For the purposes of this discussion, we disregard the scenarios $f = 0$ and $f' = 0$, since they do not arise in the exponential case.

If we have a mechanism moving P in the function of A, we can easily construct a mechanism for B, as we did for the real exponential. Indeed, add a rod r perpendicular to the real axis through A, and make B stay on r (B can slide on r). Finally, introduce in B a wheel w_i pointing to P (e.g., by introducing another rod; the pedix i of w_i stands for the constant that multiplies $f(x)$ in $B = x + if(x)$). In this way, with suitable initial conditions (but we won't treat that), $B = x+if(x)$ is constructed by guiding the length of the subtangent $P - A$. Note that, in this configuration, it is not necessary to adopt exactly P; if a mechanism guides another point P^* on the tangent to the trajectory of B, P^* should guide effectively B as P.

We can release the constraint of adopting a rod perpendicular to the real axis in A by introducing another wheel. Consider an auxiliary point $B_k = x + kf(x)$, with the constant $k \in \mathbb{C} \setminus \{0, i\}$: this point can be dynamically constructed by a linkage in the function of A and B (because $B_k = A -$

$ik(B - A)$ and we can perform algebraic operations). If we also put a wheel w_k in B_k pointing at P, we can indirectly pose another constraint to B, as we discuss in the next section. The idea is that this additional constraint can impose, under certain conditions, $B = x + if(x)$ without directly constraining the rod r to stay perpendicular to the real axis. In the complex case, we already said that the traced graph no longer suffices to define the input/output relation. Therefore, it is no longer useful to have $B - A$ perpendicular to the x-axis, and thus instead of $B = z + if(z)$ we can simply consider $B = z + f(z)$. We must note that, unlike the real case of P^* at the end of the previous paragraph, we need exactly the point P to orient the wheels on B and B_k; it is no longer sufficient to have any point of the tangent.

In the real case, the input point A can move on the real axis along a single direction. Things change when the input point is $A = z \in \mathbb{C}$ and the output is $B_k = z + kf(z)$ with $f : \mathbb{C} \to \mathbb{C}$ and $k \in \mathbb{C}$.

Consider $A = z(t)$ (thus, the complex value z is in the function of a real variable t). If we differentiate B_k w.r.t. t we get

$$\frac{d}{dt}B_k = \frac{d}{dt}\left(z + kf\right) = \frac{dz}{dt}\frac{d}{dz}\left(z + kf\right) = \frac{dz}{dt}(1 + kf').$$

As we have already introduced in Section 1.4.2 of [4], if we consider just the complex argument, we have that the direction of the wheel w_k in B_k has to be oriented as $1+kf'$ rotated by the argument of $\frac{dz}{dt}$. Our primary interest is not in the specific parameterization $z(t)$, but rather in the direction in which the point A must move. Let us introduce a wheel w in A so that, to move A in a certain direction, we have to turn w in this direction. By considering the projection of the wheel w on the plane as a complex vector, the argument of the vector represents its direction. By constraining the motion direction of A by w, we have that the argument of $\frac{dz}{dt}$ coincides with the one of w. To sum up, we have that the direction of w_k has to be $1 + kf'$ rotated by the argument of w.

Coming back to the problem of defining a device for the complex exponential, we have that $P = A - 1$. Therefore, if dz is purely real, the direction conditions in $z + kf(z)$ has to point to $z - 1$, as visible in the top of Fig. 3. (In the following figures, we put random values of k to provide an idea.)

If dz has a non null imaginary component, the direction in $z + f(z)$ has to be turned of $\arg(dz)$, as shown in the bottom of Fig. 3. As already observed, from a mechanical perspective, to know the direction of dz we can add a wheel also in z to set the direction of the motion of z.

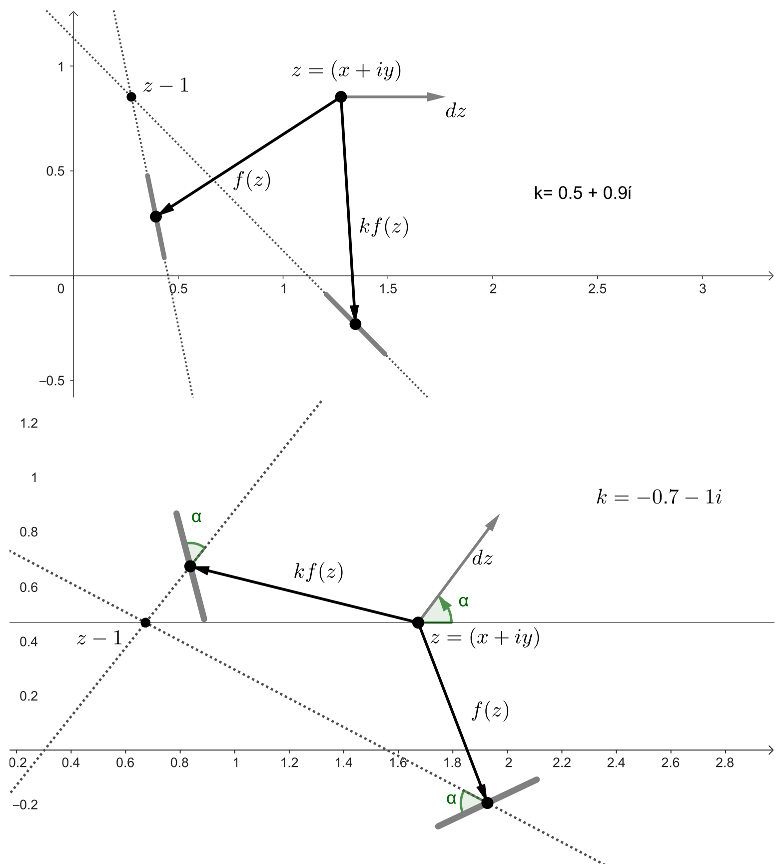

Figure 3: (Top) The schema of the machine for the complex exponential when dz is purely real. (Bottom) The machine in the general case. Note: In both cases, the gray thick segments represent wheels. In these and in all the following diagrams, the schemas represent planar machines.

4 Analytical framework

In this section we want to analytically deepen the previous ideas about the planar machine for the complex exponential. First of all let us introduce the condition of parallelism between complex values:

$$\forall\, a, b, c, d \in \mathbb{R} \quad (a + ib) \parallel (c + id) \iff ad = bc \tag{1}$$

(an obvious rewriting of the proportion $a : b = c : d$ avoiding divisions).

It is also useful to underline some properties that easily follow from (1).

$$\forall\, z, z_1, z_2 \in \mathbb{C}, z \neq 0 \quad z_1 \parallel z_2 \iff z \cdot z_1 \parallel z \cdot z_2; \tag{2}$$

$$\forall\, z, z_1, z_2 \in \mathbb{C}, z \parallel z_2 \quad z_1 + z \parallel z_2 \iff z_1 \parallel z_2. \tag{3}$$

[In general $z_1 + z \parallel z_2 + z \not\Longrightarrow z_1 \parallel z_2$ (e.g., $z_1 = 2i, z_2 = -1 + i, z = 2$).]

We aim to set the ODE $f = f'$ using a planar machine, transitioning from the real case presented in Section 2 to the complex one in Section 3.

Let $f : \mathbb{C} \to \mathbb{C}$ be a smooth function. We introduce a point $z = (x+iy)$, and consider z and $f(z) = (u+iv)$ parametrized in t (we consider $f(z(t)) = (u(t) + iv(t))$), so that $x, y, u, v : \mathbb{R} \to \mathbb{R}$ that we assume smooth. (Thus, $z' = x' + iy'$.) Having sometimes interest in considering the derivative of f with respect to z and other times with respect to t, we use the notation $f_z = \frac{d}{dz} f(z) : \mathbb{C} \to \mathbb{C}$ and $f_t = \frac{d}{dt} f(z(t)) : \mathbb{R} \to \mathbb{C}$.

For the chain rule we have $f_t = \frac{d}{dt} f = \frac{df}{dz}\frac{dz}{dt} = f_z z'$. We are interested in posing $f_z = f$, that means

$$\frac{d}{dt}(u(t) + iv(t)) = (u(t) + iv(t))(x'(t) + iy'(t)),$$

i.e.,

$$\begin{cases} u' &= ux' - vy' \\ v' &= uy' + vx' \end{cases} \tag{4}$$

We are going to prove that the complete machine of the previous section imposes (4) with certain restrictions. That means that, by setting as the initial condition $f(0) = 1$, the machine defines the complex exponential function by making the input point z moving on the complex plane. Also note that, even though we cannot define x' and y', it suffices to introduce a wheel imposing the direction of z. Furthermore, we can suppose $z' \neq 0$, excluding scenarios where the input point is stationary by changing the parametrization.

Furthermore, in the machine, we set a wheel also in an auxiliary point that, fixed any complex $k = a + ib \neq 0$, is $z + kf(z)$. Being the general case, we aim to define the equation imposed by the tangent condition in $z + kf(z)$.

The wheel in $z + kf(z)$ imposes that the derivatives with respect to t have to be parallel to $1 + kf(z)$ rotated at the angle α (the argument of $z' = x' + iy'$). Thus, the wheel direction condition implies $\frac{d}{dt}(z + kf(z)) \parallel (1 + kf(z))z'$, i.e.

$$z' + kf_t \parallel (1 + kf)z'.$$

By applying (2) with $1/z'$ (that is not null because $z' \neq 0$), we can rewrite the parallel condition as

$$1 + \frac{kf_t}{z'} \parallel 1 + kf.$$

By adding and subtracting on the left hand side kf, we get

$$1 + kf - kf + \frac{kf_t}{z'} \parallel 1 + kf$$

and we can apply (3) with the parallel value $1 + kf$:

$$\frac{kf_t}{z'} - kf \parallel 1 + kf.$$

By applying (2) with $1/k$ we get the following simpler formula:

$$\frac{f_t}{z'} - f \parallel \frac{1}{k} + f.$$

On such a formula we can apply (1):

$$\begin{aligned} &u'[x'(-b + v(a^2 + b^2)) + y'(a + u(a^2 + b^2))] + \\ &v'[y'(-b + v(a^2 + b^2)) - x'(a + u(a^2 + b^2))] = \\ &\quad (x'^2 + y'^2)(-ub - av). \end{aligned} \tag{5}$$

If we consider $a = 1, b = 0$ $(k = 1)$, we get

$$u'[x'v + y'(1 + u)] + v'[y'v - x'(1 + u)] = -v(x'^2 + y'^2). \tag{6}$$

Calling $B_1 = z + f(z)$ and, for $k \neq 1$, $B_k = z + kf(z)$ the points in which we pose the direction conditions, to impose the direction conditions in these points means to solve the system of equations (5) and (6). We can compute Cramer's determinants for the variables u', v':

$$\begin{aligned} D &= (x'^2 + y'^2)(b + ub + v(a - a^2 - b^2)); \\ D_{u'} &= (x'^2 + y'^2)(b + ub + v(a - a^2 - b^2))(ux' - vy'); \\ D_{v'} &= (x'^2 + y'^2)(b + ub + v(a - a^2 - b^2))(vx' + uy'). \end{aligned}$$

That means that the sought solution ($u' = ux' - vy'$, $v' = vx' + uy'$) is grantend when $x'^2 + y'^2 \neq 0$ and $b + ub + v(a - a^2 - b^2) \neq 0$. The first inequality holds because we hypotesized $z' \neq 0$. On the other side,

$$ub + v(a - a^2 - b^2) + b = 0 \tag{7}$$

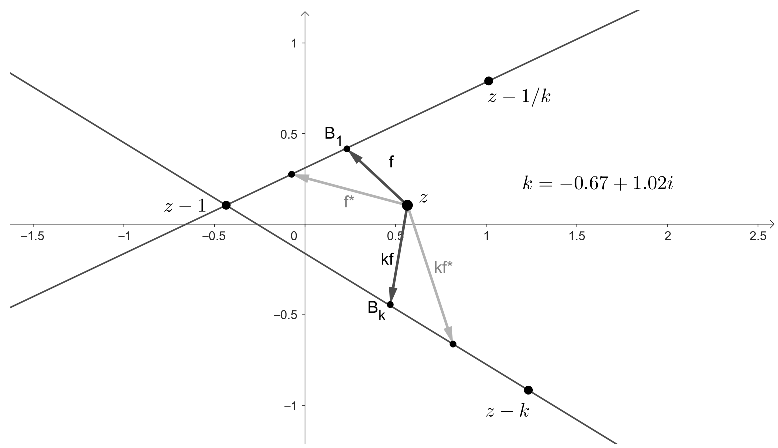

Figure 4: The tangent directions in $z + f(z)$ and $z + kf(z)$ are not sufficient to define the behavior of f if $b + ub + v(a - a^2 - b^2) = 0$.

holds along a line even if $k \neq 0$ and constitutes a singular solution. As visible in Fig. 4, such a problem arises when B_1 lies on the line through $z - 1$ and $z - 1/k$ and B_k lies on the line through $z - 1$ and $z - k$, i.e. $(a - 1)v = b(u + 1)$.

To highlight potential limitations of the construction, consider a displacement dz along the real axis—i.e., where $y' = 0$. In this case, the wheels located at B_1 and B_k both point to $z - 1$. However, under this condition, the tangent constraints do not uniquely determine the motion of B_1: indeed, both B_1 and B_k satisfy their directional requirements at every point lying on the lines through them and $z - 1$. As a result, the output function f is allowed to vary even while keeping the input z fixed—an undesirable ambiguity in the mechanism's behavior. In Fig.4, we visualize this situation plotting not only f and kf, but also alternative solutions f^* and kf^* that share the same tangent directions at the wheel points.

To avoid singular solutions, we can introduce another auxiliary point with another wheel. In addition to B_1, we can also put a wheel in B_{-1} and B_i. That defines a system of 3 equations, obtained by (5) with $k \in \{1, -1, i\}$. If we consider the system of two equations (5) with $k \in \{1, -1\}$, problems arise along $v = 0$; for $k \in \{1, i\}$, problems are on $u - v + 1 = 0$; for $k \in \{-1, i\}$ on $u + v - 1 = 0$. The intersection of these three lines is empty, thus the behavior of f is always well defined.

5 Conclusions

We introduced a planar machine for the complex exponential by generalizing, in a quite natural way, the real case. To allow readers experience the behavior of the complex exponential machine, as visible in Fig. 5, we implemented a simulation in GeoGebra https://www.geogebra.org/m/aed5qqab. Note that, to keep the construction simple, we have shown only $f(z)$ and $kf(z)$ and not the third auxiliary point, without considering problems related to singular solutions (7).

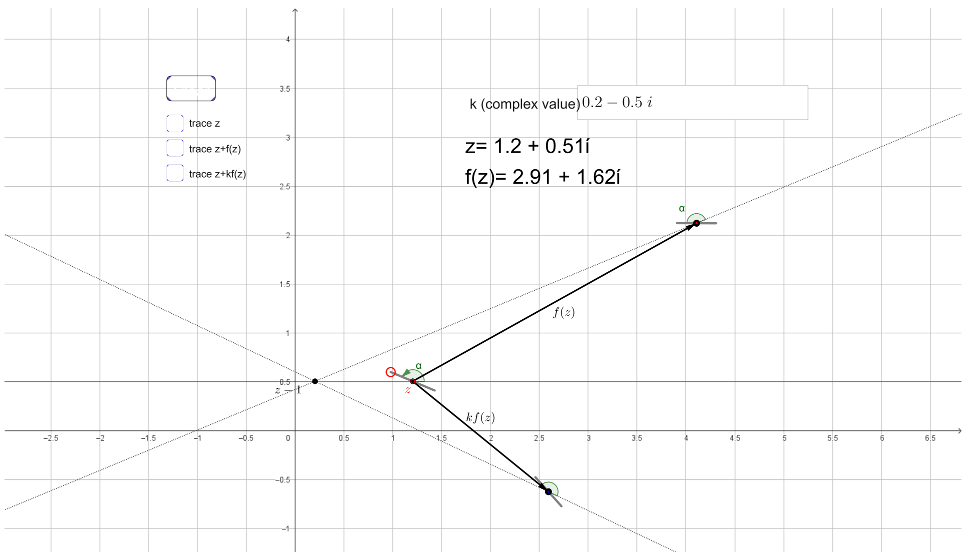

Figure 5: Simulation of a planar machine for the complex exponential.

The ability to display the trace of each point is particularly illuminating, as it visually confirms that the directions of the wheels are tangent to the curves they produce. Notably, the input point z cannot be freely dragged across the plane; instead, it must be moved along its current direction by manipulating the adjacent control point (marked with a red circle). This design underscores the pivotal role of the wheel positioned at z: its orientation is what determines the tangent directions at output points—such as $z+f(z)$ and $f + kf(z)$. However, this wheel's direction is not imposed by the system; it is manually set by the user, who actively chooses the direction along which z should travel, thereby guiding the evolution of the construction.

Although this simulation has not yet been tested in educational workshops, it appears highly promising for use in calculus classrooms. Its aim would be to introduce the behavior of complex differential equations, specifically the equation $f = f'$, not solely through analytical methods but via geometric and mechanical reasoning. For instance, by directing z along a purely real axis, the system demonstrates that f performs a scaling transformation; along a purely imaginary axis, it illustrates a rotation. These phenomena, typically expressed through analytic manipulation, become visually and geometrically evident—potentially offering novel insights into foundational results such as Euler's identity: $e^{i\pi} + 1 = 0$.

Looking ahead, we intend to develop a real prototype of the introduced machine for use in museum activities. Furthermore, our objective is to explore the role of the pivot, defined as $z - f/f'$, as a generalized analogue of the subtangent in the complex setting. This conceptual extension may provide deeper geometric intuition regarding the behavior of complex ODEs, while maintaining continuity with the real case and its historical mechanical interpretation.

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