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EXIT TIMES AND PLACES FOR A WRIGHT-FISHER PROCESS WITH AND WITHOUT JUMPS*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

First-passage problems are considered for a Wright-Fisher diffusion process $\{X(t), t \geq 0\}$, which is important in population genetics and mathematical finance. Let $\tau(x)$ be the first time that $X(t)$, starting from $X(0) = x$, is equal to 0 or to 1. We compute the expected area covered by the process in the interval $[0, \tau(x)]$. We also compute the probability that the process will take on a value smaller than or equal to 0 before a value greater than or equal to 1 in the case when there are uniform jumps according to a Poisson process. Finally, stochastic control problems called homing problems are solved explicitly in particular cases for a controlled version of the process $\{X(t), t \geq 0\}$.

Keywords: Brownian motion, first-passage time, stochastic optimal control, dynamic programming, non-linear differential equation.

MSC: 60J70, 93E20.

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1 Introduction

The one-dimensional diffusion process $\{X(t), t \geq 0\}$ defined by the stochastic differential equation

$$\mathrm{d}X(t) = \{X(t)\,[1 - X(t)]\}^{1/2}\mathrm{d}B(t), \tag{1}$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion, is a particular Wright-Fisher process (or Wright-Fisher model). It is defined in the interval $[0, 1]$.

The discrete version of the Wright-Fisher model is very important in population genetics. The diffusion limit $\{X(t), t \geq 0\}$ is also used in mathematical finance to model interest rates fluctuations ([5]) and wealth concentration ([3]). In the context of population genetics, first-passage problems for the diffusion process have been considered in Crow and Kimura [4]; see also [6] and [7].

We suppose that $X(0) = x \in (0, 1)$ and we define the first-passage time

$$\tau(x) = \inf\{t > 0 : X(t) = 0 \text{ or } 1\}. \tag{2}$$

In Section 2, we will compute the expected value of $\tau(x)$. We will also obtain the value of the expected area covered by $\{X(t), t \geq 0\}$ in the interval $[0, \tau(x)]$.

Next, in Section 3, random jumps occurring according to a Poisson process and uniformly distributed on the interval $[-1, 1]$ will be added to the model. We will compute the probability $P[X(\tau(x)) = 0]$. To do so, we must solve an integro-differential equation, subject to the appropriate boundary conditions.

In Section 4, optimal control problems known as *homing problems* will be solved explicitly and exactly for a controlled version of the Wright-Fisher process. In this type of problem, the process in controlled until a given event occurs. We will have to solve non-linear second-order differential equations.

Finally, we will end this paper with some concluding remarks in Section 5.

2 Mean exit time and mean first-passage area

Let

$$M(x; \alpha) := E\left[e^{-\alpha\tau(x)}\right], \tag{3}$$

where $\alpha > 0$. The function $M(x; \alpha)$, which is the moment-generating function of the random variable $\tau(x)$, satisfies the Kolmogorov backward equation (see, for instance, [9])

$$\frac{1}{2}x(1-x)M''(x;\alpha) = \alpha M(x\ \alpha), \tag{4}$$

subject to $M(0;\alpha) = M(1;\alpha) = 1$. Writing $M(x;\alpha)$ as follows:

$$M(x;\alpha) := E\left[1 - \alpha\tau(x) + \frac{\alpha^2}{2}\tau^2(x) - \dots\right], \tag{5}$$

we deduce that the function $m(x) := E[\tau(x)]$ (if it exists) satisfies the linear second-order ordinary differential equation (ODE)

$$\frac{1}{2}x(1-x)m''(x) = -1. \tag{6}$$

The boundary conditions are $m(0) = m(1) = 0$.

The function $m(x)$ has already been computed in other works. The general solution of Eq. (6) is

$$m(x) = -2x\ln(x) + 2(x-1)\ln(x-1) + c_1 x + c_2, \tag{7}$$

where c_1 and c_2 are arbitrary constants. The unique solution which is such that $m(0) = m(1) = 0$ is

$$m(x) = (2x-2)\ln(1-x) - 2x\ln(x). \tag{8}$$

This function is shown in Figure 1.

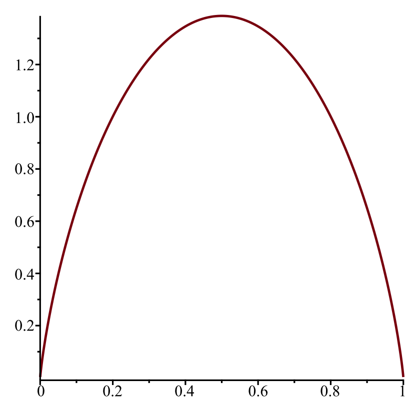

Figure 1: Function $m(x)$ defined in Eq. (8) for $x \in [0,1]$.

Remark 1. *In the application to population genetics, the function $m(x)$ is the expected time to* fixation *(if $X(\tau(x)) = 1$) or* extinction *(if $X(\tau(x)) = 0$) for the neutral Wright-Fisher model without mutation.*

Next, we consider the area covered by $\{X(t), t \geq 0\}$ in the interval $[0, \tau(x)]$. Let

$$A(x) := E\left[\int_0^{\tau(x)} X(t)\,\mathrm{d}t\right]. \tag{9}$$

The function $A(x)$ satisfies the ODE (see [2])

$$\frac{1}{2}\,x\,(1-x)\,A''(x) = -x, \tag{10}$$

subject to the boundary conditions $A(0) = A(1) = 0$. We find that the general solution of the above ODE is

$$A(x) = 2\,(x-1)\,\ln(x-1) + d_1\,x + d_2. \tag{11}$$

Moreover, the solution that satisfies the boundary conditions is

$$A(x) = 2\,(x-1)\,\ln(x-1) - 2\,x + (-2\,i\,\pi + 2)\,x + 2\,i\,\pi, \tag{12}$$

which can be rewritten as follows:

$$A(x) = 2\,(x-1)\,\ln(1-x) \quad \text{for } 0 \leq x \leq 1. \tag{13}$$

See Figure 2.

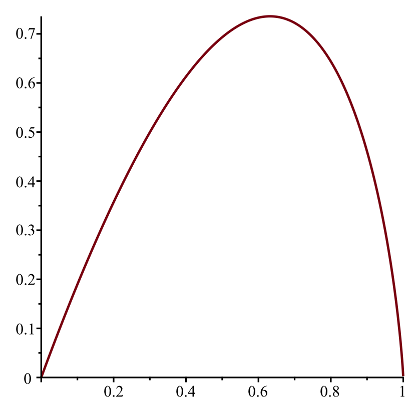

Figure 2: Function $A(x)$ defined in Eq. (13) for $x \in [0, 1]$.

3 Uniform Poissonian jumps

In this section, we add jumps to the Wright-Fisher process. We consider the jump-diffusion process $\{X_1(t), t \geq 0\}$ defined by

$$X_1(t) = X_1(0) + \int_0^t \{X_1(s)[1 - X_1(s)]\}^{1/2} \mathrm{d}B(s) + \sum_{i=1}^{N(t)} Y_i, \quad (14)$$

where $\{N(t), t \geq 0\}$ is a Poisson process with rate λ and $Y_1, Y_2, \ldots$ are independent random variables that are uniformly distributed on the interval $[-1, 1]$:

$$f_{Y_i}(y) = \frac{1}{2} \quad \text{if } -1 \leq y \leq 1, \quad (15)$$

for $i = 1, 2, \ldots$ The standard Brownian motion $\{B(t), t \geq 0\}$ as well as the random variables $\{Y_1, Y_2, \ldots\}$ are assumed to be independent of the Poisson process.

The first-passage time $\tau(x)$ is replaced by

$$\tau_1(x) = \inf\{t > 0 : X_1(t) \leq 0 \text{ or } X_1(t) \geq 1 \mid X_1(0) = x\}. \quad (16)$$

It can be shown (see, for example, [8] or [10]) that the moment-generating function $M_1(x; \alpha)$ of $\tau_1(x)$ satisfies the integro-differential equation (IDE)

$$\frac{1}{2} x (1 - x) M_1''(x; \alpha) + \lambda \left\{ \frac{1}{2} \int_{-1}^{1} M_1(x + y; \alpha) \, \mathrm{d}y - M_1(x; \alpha) \right\} = \alpha M_1(x; \alpha). \quad (17)$$

Moreover, we can write that $M_1(x; \alpha) = 1$ if $x \leq 0$ or $x \geq 1$.

We have

$$\begin{aligned} \int_{-1}^{1} M_1(x + y; \alpha) \, \mathrm{d}y &= \int_{-1}^{-x} 1 \, \mathrm{d}y + \int_{-x}^{1-x} M_1(x + y; \alpha) \, \mathrm{d}y + \int_{1-x}^{1} 1 \, \mathrm{d}y \\ &= 1 + \int_{-x}^{1-x} M_1(x + y; \alpha) \, \mathrm{d}y. \end{aligned} \quad (18)$$

In the case of the expected value $m_1(x)$ of the random variable $\tau_1(x)$, we find that

$$\frac{1}{2} x (1 - x) m_1''(x) + \lambda \left\{ \frac{1}{2} \int_{-1}^{1} m_1(x + y) \, \mathrm{d}y - m_1(x) \right\} = -1, \quad (19)$$

together with $m_1(x) = 0$ if $x \leq 0$ or $x \geq 1$. This time,

$$\begin{aligned} \int_{-1}^{1} m_1(x+y)\,\mathrm{d}y &= \int_{-1}^{-x} 0\,\mathrm{d}y + \int_{-x}^{1-x} m_1(x+y)\,\mathrm{d}y + \int_{1-x}^{1} 0\,\mathrm{d}y \\ &= \int_{-x}^{1-x} m_1(x+y)\,\mathrm{d}y. \end{aligned} \tag{20}$$

Finally, we define

$$p_1(x) = P[X_1(\tau_1(x)) \leq 0]. \tag{21}$$

This function satisfies the IDE

$$\frac{1}{2}\,x\,(1-x)\,p_1''(x) + \lambda \left\{ \frac{1}{2} \int_{-1}^{1} p_1(x+y)\,\mathrm{d}y - p_1(x) \right\} = 0, \tag{22}$$

and is such that $p_1(x) = 1$ if $x \leq 0$ and $p_1(x) = 0$ if $x \geq 1$. We calculate

$$\begin{aligned} \int_{-1}^{1} p_1(x+y)\,\mathrm{d}y &= \int_{-1}^{-x} 1\,\mathrm{d}y + \int_{-x}^{1-x} p_1(x+y)\,\mathrm{d}y + \int_{1-x}^{1} 0\,\mathrm{d}y \\ &= 1 - x + \int_{-x}^{1-x} p_1(x+y)\,\mathrm{d}y, \end{aligned} \tag{23}$$

so that

$$\frac{1}{2}\,x\,(1-x)\,p_1''(x) + \lambda \left\{ \frac{1}{2} \left[1 - x + \int_{-x}^{1-x} p_1(x+y)\,\mathrm{d}y \right] - p_1(x) \right\} = 0. \tag{24}$$

We can write that

$$\int_{-x}^{1-x} p_1(x+y)\,\mathrm{d}y = \int_{0}^{1} p_1(z)\,\mathrm{d}z \tag{25}$$

(a constant). Hence, differentiating Eq. (24) with respect to x, we find that the function $p_1(x)$ satisfies the third-order linear ODE

$$\frac{1}{2}\,x\,(1-x)\,p_1'''(x) + \frac{1}{2}\,(1-2\,x)\,p_1''(x) - \lambda \left(\frac{1}{2} + p_1'(x) \right) = 0 \tag{26}$$

(which is actually a second-order linear ODE for $\rho(x) := p_1'(x)$).

We can now state the following proposition.

Proposition 1. *The function $p_1(x)$ defined in Eq. (21) satisfies the ODE (26) for $0 < x < 1$. Moreover, we have $p_1(0) = 1$ and $p_1(1) = 0$. Finally, by symmetry, we can write that $p_1(0.5) = 1/2$.*

Making use of the software program *Maple*, we can find the general solution of Eq. (26):

$$\begin{aligned} p_1(x) &= k_1 \frac{x^{\frac{1}{2}+\frac{\kappa}{2}} {}_2F_1\left(-\frac{1}{2}-\frac{\kappa}{2}, \frac{1}{2}-\frac{\kappa}{2}; 1-\kappa; x^{-1}\right)}{\frac{1}{2}+\frac{\kappa}{2}} \\ &\quad + k_2 \frac{x^{\frac{1}{2}-\frac{\kappa}{2}} {}_2F_1\left(-\frac{1}{2}+\frac{\kappa}{2}, \frac{1}{2}+\frac{\kappa}{2}; 1+\kappa; x^{-1}\right)}{\frac{1}{2}-\frac{\kappa}{2}} \\ &\quad - \frac{x}{2} + k_3, \end{aligned} \tag{27}$$

where ${}_2F_1$ is a generalized hypergeometric function (see [1]),

$$\kappa := \sqrt{-8\,\lambda + 1}, \tag{28}$$

and k_i is an arbitrary constant, for $i = 1, 2, 3$. There is a unique solution that satisfies the conditions in Proposition 1. The exact mathematical expression is quite involved. If we set $\lambda = 1$, we can give the following approximate expression for $p_1(x)$:

$$\begin{aligned} p_1(x) &\approx -(0.1474 + 0.5246\,i)\, x^{0.5+1.3229\,i} \\ &\quad \times {}_2F_1\left(\frac{1}{2} - \frac{i}{2}\sqrt{7}, -\frac{1}{2} - \frac{i}{2}\sqrt{7}; 1 - i\sqrt{7}; x^{-1}\right) \\ &\quad -(0.0082 + 0.0023\,i)\, x^{0.5-1.3229\,i} \\ &\quad \times {}_2F_1\left(\frac{1}{2} + \frac{i}{2}\sqrt{7}, -\frac{1}{2} + \frac{i}{2}\sqrt{7}; 1 + i\sqrt{7}; x^{-1}\right) \\ &\quad + 0.7500 - 0.5\,x \end{aligned} \tag{29}$$

for $0 \le x \le 1$.

Remark 2. *We can check that the function defined in the above equation does indeed satisfy the IDE (22) with* $\lambda = 1$.

Now, when there are no jumps, the function $p_0(x)$ that corresponds to $p_1(x)$ satisfies the simple ODE

$$\frac{1}{2}\, x\,(1-x)\, p_0''(x) = 0 \quad \Longrightarrow \quad p_0''(x) = 0 \quad \text{for } 0 < x < 1. \tag{30}$$

The unique solution such that $p_0(0) = 1$ and $p_0(1) = 0$ is the straight line $p_0(x) = 1 - x$.

In Figure 3, the functions $p_1(x)$ and $p_0(x)$ are displayed for $0 \le x \le 1$.

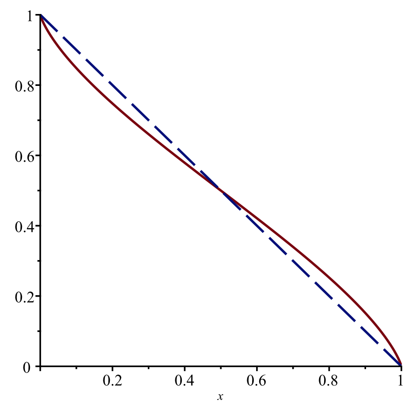

Figure 3: Function $p_1(x)$ defined in Eq. (29) (full line) and $p_0(x) = 1 - x$ for $x \in [0, 1]$.

4 Homing problems

In this section, we consider a controlled version of the Wright-Fisher process $\{X(t), t \geq 0\}$. We define the controlled process $\{X_u(t), t \geq 0\}$ by

$$\mathrm{d}X_u(t) = b[X_u(t)]\, u[X_u(t)]\, \mathrm{d}t + \{X_u(t)\,[1 - X_u(t)]\}^{1/2} \mathrm{d}B(t), \tag{31}$$

where $b(\cdot) \neq 0$, and the control variable $u(\cdot)$ is assumed to be a continuous function.

Our objective is to find the value $u^*[X_u(t)]$ of $u[X_u(t)]$ that minimizes the expected value of the cost function

$$J(x) = \int_0^{\tau(x)} \left\{ \frac{1}{2} q[X_u(t)]\, u^2[X_u(t)] + r[X_u(t)] + \theta \right\} \mathrm{d}t, \tag{32}$$

where $q(\cdot) > 0$, $r(\cdot)$ is a real function, and θ is a real parameter.

Remark 3. *This type of problem, in which the optimizer controls a stochastic process until a given event takes place, is called a* homing problem*. If $r(\cdot) = 0$ and $\theta > 0$, the aim is to make the controlled process leave the interval $(0, 1)$ as soon as possible, while taking the quadratic control costs into account.*

Let $F(x)$ denote the *value function*:

$$F(x) := \inf_{\substack{u[X_u(t)] \\ 0 \leq t < \tau(x)}} E[J(x)]. \tag{33}$$

Using dynamic programming, we can prove the following proposition (see [11]).

Proposition 2. *The value function $F(x)$ satisfies the non-linear ODE*

$$\theta + r(x) - \frac{b^2(x)}{2\,q(x)}\,[F'(x)]^2 + \frac{1}{2}\,x\,(1-x)\,F''(x) = 0. \tag{34}$$

The equation is valid for $0 < x < 1$, and we have the boundary conditions $F(0) = F(1) = 0$. Furthermore, the optimal control can be expressed in terms of the value function as follows:

$$u^*(x) = -\frac{b(x)}{q(x)}\,F'(x). \tag{35}$$

Whittle [11] has proved that if the relation (in our case)

$$x\,(1-x) = \delta\,\frac{b^2(x)}{q(x)} \tag{36}$$

holds for a constant $\delta > 0$, then Eq. (34) can be transformed into a linear ODE by defining

$$\Phi(x) = e^{-F(x)/\delta}. \tag{37}$$

Here, we will first solve a problem in which the infinitesimal variance $x\,(1-x)$ is *not* proportional to $b^2(x)/q(x)$.

Assume that $\theta = 0$ and

$$r(x) = x, \quad b(x) = \sqrt{x} \quad \text{and} \quad q(x) \equiv 1. \tag{38}$$

Then, the aim is to minimize the expected area covered by the controlled process in the interval $[0, \tau(x)]$ (taking the control costs into account). Note that $b^2/q(x) = x$ is not proportional to $x\,(1-x)$.

With the above choices, Eq. (34) becomes

$$x - x\,[F'(x)]^2 + \frac{1}{2}\,x\,(1-x)\,F''(x) = 0. \tag{39}$$

Maple gives us the following solution of Eq. (39):

$$F(x) = \int \tanh\left[2\,\ln(x-1) + C_1\right]\,\mathrm{d}x + C_2. \tag{40}$$

To satisfy the boundary condition $F(0) = 0$, we can write that

$$F(x) = \int_0^x \tanh\left[2\,\ln(x-1) + C_1\right]\,\mathrm{d}x. \tag{41}$$

Finally, evaluating numerically the integral between 0 and 1 for various values of the constant C_1, we find that in order to obtain $F(1) = 0$, we must take $C_1 \approx 1.53676$, so that

$$F(x) \approx \int_0^x \tanh\left[2\,\ln(x-1) + 1.53676\right] \mathrm{d}x. \tag{42}$$

From Eq. (35), we deduce that the optimal control is given by

$$u^*(x) \approx -\sqrt{x}\tanh\left[2\,\ln(x-1) + 1.53676\right] \quad \text{for } 0 < x < 1. \tag{43}$$

We present the functions $F(x)$ and $u^*(x)$ in Figures 4 and 5, respectively.

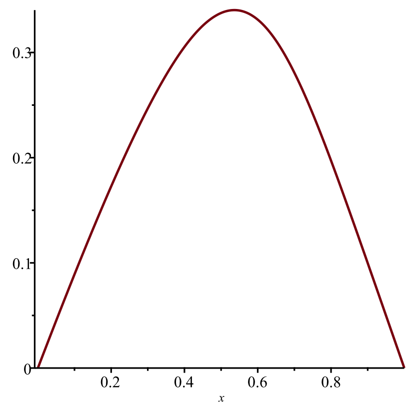

Figure 4: Value function $F(x)$ defined in Eq. (42) for x in the interval $[0, 1]$.

4.1 Generalization

A more general definition of the Wright-Fisher diffusion process is

$$\mathrm{d}X(t) = \gamma_1\, X(t)\,[1 - X(t)] + \gamma_2 - (\gamma_2 + \gamma_3)\, X(t) + \{X(t)\,[1 - X(t)]\}^{1/2}\mathrm{d}B(t), \tag{44}$$

where γ_i is a non-negative constant, for $i = 1, 2, 3$. Then, Eq. (34) becomes

$$\begin{aligned} 0 \quad = \quad & \theta + r(x) - \frac{b^2(x)}{2\,q(x)}\,[F'(x)]^2 + [\gamma_1\, x\,(1-x) + \gamma_2 - (\gamma_2 + \gamma_3)\, x]\, F'(x) \\ & + \frac{1}{2}\, x\,(1-x)\, F''(x). \end{aligned} \tag{45}$$

Next, suppose that we add the following terminal cost $K[X(\tau(x))]$ to the cost function $J(x)$ defined in Eq. (32):

$$K[X(\tau(x))] = \begin{cases} 0 & \text{if } X(\tau(x)) = 0, \\ K_0 & \text{if } X(\tau(x)) = 1, \end{cases} \tag{46}$$

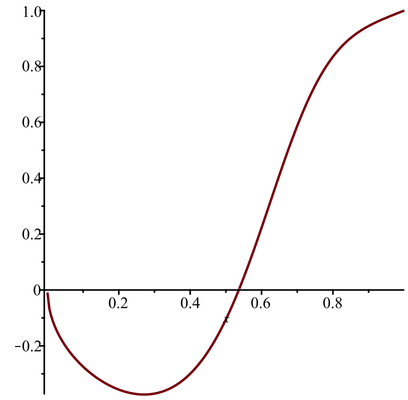

Figure 5: Optimal control $u^*(x)$ defined in Eq. (43) for x in the interval $(0, 1)$.

where $K_0 > 0$. This implies that the optimizer must try to leave the interval $[0, 1]$ through the origin.

Assume that $\gamma_2 = 0$ and $\gamma_1 = \gamma_3 = 1$. Moreover, we take $\theta = 0$,

$$r(x) = x, \quad b(x) = \sigma\sqrt{x\,(1-x)} \quad \text{and} \quad q(x) \equiv 1, \tag{47}$$

so that we must solve the second-order non-linear ODE

$$0 = x - \sigma^2\,\frac{x\,(1-x)}{2}\,[F'(x)]^2 - x^2\,F'(x) + \frac{1}{2}\,x\,(1-x)\,F''(x), \tag{48}$$

subject to the boundary conditions $F(0) = 0$ and $F(1) = K_0$.

Remark 4. *Note that we could linearize Eq. (48) by setting* $\Phi(x) = e^{-\sigma^2 F(x)}$.

If $\sigma = 1$, we find that the following function is a solution of Eq. (48) that satisfies the condition $F(0) = 0$:

$$F(x) = -\ln\left(\frac{1 - e^{2(1-x)}}{(e^2-1)\,(x-1)}\right). \tag{49}$$

We have (making use of L'Hospital's rule)

$$F(1) = -\ln(2) + \ln\left(e^2 - 1\right) \approx 1.161 > 0. \tag{50}$$

Hence, if we choose the constant $K_0 = -\ln(2) + \ln\left(e^2 - 1\right)$ in Eq. (46), we can state that the function $F(x)$ defined in Eq. (49) is indeed the value function in the problem considered.

The optimal control is given by

$$u^*(x) = \frac{\left(2\, e^{-2\,x+2} x - e^{-2\,x+2} - 1\right) x}{e^{-2\,x+2} - 1} \quad \text{for } 0 < x < 1. \tag{51}$$

The value function $F(x)$ and the optimal control are shown in Figures 6 and 7, respectively.

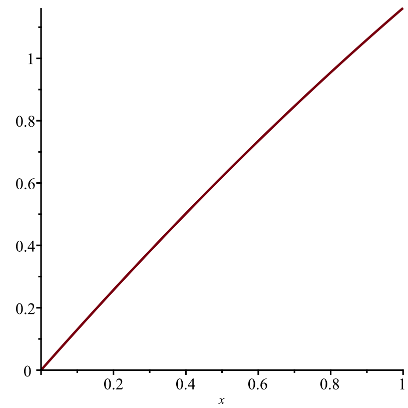

Figure 6: Value function $F(x)$ defined in Eq. (49) for x in the interval $[0, 1]$.

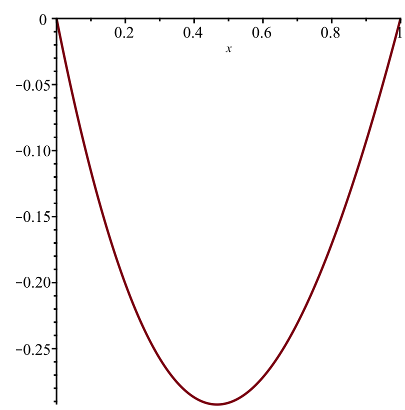

Figure 7: Optimal control $u^*(x)$ defined in Eq. (51) for x in the interval $(0, 1)$.

Finally, if $\sigma = \sqrt{2}$, the solution of Eq. (48) can be written as follows:

$$\begin{aligned} F(x) \quad &= \quad x - \frac{1}{2} \ln \left[c_1 \left(2\,\mathrm{Ei}_1(1, 2 - 2\,x)\,(x-1) + e^{2(x-1)} \right) + c_2\,(1-x) \right] \\ &\quad + \frac{1}{2} \ln \left[(1-x)/2 \right], \end{aligned} \tag{52}$$

where $\mathrm{Ei}_1(\cdot)$ is an exponential integral:

$$\mathrm{Ei}_1(z) := \int_1^\infty \frac{e^{-xz}}{x}\,\mathrm{d}x. \tag{53}$$

It turns out that the solution that satisfies the boundary conditions $F(0) = 0$ and $F(1) = K_0 = 1$ is simply $F(x) = x$ (obtained by taking $c_1 = 0$ and $c_2 = 1/2$), so that

$$u^*(x) = -4x(1-x) \quad \text{for } 0 < x < 1. \tag{54}$$

See Figure 8.

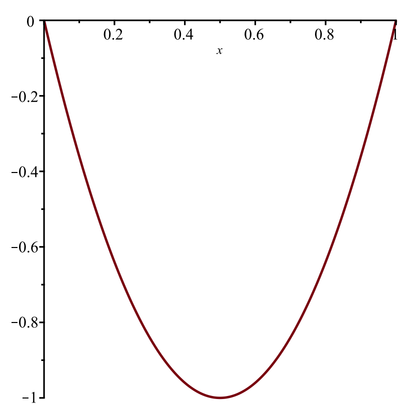

Figure 8: Optimal control $u^*(x)$ defined in Eq. (54) for x in the interval $(0, 1)$.

5 Concluding remarks

In this paper, various first-passage problems were considered and solved explicitly and exactly for a diffusion process that is very important in population genetics. The Wright-Fisher process is also used in financial mathematics.

In Section 3, jumps that occur according to a Poisson process were added to the model. Nowadays, jumps are often added to diffusion processes in order to better model phenomena such as the evolution of financial indices. Indeed, ordinary diffusion processes are unable to reproduce the rapid increases or decreases that are sometimes observed, for instance when there is a stock market crash. A practical implication of adding jumps is that a simple problem such as determining the probability of a one-dimensional Wright-Fisher process reaching 0 before 1 becomes much more difficult. Indeed, in the absence of jumps, one has to solve an elementary ordinary differential equation. With jumps, the differential equation becomes an integro-differential equation (when the jump size is a continuous random variable) or a difference-differential equation (when the jump size is a discrete random variable).

In Section 4, we studied homing problems for a controlled version of the Wright-Fisher process. These problems are usually difficult to solve explicitly, because we need to find the solution of a non-linear second-order differential equation. If the relation in Eq. (36) holds, then it is possible to linearize the differential equation. However, even this simplified problem is often arduous.

We could try to solve problems for other jump distributions. In particular, the case when the jumps are exponentially distributed is often treated for other jump-diffusion processes. In the case of the stochastic control problems, our aim was to find exact analytical expressions for the value function (and hence for the optimal control). When it is not possible to do so, we could make use of numerical methods or simulations.

Dedication. I, Mario Lefebvre, would like to dedicate this paper to my dear friend Professor Liliana Restuccia, a distinguished scientist, on the occasion of her seventieth birthday.

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