Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 17, No. 3/2025

ON THE EXISTENCE OF SOLUTIONS FOR A CLASS OF HILFER-HADAMARD FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS*

Aurelian Cernea†

Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

DOI 10.56082/annalsarscimath.2025.3.29

Abstract

A nonlocal integro-multi-point boundary value problem associated to a Hilfer-Hadamard fractional integro-differential inclusion is studied. The existence of solutions is established in the case where the set-valued maps have non-convex values.

Keywords: differential inclusion, fractional derivative, boundary value problem.

MSC: 34A60, 34A12, 34A08.

1 Introduction

The recent literature abounds in papers concerning the study of systems governed by fractional order derivatives. That is because this kind of system proved to be more realistic models than the ones using classical derivatives (see [2, 9, 13–15] etc.).

*Accepted for publication on June 29, 2025

†acernea@fmi.unibuc.ro, Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest and Academy of Romanian Scientists, Ilfov 3, 050044 Bucharest, Romania

In [11], Hilfer introduced a generalization of both Riemann-Liouville and Caputo fractional derivatives. In fact, this derivative is an interpolation between Riemann-Liouville and Caputo derivatives. Properties and applications of Hilfer fractional derivative may be found in [12]. The modification of Hilfer fractional derivative resulted in the concept of Hilfer-Hadamard fractional derivative. Similarly, the Hilfer-Hadamard fractional derivative covers the cases of Riemann-Liouville-Hadamard and Caputo-Hadamard fractional derivatives.

In this note we consider the following problem

$$D_{HH}^{\alpha,\beta}x(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([1,T]) \tag{1}$$

with boundary conditions of the form

$$x(1) = 0, \quad \sum_{i=1}^{m} \theta_i x(\xi_i) = \lambda I_H^{\delta} x(\eta), \tag{2}$$

where $D_{HH}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha \in (1,2]$ and type $\beta \in [0,1]$, $\xi_i, \eta \in (1,T)$, $\theta_i, \lambda \in \mathbf{R}$, $i = \overline{1,m}$, I_H^{δ} is the Hadamard fractional integral of order $\delta > 0$, $F : [1,T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $V : C([1,T],\mathbf{R}) \to C([1,T],\mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [1,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

The starting point of our study is a recent paper [16], where problem (1)-(2) is studied in the single-valued case; namely, the right-hand side in (1) is given by a single-valued map. Existence and uniqueness results are provided by using well-known fixed point theorems: Banach, Schaefer, Leray-Schauder and Krasnoselskii.

Our goal is to provide an extension of the study in [16] to the set-valued framework. Moreover, our result may be viewed, also, as a generalization to the case when the right-hand side contains a nonlinear Volterra integral operator. Even if the method we use here is known in the theory of differential inclusions (e.g., [4–7] etc.) it is largely ignored by the authors that are dealing with such problems in favor of fixed point approaches, most probably, because it is much easier to handle the applications of classical fixed point theorems. The approach presented here avoids the application of fixed point theorems and takes into account the case where the values of F are not convex; instead, this set-valued map is assumed to be Lipschitz in state variables. We establish an existence result for problem (1)-(2) by using Filippov's technique (see [10]); namely, the existence of solutions is obtained by starting from a pair of given "quasi" solutions. In addition, the result provides an estimate between the "quasi" solutions and the solutions obtained.

We mention that in the late years several papers in the literature are devoted to the exisence of solutions of fractional differential inclusions governed by Hilfer-Hadamard fractional derivative ([1,8,16,17] etc.). We, also, note that a similar result may be found in our previous paper [8], where the differential inclusion (1) is studied with boundary conditions of the form

$$x(1) = 0, \quad x(T) = \sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i I_H^{\varphi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k D_H^{\omega_k} x(\mu_k), \tag{3}$$

where D_H^{ω} denotes the Hadamard fractional derivative of order $\omega > 0$. Obviously, it is not possible to obtain the boundary conditions (2) and (3) one from another and, therefore, the two problems are different.

The paper is organized as follows. In Section 2 we recall some preliminary results that we need in the sequel, and in Section 3 we prove our main results.

2 Preliminaries

Let (X, d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the compact subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \; d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

Let $I = [1, T]$, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t \in I} |x(t)|$ and $L^1(I, \mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_1^T |u(t)|dt$.

Definition 1. *a) The Hadamard fractional integral of order $q > 0$ of a Lebesgue integrable function $f : [1, \infty) \to \mathbf{R}$ is defined by*

$$I_H^q f(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\ln \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds$$

provided the integral exists and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$.

b) The Hadamard fractional derivative of order $q > 0$ of a function $f : [1, \infty) \to \mathbf{R}$ is defined by

$$D_H^q f(t) = t \frac{d^n}{dt^n} (I_H^{n-q} f)(t),$$

provided the integral exists and $n = [q] + 1$, $[q]$ is the integer part of q.

c) The Hilfer-Hadamard fractional derivative of order $\alpha \in (n-1, n)$, $n \geq 2$ and type $\beta \in [0, 1]$ of a function $f \in L^1(I, \mathbf{R})$ is defined by

$$D_{HH}^{\alpha,\beta} f(t) = (I_H^{\beta(n-\alpha)} D_H^\gamma f)(t), \quad \gamma = \alpha + n\beta - \alpha\beta.$$

When $\beta = 0$ the Hilfer-Hadamard fractional derivative gives Hadamard fractional derivative and when $\beta = 1$ the Hilfer-Hadamard fractional derivative gives Caputo-Hadamard fractional derivative.

In our case $n = [\alpha] + 1 = 2$ and $\gamma = \alpha + (2 - \alpha)\beta$. The next result is proved in [16].

Lemma 1. *Let $p(.) \in C(I, \mathbf{R})$ and assume that $\Lambda \neq 0$, where*

$$\Lambda = \sum_{i=1}^m \theta_i (\ln \xi_i)^{\gamma-1} - \lambda \frac{\Gamma(\gamma)}{\Gamma(\gamma+\delta)} (\ln \eta)^{\gamma+\delta-1}.$$

Then, the solution of problem $D_{HH}^{\alpha,\beta} x(t) = p(t)$ with boundary conditions (2) is given by

$$x(t) = I_H^\alpha p(t) + \frac{(\ln t)^{\gamma-1}}{\Lambda} [\Lambda I_H^{\alpha+\delta} p(\eta) - \sum_{i=1}^m \theta_i I_H^\alpha p(\xi_i)], \quad t \in [1, T]. \tag{4}$$

Definition 2. *$x(.) \in C(I, \mathbf{R})$ is called a solution of a problem (1)-(2) if there exists $p(.) \in L^1(I, \mathbf{R})$ such that $p(t) \in F(t, x(t), V(x)(t))$ a.e. (I) and $x(.)$ is given by (4).*

Remark 1. *If we denote*

$$\mathcal{R}(t,s) = \frac{1}{\Gamma(\alpha)} \left(\ln \frac{t}{s} \right)^{\alpha-1} \frac{\chi_{[1,t](s)}}{s} + \frac{\lambda}{\Lambda} \frac{(\ln t)^{\gamma-1}}{\Gamma(\alpha+\delta)} \left(\ln \frac{\eta}{s} \right)^{\alpha+\delta-1} \frac{\chi_{[1,\eta]}(s)}{s} - \sum_{i=1}^m \frac{(\ln t)^{\gamma-1}}{\Lambda} \frac{\theta_i}{\Gamma(\alpha)} \left(\ln \frac{\xi_i}{s} \right)^{\alpha-1} \frac{\chi_{[1,\xi_i]}(s)}{s},$$

where $\chi_A(.)$ denotes the characteristic function of the set A, then the solution $x(.)$ in (4) may be put as $x(t) = \int_1^T \mathcal{R}(t,s) p(s) ds$.

Moreover,

$$|\mathcal{R}(t,s)| \leq \tfrac{(\ln T)^{\alpha-1}}{\Gamma(\alpha)} + |\tfrac{\lambda}{\Lambda}|\tfrac{(\ln T)^{\gamma-1}}{\Gamma(\alpha+\delta)}(\ln \eta)^{\alpha+\delta-1} + \tfrac{(\ln T)^{\alpha-1}}{|\lambda|\Gamma(\alpha)}\textstyle\sum_{i=1}^{m}|\theta_i|(\ln \xi_i)^{\alpha-1} =: M \quad \forall\, t,s \in I.$$

Also, we need a variant of Kuratowski and Ryll-Nardzewski selection theorem concerning measurable set-valued maps, that may be found in [2].

Lemma 2. *Consider X a separable Banach space, B is the closed unit ball in X, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X, L : I \to \mathbf{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.(I),$$

then the set-valued map $t \to H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

3 Main results

In order to prove our results we need the following hypotheses.

Hypothesis H1 i) *$F(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I,(0,\infty))$ such that, for almost all $t \in I$, $F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t,x_1,y_1),F(t,x_2,y_2)) \leq L(t)(|x_1-x_2|+|y_1-y_2|)\ \forall\, x_1,x_2,y_1,y_2 \in \mathbf{R}.$$

iii) *$k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that $\forall x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ is measurable.*

iv) *$|k(t,s,x)-k(t,s,y)| \leq L(t)|x-y| \quad a.e.\ (t,s) \in I \times I, \quad \forall\, x,y \in \mathbf{R}.$*

We use next the following notations

$$M(t) := L(t)(1+\int_1^t L(u)du), \quad t \in I, \quad K = \int_1^T M(t)dt.$$

Theorem 1. *Let $\alpha \in (1,2]$, $\beta \in [0,1]$ and $\Lambda \neq 0$. Assume that Hypothesis H1 is satisfied and $MK < 1$. Let $y(.) \in C(I,\mathbf{R})$ be such that $y(1) = 0$, $\sum_{i=1}^{m}\theta_i y(\xi_i) = \lambda I_H^{\delta} y(\eta)$ and there exists $q(.) \in L^1(I,\mathbf{R}_+)$ with $d(D_{HH}^{\alpha,\beta}y(t),F(t,y(t),V(y)(t))) \leq q(t)$ a.e. (I).*

Then there exists $x(.) : I \to \mathbf{R}$ a solution of problem (1)-(2) satisfying for all $t \in I$

$$|x(t)-y(t)| \leq \frac{M}{1-MK}||q(.)||_1.$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and the assumption in the statement of the theorem may be rewritten as

$$F(t, y(t), V(y)(t)) \cap \{D_{HH}^{\alpha,\beta} y(t) + q(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 2 that there exists a measurable selection $p_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|p_1(t) - D_{HH}^{\alpha,\beta} y(t)| \leq q(t) \quad a.e.\ (I) \tag{5}$$

Define $x_1(t) = \int_1^T \mathcal{R}(t,s)p_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq M \int_1^T q(t)dt.$$

Next we construct, by induction, the sequences $x_n(.) \in C(I, \mathbf{R})$ and $p_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ such that

$$x_n(t) = \int_1^T \mathcal{R}(t,s)p_n(s)ds, \quad t \in I, \tag{6}$$

$$p_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\,(I), \tag{7}$$

$$|p_{n+1}(t) - p_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \int_1^t L(s)|x_n(s) - x_{n-1}(s)|ds)\ a.e.\,(I) \tag{8}$$

Let us note that from (5)-(8) it follows for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq M(MK)^n \int_1^T q(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \leq \int_1^T |\mathcal{R}(t,t_1)|.|p_{n+1}(t_1) - p_n(t_1)|dt_1 \leq$$

$$M \int_1^T L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \int_1^{t_1} L(s)|x_n(s) - x_{n-1}(s)|ds]dt_1 \leq M$$

$$\int_1^T L(t_1)(1 + \int_1^{t_1} L(s)ds)dt_1 \cdot M^n K^{n-1} \int_1^T q(t)dt = M(MK)^n \int_1^T q(t)dt.$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$ hence converging uniformly to some $x(.) \in C(I, \mathbf{R})$. Hence, by (8), for almost all $t \in I$, the sequence $\{h_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $h(.)$ be the pointwise limit of $h_n(.)$.

At the same time, one has

$$\begin{aligned}&|x_n(t) - y(t)| \leq |x_1(t) - y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq \\ &M \textstyle\int_1^T q(t)dt + \sum_{i=1}^{n-1} (M \int_1^T q(t)dt)(MK)^i \leq \frac{M \int_1^T q(t)dt}{1-MK}.\end{aligned} \tag{9}$$

On the other hand, from (5), (8) and (9) we obtain for almost all $t \in I$

$$\begin{aligned}&|p_n(t) - D_{HH}^{\alpha,\beta} y(t)| \leq \textstyle\sum_{i=1}^{n-1} |p_{i+1}(t) - p_i(t)| + |p_1(t) - D_{HH}^{\alpha,\beta} y(t)| \leq \\ &L(t) \frac{M \int_1^T q(t)dt}{1-MK} + q(t).\end{aligned}$$

Thus the sequence $p_n(.)$ is integrably bounded and therefore $p(.) \in L^1(I, \mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (6), (7) we deduce that $x(.)$ is a solution of (1)-(2). Finally, passing to the limit in (9) we obtained the desired estimate on $x(.)$.

Now we realize the construction in (6)-(9). Assume that for some $S \geq 1$ we already constructed $x_n(.) \in C(I, \mathbf{R})$ and $p_n(.) \in L^1(I, \mathbf{R})$, $n = 1, 2, ...S$ satisfying (6), (8) for $n = 1, 2, ...S$ and (7) for $n = 1, 2, ...S - 1$. The set-valued map $t \to F(t, x_S(t), V(x_S)(t))$ is measurable. Moreover, the map $t \to L(t)(|x_S(t) - x_{S-1}(t)| + \int_1^t L(u)|x_S(u) - x_{S-1}(u)|du)$ is measurable. By the lipschitzianity of $F(t, .)$ we have that for almost all $t \in I$

$$\begin{aligned}&F(t, x_S(t), V(x_S)(t)) \cap \{h_S(t) + L(t)(|x_S(t) - x_{S-1}(t)| + \\ &\textstyle\int_1^t L(u)|x_S(u) - x_{S-1}(u)|du)[-1, 1]\} \neq \emptyset.\end{aligned}$$

Lemma 2 yields that there exist a measurable selection $p_{S+1}(.)$ of $F(., x_S(.), V(x_S)(.))$ such that for almost all $t \in I$

$$|p_{S+1}(t) - p_S(t)| \leq L(t)(|x_S(t) - x_{S-1}(t)| + \int_1^t L(u)|x_S(u) - x_{S-1}(u)|du).$$

We define $x_{S+1}(.)$ as in (6) with $n = S + 1$. Thus $p_{S+1}(.)$ satisfies (7) and (8) and the proof is complete. $\square$

Taking $y(.) = 0$ and $q(.) = L(.)$ in Theorem 1 we obtain the next existence result for problem (1)-(2).

Corollary 1. *Let $\alpha \in (1, 2]$, $\beta \in [0, 1]$ and $\Lambda \neq 0$. Assume that Hypothesis H1 is satisfied, $d(0, F(t, 0, 0)) \leq L(t)$ a.e. (I) and $MK < 1$. Then there exists $x(.)$ a solution of problem (1)-(2) satisfying for all $t \in I$ $|x(t)| \leq \frac{M}{1-MK}||L(.)||_1$.*

If F does not depend on the last variable, Hypothesis H1 becames

Hypothesis H2 i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ *measurable.*

ii) *There exists* $L(.) \in L^1(I, (0,\infty))$ *such that, for almost all* $t \in I$, $F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x_1), F(t,x_2)) \leq L(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathbf{R}.$$

Denote $L_0 = \int_1^T L(t)dt$.

Corollary 2. *Let* $\alpha \in (1,2]$, $\beta \in [0,1]$ *and* $\Lambda \neq 0$. *Assume that Hypothesis H2 is satisfied,* $d(0, F(t,0)) \leq L(t)$ *a.e.* (I) *and* $ML_0 < 1$. *Then there exists* $x(.)$ *a solution of the fractional differential inclusion*

$$D_{HH}^{\alpha,\beta} x(t) \in F(t, x(t)) \quad a.e.\ (I), \tag{10}$$

with boundary conditions (2) satisfying for all $t \in I$

$$|x(t)| \leq \frac{ML_0}{1 - ML_0}.$$

Remark 2. *If in (1),* F *is a single-valued map, Corollary 2 provides a generalization to the set-valued framework of Theorem 10 in [16] whose proof is done using Banach's contraction principle.*

Example 1. *Let us consider the problem*

$$D_{HH}^{\frac{3}{2},\frac{1}{2}} x(t) \in \left[-\frac{19}{200} \frac{|x(t)|}{1 + |x(t)|}, 0 \right] \cup \left[0, \left(\frac{19}{200} \right)^2 \frac{|\int_1^t x(s)ds|}{1 + \frac{19}{200} |\int_1^t x(s)ds|} \right] \tag{11}$$

a.e. $([1, \frac{9}{2}])$ *with nonlocal integral boundary conditions as in [16]; namely,*

$$x(1) = 0, \quad \frac{6}{29} I_H^{\frac{5}{2}} x\left(\frac{5}{2}\right) = \frac{1}{11} x\left(\frac{3}{2}\right) + \frac{2}{13} x(2) + \frac{3}{17} x(3) + \frac{4}{19} x\left(\frac{7}{2}\right) + \frac{5}{23} x(4). \tag{12}$$

In this case, $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $T = \frac{9}{2}$, $m = 5$, $\theta_1 = \frac{1}{11}$, $\theta_2 = \frac{2}{13}$, $\theta_3 = \frac{3}{17}$, $\theta_4 = \frac{4}{19}$, $\theta_5 = \frac{5}{23}$, $\xi_1 = \frac{3}{2}$, $\xi_2 = 2$, $\xi_3 = 3$, $\xi_4 = \frac{7}{2}$, $\xi_5 = 4$, $\lambda = \frac{6}{29}$, $\delta = \frac{5}{2}$, $\eta = \frac{5}{2}$ *and* $\gamma = \frac{7}{4}$.

Define $F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *by*

$$F(t,x,y) = [-\frac{19}{200} \cdot \frac{|x|}{1+|x|}, 0] \cup [0, \frac{19}{200} \cdot \frac{|y|}{1+|y|}]$$

and $k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *by* $k(t,s,x) = \frac{19}{200}x$.

Since

$$\sup\{|u|; \quad u \in F(t,x,y)\} \leq \frac{19}{200} \quad \forall\, t \in [1,4.5],\ x,y \in \mathbf{R},$$

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2)) \leq \frac{19}{200}|x_1-x_2| + \frac{19}{200}|y_1-y_2|,$$

$\forall\ x_1,x_2,y_1,y_2 \in \mathbf{R}$. *By standard computations (e.g., [6])* $L(t) \equiv \frac{19}{200}$, $K \approx 0.44305625$, $|\Lambda| \approx 1.15231$, $M \approx 2.250815481$*; therefore,* $MK \approx 0.9972378 < 1$. *So, we may apply Corollary 2 in order to obtain the existence of a solution for problem (11)-(12).*

References

[1] B. Ahmad and S.K. Ntouyas, Hilfer-Hadamard fractional boundary value problems with nonlocal mixed boundary conditions, *Fractal Fract.* 5(4) (2021), 195.

[2] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, 1990.

[3] D. Băleanu, K. Diethelm, E. Scalas and J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, 2012.

[4] A. Cernea, Filippov lemma for a class of Hadamard-type fractional differential inclusions, *Fract. Calc. Appl. Anal.* 18 (2015), 163-171.

[5] A. Cernea, On a fractional integro-differential inclusion of Caputo-Katugampola type, *Bull. Math. Anal. Appl.* 11 (2019), 22-27.

[6] A. Cernea, On some fractional integro-differential inclusions with Erdélyi-Kober fractional integral boundary conditions, *Mem. Differ. Equations Math. Phys.* 79 (2020), 15-26.

[7] A. Cernea, On the solutions of a certain boundary value problem associated to a fractional differential inclusion, *Discuss. Math., Differ. Incl., Control Optim.* 41 (2021), 5-17.

[8] A. Cernea, A note on a Hilfer-Hadamard fractional integro-differential inclusion. In *2022 Proceedings of International E-Conference on Mathematical and Statistical Sciences: A Selçuk Meeting*, 2022, 42-47.

[9] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, 2016.

[10] A.F. Filippov, Classical solutions of differential equations with multi-valued right hand side, *SIAM J. Control* 5 (1967), 609-621.

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, 2000.

[12] R. Hilfer, Y. Luchko and Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivative, *Fract. Calc. Appl. Anal.* 12 (2009), 299-318.

[13] A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.

[14] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, 1993.

[15] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[16] C. Promsakon, S.K. Ntouyas and J. Tariboon, Hilfer-Hadamard non-local integro-multipoint fractional boundary value problems, *J. Funct. Spaces* 2021 (2021), 8031524.

[17] W. Saengthong, E. Thailert and S.K. Ntouyas, Existence and uniqueness of solution for system of Hilfer-Hadamard sequential fractional differential equations with two point boundary conditions, *Adv. Differ. Equations* 2019 (2019), 525.