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HYDRODYNAMICS OF SUPERFLUID HELIUM WITH SUPERFLUID ENTROPY*

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Dedicated to Liliana Restuccia on the occasion of her 70th anniversary. M.S. and L. are very grateful to Liliana for the long conversations, a sign of a long and fruitful friendship both at work and in their personal lives, and for her always helpful advice and suggestions. It was an honor to work with her.

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Abstract

In this paper, we consider an extension of the two-fluid model for superfluid helium. Over the years, two kinds of models have been proposed to describe the observations on helium II: the two-fluid model, in which the specific entropy of the superfluid is assumed to be zero, and the extended one-fluid model, derived from extended thermodynamics. Since the statement that the entropy of the superfluid fraction vanishes has not been demonstrated theoretically, in this paper we generalize the standard (Landau and Tisza) two-fluid model allowing that a small amount of entropy is carried by the superfluid component.

Keywords: superfluid helium, one-fluid model, two-fluid model.
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1 Introduction

Liquid helium, for temperatures below *lambda point*, exhibits peculiar properties, which are a consequence of important effects of quantum nature [5,7]. It is found experimentally that liquid helium II has an extremely low viscosity and a very high thermal conductivity: in fact, it has an extraordinary ability to flow without friction through a fine capillary and it is unable to boil. Among its typical effects we recall the thermal-mechanical phenomena where a pressure difference causes a temperature difference, the fountain effect, and the capability of propagating a temperature wave known as the second sound.

Two kinds of models have been proposed to describe the observations on helium II: the two-fluid model, initially proposed by Tisza [16] and Landau [6], and widely used since then, and an extended one-fluid model, derived from extended thermodynamics. This latter considers a vectorial internal degree of freedom, identified as the heat flux $\mathbf{q}$. In the two-fluid model, helium II is considered as a mixture of a normal component and a superfluid component, each of them having their own velocity, $\mathbf{v}_n$ and $\mathbf{v}_s$. Both models may be related to each other [10] because the fluid velocity $\mathbf{v}$ and the heat flux $\mathbf{q}$ are closely related to $\mathbf{v}_n$ and $\mathbf{v}_s$.

The one-fluid model

A thermodynamic formalism, known as Extended Thermodynamics (in short, E.T.) [3,4,8,12] was developed, in order to describe rapid phenomena or materials in which the relaxation times of some fluxes are long. This theory, in fact, uses the dissipative fluxes, besides the traditional variables, as independent fields. An extended approach to thermodynamics is required in helium II because the relaxation time of heat flux is comparable with the evolution times of the other variables. This field cannot therefore be expressed by means of a constitutive equation as a dependent variable. This point of view is confirmed by the fact that the thermal conductivity of helium II cannot be measured. In analogy with heat transport problem, using E.T., the relative motion of the excitations is well described by the dynamics of the heat flux. For this reason, in the one-fluid model of liquid helium II, it is rather natural to select as fundamental fields the density ρ, the velocity $\mathbf{v}$, the absolute temperature T and the internal energy flux per unit mass, briefly called here heat flux $\mathbf{q}$ [9].

The set of field equations is given by the balance equation of mass, momentum and energy complemented by an evolution equation for the heat flux, as [10]

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\nabla \cdot \mathbf{v} + \zeta \nabla T + \nabla \cdot \mathbf{P_q} = -\sigma_{\mathbf{q}}, \tag{1}$$

where $\mathbf{q}$ is the heat flux, $\mathbf{v}$ the velocity of the fluid, T the temperature (linked to the internal energy), $\mathbf{P_q}$ a non equilibrium stress tensor and $\sigma_{\mathbf{q}}$ the production term. In E.T. the positive coefficient ζ is defined as the ratio between the relaxation time of the heat flux and its thermal conductivity. In liquid helium II this ratio is finit and it is linked to the second sound velocity.

The two-fluid model

The two-fluid model regards liquid helium II as a two components mixture: a normal component with normal viscosity and thermal conductivity and a superfluid component with zero entropy and viscosity. The superfluid component is absent at the *lambda point*, while the normal component is absent at zero temperature. This theory was inspired by considerations of quantum statistical mechanics.

In the two-fluid model, ρ_n and ρ_s are the mass densities of normal and superfluid components, such that $\rho_n + \rho_s = \rho$, with ρ being the total, observable, mass density of the fluid. Both ρ_n and ρ_s depend on the temperature T in such a way that $\rho_s = 0$ at $T = T_\lambda$ (with T_λ the lambda transition temperature) and $\rho_s = \rho$ in the limit when T tends to zero. The behaviour of ρ_n is the opposite one, namely, $\rho_n = \rho$ at $T = T_\lambda$ and $\rho_n = 0$ in the limit when T tends to zero. The behaviour of ρ_s in terms of T is $\rho_s = \rho[(T_\lambda - T)/T_\lambda]^{1/3}$. Furthermore, these components are assumed to move with respective speeds $\mathbf{v}_n$ and $\mathbf{v}_s$, related to the barycentric speed $\mathbf{v}$ as

$$\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s = \rho \mathbf{v}. \tag{2}$$

Landau's evolution equations

To describe the evolution of the system, evolution equations for the variables are needed and their consistency with the second law of thermodynamics must be examined. We shall present here the Landau evolution equations, that describe the behaviour of liquid He II in the zero temperature limits, where dissipative phenomena are absent.

The density ρ and the mass flux $\rho\mathbf{v}$ must satisfy the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3}$$

and the balance of the total momentum

$$\frac{\partial(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s)}{\partial t} + \nabla \cdot (\rho_n \mathbf{v}_n \mathbf{v}_n + \rho_s \mathbf{v}_s \mathbf{v}_s + p\mathbf{U}) = \mathbf{0}, \tag{4}$$

with p the thermodynamic pressure and $\mathbf{U}$ the identity tensor.

Besides the general balance equations 3 and 4, a particular equation describing the evolution of $\mathbf{v}_s$ is needed. In order that $\nabla \times \mathbf{v}_s = 0$ is always zero, the time derivative of $\mathbf{v}_s$ must be the gradient of some scalar function. In Landau's model it is taken

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \cdot \left(\frac{1}{2}\mathbf{v}_s^2 + \mu\right) = \mathbf{0}, \tag{5}$$

with μ some scalar, identified a posteriori with the chemical potential.

Since $\nabla\mu = \frac{1}{\rho}\nabla p - s\nabla T$ (with s the specific entropy, i.e. the entropy per unit mass), equation 5 may be written as

$$\rho_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s\right) = -\frac{\rho_s}{\rho}\nabla p + \rho_s s \nabla T. \tag{6}$$

This has the form of the Euler equation but with the additional term in ∇T.

Landau completed these equations imposing that in HeII the entropy is conserved, and wrote for the entropy density ρs the following balance equation

$$\frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho s \mathbf{v}_n) = 0. \tag{7}$$

Equations 3–7 describe the whole evolution of the system; in them ρ_s, ρ_n, μ and s are functions not only of the usual thermodynamic variables p and T, but also of the squared of the counterflow velocity $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s$.

Energy and entropy are not independent of each other, but related by a fundamental equation of state. For the specific energy e_0 in a coordinate system in which the velocity of the superfluid component is zero, Landau wrote

$$d(\rho e_0) = \mu d\rho + T d(\rho s) + \mathbf{V}_{ns} \cdot d\mathbf{j}_0, \tag{8}$$

with the momentum $\mathbf{j}_0$ in such system being $\mathbf{j}_0 = \rho_n \mathbf{V}_{ns}$. Incidentally, 8 implies for the pressure p the relation

$$p = -\rho e_0 + \rho s T + \rho\mu + \rho_n \nabla V_{ns}^2. \tag{9}$$

Differentiating 9 and using 8 one obtains

$$d\mu = -s dT + \frac{1}{\rho} dp - \frac{\rho_n}{\rho}\mathbf{V}_{ns} \cdot d\mathbf{V}_{ns}. \tag{10}$$

Note that the velocities $\mathbf{v}_s$ and $\mathbf{v}_n$ are not directly observable, but they may be obtained from $\mathbf{v}$ and $\mathbf{q}$. In Landau model $\mathbf{q}$ is related to $\mathbf{v}_n$ and $\mathbf{v}$ in this way:

$$\mathbf{q} = \rho s T(\mathbf{v}_n - \mathbf{v}), \tag{11}$$

with s the specific entropy, that in conventional Landau's model is ascribed only to the normal component. Indeed, since the specific entropy of the superfluid is assumed to be zero (in the conventional two-fluid model), the only flow of heat in helium at rest is related to the motion of the normal component. The barycentric contribution in $\mathbf{v}$ is subtracted in order that $\mathbf{q}$ does not contain the purely convective contribution of the global motion of the system. If $\mathbf{v}$ is written in terms of $\mathbf{v}_n$ and $\mathbf{v}_s$ 11 becomes

$$\mathbf{q} = \rho_s T s \mathbf{V}_{ns}, \tag{12}$$

where $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ denotes the so-called counterflow velocity.

Though the system cannot be physically decomposed into these separate components, ρ_n, ρ_s, $\mathbf{v}_n$ and $\mathbf{v}_s$ may be mathematically obtained from some specific experiments. In the last years there has been much progress in visualization techniques trying to visualize the flow of the normal component through its effects on very small particles, which are drifted by normal flow.

Aims of the paper

This paper generalizes the standard (Landau and Tisza) two-fluid model allowing that a small amount of entropy is carried by the superfluid component. Indeed, as observed by Puttermann [13], the statement that the entropy of the superfluid fraction vanishes has not been demonstrated theoretically, so that it must be regarded as an additional postulate. The experimental results [2, 15] simply tell us that entropy carried by superfluid component is smaller than 2% with respect to entropy carried by normal component. Here a more general two-fluid model, where also the superfluid component carry a small amount of entropy, is presented (see also [1, 14], where theoretical microscopic motivations in favor of a small superfluid entropy are advanced).

The structure of the paper is the following: in Section 2 we present a generalized model for superfluid helium that take into account of the presence of an amount of superfluid entropy, first for a general fluid mixture (sect. 2.1) and then for superfluid helium (sect. 2.2). In Section 2.3 we present the new field equations and in Section 2.4 we compare them with the Landau model, while in Section 2.5 we consider how the velocity of second sound is affected by the presence of the superfluid entropy. Section 3 contains our conclusions.

2 The two-fluid model with superfluid entropy

2.1 Müller rational theory of fluid mixtures

The phenomenological model of helium as a two-fluid mixture can be viewed in the general frame of the rational theory of fluid mixtures [11]. This theory allows the determination of very strong constraints on the functional form of constitutive equations through a systematic exploitation of second law of thermodynamics and material objectivity principle. The following

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = \tau_\alpha, \tag{13}$$

$$\frac{\partial \rho_\alpha \mathbf{v}_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \mathbf{v}_\alpha + \mathbf{P}_\alpha) = \mathbf{m}_\alpha, \tag{14}$$

$$\frac{\partial \rho \epsilon}{\partial t} + \nabla \cdot (\rho \epsilon \mathbf{v} + \mathbf{q}) = \mathbf{P} : \nabla \mathbf{v}, \tag{15}$$

are respectively the balance equations of mass and momentum of the components and the balance equation of energy of the whole mixture; ρ_α and $\mathbf{v}_\alpha$ are the partial densities and velocities, $\rho = \sum_\alpha \rho_\alpha$ is the total density, $\mathbf{v}$ the barycentric velocity, ϵ is the specific internal energy and $\mathbf{q}$ the internal energy flux of the mixture. In these equations τ_α and $\mathbf{m}_\alpha$ are the mass and the momentum production of two components. Further $\mathbf{P}_\alpha$ and $\mathbf{P}$ denote respectively the stresses on the single components and on the mixture.

In this work, for sake of simplicity, we shall not consider viscosity effects, that in helium II are very mild. Thus, the partial stresses are $\mathbf{P}_\alpha = p_\alpha \mathbf{U}$, p_α being the partial pressures, and $\mathbf{U}$ the identity tensor. The stress $\mathbf{P}$ on the whole mixture is decomposed in an intrinsic term and a part due to the relative motion between the two components

$$\mathbf{P} = \sum_\alpha \left(p_\alpha \mathbf{U} + \rho_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha \right), \tag{16}$$

where $\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}$ are the velocities relative to the barycentric velocity.

The laws of conservation of mass and momentum imply

$$\sum_\alpha \tau_\alpha = 0 \qquad \text{and} \qquad \sum_\alpha \mathbf{m}_\alpha = 0. \tag{17}$$

Thus, from 13–14 we obtain immediately the balance equation of mass and momentum of the mixture

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{18}$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + \mathbf{P}) = 0. \tag{19}$$

In 15 the specific internal energy ϵ is sum of the specific energies of the two components, which are decomposed into an intrinsic part and a term depending on the relative motion [11]

$$\epsilon = \sum_\alpha \frac{\rho_\alpha}{\rho}\left(\epsilon_\alpha + \frac{1}{2}u_\alpha^2\right). \tag{20}$$

The internal energy flux $\mathbf{q}$ of the mixture is defined as

$$\mathbf{q} = \sum_\alpha \mathbf{q}_\alpha + \rho_\alpha\left(\epsilon_\alpha + \frac{1}{2}u_\alpha^2\right)\mathbf{u}_\alpha - \mathbf{u}_\alpha \cdot \mathbf{P}_\alpha, \tag{21}$$

with $\mathbf{q}_\alpha$ the partial energy fluxes. It is useful to decompose it into an intrinsic part and a term depending on the relative motion [11]

$$\mathbf{q} = \mathbf{q}^I + \frac{1}{2}\sum_\alpha \rho_\alpha u_\alpha^2 \mathbf{u}_\alpha, \tag{22}$$

with

$$\mathbf{q}^I = \sum_\alpha \mathbf{q}_\alpha + \rho_\alpha\epsilon_\alpha\mathbf{u}_\alpha - \mathbf{u}_\alpha \cdot \mathbf{P}_\alpha \tag{23}$$

and $\mathbf{q}_\alpha$ the partial energy fluxes.

Equations 13–15 must be completed by the second law of thermodynamics. Introducing the specific entropy s, defined as

$$\rho s = \sum_\alpha \rho_\alpha s_\alpha \tag{24}$$

(with s_α the partial entropies), and the entropy flux $\mathbf{\Phi}_s$, this law is traduced in the following inequality

$$\frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho s\mathbf{v} + \mathbf{\Phi}_s) \geq 0. \tag{25}$$

Assuming that the constitutive quantities are linear in the vectorial and tensorial quantities, using the principle of material objectivity and the second law of thermodynamics, some restrictions for the constitutive quantities are found [11]. In particular, the entropy flux $\mathbf{\Phi}_s$ is given by

$$\mathbf{\Phi}_s = \frac{1}{T}\left(\mathbf{q}^I + \sum_\alpha \rho_\alpha \mu_\alpha^I \mathbf{u}_\alpha\right), \tag{26}$$

with $\mathbf{q}^I$ expressed by 23, and where μ_α^I represent the intrinsic chemical potentials of the partial components. Further the following expressions are derived for the momentum productions of the two components of the mixture

$$\mathbf{m}_\alpha - \tau_\alpha \mathbf{v}_\alpha = \sum_\beta \left(\frac{\partial p_\alpha}{\partial \rho_\beta} - \rho_\alpha \frac{\partial \mu_\alpha^I}{\partial \rho_\beta} \right) \nabla \rho_\beta + m_u^\alpha \mathbf{u} + m_T^\alpha \nabla T. \tag{27}$$

2.2 Application to liquid helium II

According to the Nernst-Planck formulation of third law of thermodynamics, the entropy of the whole mixture is zero for $T = 0$. Because the normal component is absent at this temperature, we conclude that $s_s = 0$ for $T = 0$. Landau assumed that this condition is valid also for $T \neq 0$, so that he associated a nonzero entropy only to the normal component. Here, we will consider a non convectional two-fluid model, where the entropy is expressed by

$$\rho s = \rho_n s_n + \rho_s s_s, \tag{28}$$

with $s_s \neq 0$.

Now we will particularize equations 16–27, recalled in the previous subsection, to the case of superfluid He II.

As $\rho_s \mathbf{u}_s + \rho_n \mathbf{u}_n = 0$, here we consider only the superfluid relative velocity $\mathbf{u}_s = \mathbf{v}_s - \mathbf{v}$, that in the following will be denoted simply as $\mathbf{u}$. This relative velocity $\mathbf{u}$ is linked to the counterflow velocity $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ by the relation $\mathbf{u} = -\frac{\rho_n}{\rho} \mathbf{V}_{ns}$. With these positions, the stress $\mathbf{P}$ on the whole mixture is written

$$\mathbf{P} = p\mathbf{U} + \frac{\rho \rho_s}{\rho_n} \mathbf{u}\mathbf{u} = p\mathbf{U} + \frac{\rho_s \rho_n}{\rho} \mathbf{V}_{ns} \mathbf{V}_{ns}, \tag{29}$$

where $p = p_n + p_s$ is the total pressure.

The specific internal energy ϵ, form equation 20, is written as

$$\epsilon = \epsilon^I + \frac{1}{2} \frac{\rho_s}{\rho_n} u^2 = \epsilon^I + \frac{1}{2} \frac{\rho_s \rho_n}{\rho^2} V_{ns}^2, \tag{30}$$

where $\rho \epsilon^I = \rho_s \epsilon_s + \rho_n \epsilon_n$.

The internal energy flux $\mathbf{q}$ of the mixture is expressed by 22, and can be decomposed in the following way

$$\mathbf{q} = \mathbf{q}^I - \frac{\rho_n \rho_s (\rho_n - \rho_s)}{2\rho^2} V_{ns}^2 \mathbf{V}_{ns}, \tag{31}$$

with

$$\mathbf{q}^I = \mathbf{q}_s + \mathbf{q}_n - \frac{\rho_s \rho_n}{\rho} \left(\epsilon_s - \epsilon_n + \frac{p_s}{\rho_s} - \frac{p_n}{\rho_n} \right) \mathbf{V}_{ns}. \tag{32}$$

The entropy flux $\mathbf{\Phi}_s$ is given by

$$\mathbf{\Phi}_s = \frac{1}{T}\left(\mathbf{q}^I - \frac{\rho_s\rho_n}{\rho}(\mu_n^I - \mu_s^I)\mathbf{V}_{ns}\right), \tag{33}$$

where μ_n^I and μ_s^I represent the intrinsic chemical potentials of the normal and superfluid components, connected with the chemical potentials by the relation [11]

$$\mu_n - \mu_s = \mu_n^I - \mu_s^I - \frac{1}{2}V_{ns}^2. \tag{34}$$

The experimental observations show a very free and easy transition from the superfluid component to the normal component and viceversa, thus showing the complete absence of dynamical interaction between the two components. This leads to an important mathematical consequence: in superfluid helium $\mathbf{m}_s$ is caused only by the mass exchange between the two components, i.e.

$$\mathbf{m}_s - \tau_s\mathbf{v}_s = 0. \tag{35}$$

Then, from mass and momentum conservation laws ($\tau_s + \tau_n = 0$, $\mathbf{m}_s + \mathbf{m}_n = 0$), we obtain

$$\mathbf{m}_n - \tau_n\mathbf{v}_n = \tau_n(\mathbf{v}_s - \mathbf{v}_n) = -\tau_n\mathbf{V}_{ns} = \frac{\rho}{\rho_n}\tau_n\mathbf{u}. \tag{36}$$

From this, taking into account 27, we get $m_T^s = m_T^n = 0$, $m_u^s = 0$, $m_u^n = \frac{\rho}{\rho_n}\tau_n$, and the important relationship

$$\frac{\partial p^\alpha}{\partial\rho_\beta} = \rho_\alpha\frac{\partial\mu_\alpha^I}{\partial\rho_\beta} \qquad (\alpha = s, n). \tag{37}$$

From 37 several important consequences can be deduced; mixtures obeying the previous condition were called by Müller simple mixtures [11].
We deduce therefore that, if in a mixture there is no dynamical interaction between the two components, as it is the case for helium, the mixture is necessarily simple.

The second law of thermodynamics, when applied to simple mixtures, leads to the following functional forms for the constitutive quantities

$$p_\alpha = p_\alpha(\rho_\alpha, T), \quad \mu_\alpha^I = \mu_\alpha^I(\rho_\alpha, T), \quad \epsilon_\alpha = \epsilon_\alpha(\rho_\alpha, T), \quad s_\alpha = s_\alpha(\rho_\alpha, T). \tag{38}$$

Moreover, in a simple mixture, the intrinsic chemical potentials coincide with the partial specific free enthalpies, and there is a Gibbs equation for each component of the mixture

$$\mu_\alpha^I = \epsilon_\alpha - Ts_\alpha + \frac{p_\alpha}{\rho_\alpha}, \qquad Tds_\alpha = d\epsilon_\alpha - \frac{p_\alpha}{\rho_\alpha^2}d\rho_\alpha. \tag{39}$$

From this, one gets

$$d\mu_\alpha^I = \frac{1}{\rho_\alpha} dp_\alpha - s_\alpha dT \tag{40}$$

and the Gibbs-Duhem equation and the Gibbs equation for the whole mixture are

$$\rho(\epsilon^I - Ts) = \rho_s \mu_s^I + \rho_n \mu_n^I - p, \tag{41}$$

$$Td(\rho s) = d(\rho \epsilon^I) - \mu_s^I d\rho_s - \mu_n^I d\rho_n. \tag{42}$$

Another very important consequence of the complete absence of dynamical interaction between normal and superfluid components is that the chemical equilibrium between the two phases is reached instantaneously, therefore this process is completely reversible. As a consequence the production of entropy during this process must be exactly zero. This implies that, using 34

$$\mu_n = \mu_s \quad \Rightarrow \quad \mu_n^I - \mu_s^I - \frac{1}{2} V_{ns}^2 = 0. \tag{43}$$

Equation 43 has several important consequences. First observe that in this hypothesis the Gibbs-Duhem equation and Gibbs equation for helium can be written

$$\mu_s^I = \epsilon^I - Ts + \frac{p}{\rho} - \frac{1}{2} \frac{\rho_n}{\rho} V_{ns}^2, \tag{44}$$

$$Td(\rho s) = d(\rho \epsilon^I) - \mu_s^I d\rho - \frac{1}{2} V_{ns}^2 d\rho_n, \tag{45}$$

from which we get

$$d\mu_s^I = -sdT + \frac{1}{\rho} dp - \frac{1}{2} \frac{\rho_n}{\rho} dV_{ns}^2. \tag{46}$$

As one can see from equations 44–46, the chemical potential μ_s^I, the energy ϵ and the entropy s of helium depend not only on the equilibrium quantities, but also on the counterflow velocity $\mathbf{V_{ns}}$. Hence they are non-equilibrium quantities.

Introducing the total internal energy ϵ, from 44 and 45 we get also

$$\mu_s^I = \epsilon - Ts + \frac{p}{\rho} - \frac{\rho_s \rho_n}{\rho^2} V_{ns}^2 \tag{47}$$

$$Tds = d\epsilon - \frac{p}{\rho^2} d\rho - \frac{1}{2} \frac{\rho_s}{\rho_n} du^2. \tag{48}$$

Using equations 39 and 43, one can write the internal energy flux 32 in the following way

$$\mathbf{q} = \lambda\nabla T - \frac{\rho_n\rho_s}{\rho}\left[T(s_s - s_n)\mathbf{V}_{ns} + \frac{\rho_s}{\rho}V_{ns}^2\mathbf{V}_{ns}\right], \tag{49}$$

where we have put $\mathbf{q}_s + \mathbf{q}_n = \lambda\nabla T$ (being λ the thermal conductivity), so the entropy flux can be written as

$$\mathbf{\Phi}_s = \frac{\lambda}{T}\nabla T + \rho(s - s_s)(\mathbf{v}_n - \mathbf{v}). \tag{50}$$

A separate discussion must be made for the thermal conductivity λ. As we said in the introduction, experiments show that the thermal conductivity of helium is extremely high. As mentioned in Sect. 1, it was precisely these experimental motivations that led to the formulation of the one-fluid model of helium II, deduced from extended thermodynamics. However, in the conventional two-fluid model, the thermal conductivity λ is maintained to explain the attenuation of the second sound and is related only to the normal component. Instead, the attenuation of the second sound in the one-fluid model is attributed to the presence of dissipative terms dependent on the heat flow gradient. Postponing the comparison between the two models (two-fluid and one-fluid) in the presence of dissipation to a subsequent work, in the remainder of this work we will assume $\lambda = 0$.

Therefore, we are in the total absence of dissipative phenomena and the entropy is conserved. From equation 25 we get

$$\frac{\partial(\rho s)}{\partial t} + \nabla\cdot(\rho(s - s_s)\mathbf{v}_n) = 0. \tag{51}$$

2.3 Field equations

To determine the system of field equations, we recall that the left hand side of equation 43 is a function of ρ, ρ_s, T and V_{ns}. Its vanishing provides an algebraic dependence between these fields, so that one of them can be expressed in terms of the remaining fields. As a consequence of the reduction in the number of independent fields, to establish the field equations, we shall use only the equations of balance of mass and momentum, obtained immediately from 13–14

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{52}$$

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v} + \mathbf{P}) = 0, \tag{53}$$

the balance equation for the energy of the whole system 15, and the momentum balance equation for the superfluid component.

$$\frac{\partial \rho\epsilon}{\partial t} + \nabla \cdot (\rho\epsilon \mathbf{v} + \mathbf{q}) = \mathbf{P} : \nabla \mathbf{v}, \tag{54}$$

$$\frac{\partial \rho_s \mathbf{v}_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s \mathbf{v}_s + \mathbf{P}_s) = \mathbf{m}_s. \tag{55}$$

These equations must be complemented by the constitutive quantities 29, 31, 35, 36, and 49.

The equation of balance of mass for the superfluid component which has been neglected can be used, once the fundamental fields are known, to determine the mass production τ_s of the superfluid.

The evolution equations of velocities of two components are

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s + \frac{1}{\rho_s}\nabla p_s = 0, \tag{56}$$

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n + \frac{1}{\rho_n}\nabla p_n = \frac{1}{\rho_n}(\mathbf{m}_n - \tau_n \mathbf{v}_n) = -\frac{1}{\rho_n}\tau_n \mathbf{V}_{ns}. \tag{57}$$

Note that the right-hand side of equation 57 depends on the mass production τ_n. This term is usually neglected in the most literature. However, it is know that normal and superfluid components can easily transform one in the other when the temperature changes, and the mass supply $\tau_n (= -\tau_s)$ of the normal component, in general, cannot be neglected.

Here, to overcome this problem, we will follow the Landau suggestion and use as field variable the total velocity $\mathbf{v}$ instead of $\mathbf{v}_n$.

Now, by using 40, we obtain

$$\frac{1}{\rho_s}\nabla p_s = \nabla \mu_s^I + s_s \nabla T, \qquad \frac{1}{\rho_n}\nabla p_n = \nabla \mu_n^I + s_n \nabla T, \tag{58}$$

from which

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s + \nabla \mu_s^I + s_s \nabla T = 0; \tag{59}$$

then, taking into account Eq.45, we obtain

$$\nabla \mu_s^I = -s\nabla T + \frac{1}{\rho}\nabla p - \frac{1}{2}\frac{\rho_n}{\rho}\nabla V_{ns}^2. \tag{60}$$

Substituting in 56, we get the following evolution equation for the superfluid velocity

$$\rho_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s \right) - \rho_s (s - s_s)\nabla T + \frac{\rho_s}{\rho}\nabla p - \frac{1}{2}\frac{\rho_s \rho_n}{\rho}\nabla V_{ns}^2 = 0. \tag{61}$$

We can also obtain an evolution equation of the velocity of the normal component. One arrives to the following equation

$$\rho_n \left(\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n \right) + \frac{\rho_n}{\rho} \nabla p - \rho_n (s - s_n) \nabla T + \frac{\rho_s \rho_n}{2\rho} \nabla V_{ns}^2 = -\tau_n \mathbf{V}_{ns}, \tag{62}$$

that however always contain the unknown quantity $-\tau_n \mathbf{V}_{ns} = \tau_s \mathbf{V}_{ns}$.

Note that both equations 61 and 62 contain a term proportional to the temperature gradient. Observing that $\rho_n(s - s_n) = -\rho_s(s - s_s)$, we recognize that the linear term in the temperature gradient appears in both equations with an opposite sign, meaning that it transfers motion between both components.

2.4 Comparison with Landau model

As mentioned above, Landau, in its derivation of HeII evolution equations, assumed the conservation of the entropy, so imposed zero all the dissipative coefficients. Further he associated a nonzero entropy only to the normal component; therefore he assumed zero the entropy of the superfluid component ($s_s = 0$), i.e.

$$s_{Landau} = \frac{\rho_n}{\rho} s_n. \tag{63}$$

Under these hypotheses the expression 49 of heat flux $\mathbf{q}$ becomes

$$\mathbf{q} = \left(\rho T s - \rho \frac{\rho_n}{\rho_s} (v_n - v)^2 \right) (\mathbf{v}_n - \mathbf{v}), \tag{64}$$

so that, we see that its linear part coincides with the expression of the heat flux postulated by Landau, while the total entropy flux can be written as

$$\rho s \mathbf{v} + \mathbf{\Phi}_s = \rho s \mathbf{v} + \rho s (\mathbf{v}_n - \mathbf{v}) = \rho s \mathbf{v}_n, \tag{65}$$

that is identical with the expression of the Landau entropy flux. Note that the balance law for entropy, postulated by Landau, can be deduced immediately from the two fluid model discussed above, when the entropy of the superfluid component, in the entropy balance equation 51, is put equal to zero.

Consider now eq. 9, where e_0 is the specific energy of the mixture in the frame where the superfluid center of mass is at rest, and compare it with 44, we deduce that the energy density ρe_0 is connected to the intrinsic internal energy density $\rho \epsilon^I$ appearing in eq 44, through the relation

$$\rho e_0 = \rho \epsilon^I + \frac{1}{2} \rho_n V_{ns}^2. \tag{66}$$

In order to compare the model deduced in previous section with Landau model, we observe, first, that mass and momentum balance equations are identical with the first two Landau equations. The evolution equation for the superfluid component 59 (assuming $s_s = 0$) becomes

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s + \nabla \mu_s^I = 0, \tag{67}$$

recalling that it is $(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = \mathbf{v}_s \times (\nabla \times \mathbf{v}_s) + \nabla \left(\frac{1}{2}v_s^2\right)$, one can deduce, by taking the curl of this quantity, that $\frac{\partial}{\partial t}(\nabla \times \mathbf{v}_s) = 0$; therefore, if $\nabla \times \mathbf{v}_s$ is initially zero, it remains zero at any instant of time. Putting therefore $\nabla \times \mathbf{v}_s = 0$ in Eq. 67, the latter equation can be written in the form postulated by Landau

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \cdot \left(\frac{v_s^2}{2} + \mu_s^I\right) = 0. \tag{68}$$

Comparing this equation with Landau equation for the velocity of the superfluid component 5, we are lead to identify the chemical potential μ introduced by Landau with the intrinsic part μ_s^I of the chemical potential of the superfluid component.

2.5 Second sound (temperature wave)

We will show in this subsection as the presence of a non zero superfluid entropy modify the second sound velocity.

To consider wave propagation of ρ and s in the superfluid, we start from the balance equation of total mass and total momentum, omitting nonlinear terms and viscosity

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{v}, \tag{69}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p, \tag{70}$$

combining them and neglecting second order terms, one gets

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p. \tag{71}$$

To obtain the second equation, one writes an equation for the relative velocity $\mathbf{u} = \mathbf{v}_s - \mathbf{v}$, by using 61 and 70. Neglecting nonlinear and dissipative terms, one gets

$$\frac{\partial \mathbf{u}}{\partial t} = -(s - s_s)\nabla T. \tag{72}$$

This equation must be combined with the evolution equation of entropy 51, so that one gets

$$\frac{\partial s}{\partial t} = -\frac{\rho_s}{\rho}(s - s_s)\nabla \cdot \mathbf{u}. \tag{73}$$

Combining them one arrives to

$$\frac{\partial^2 s}{\partial t^2} = \frac{\rho_s}{\rho_n}(s - s_s)^2 \nabla^2 T. \tag{74}$$

Expressing p and T in 71 and 74 in terms of ρ and s, it is found

$$\begin{aligned} \frac{\partial^2 \rho}{\partial t^2} &= \left(\frac{\partial p}{\partial \rho}\right)_s \nabla^2 \rho + \left(\frac{\partial p}{\partial s}\right)_\rho \nabla^2 s \\ \frac{\partial^2 s}{\partial t^2} &= \frac{\rho_s}{\rho_n}(s - s_s)^2 \left[\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \rho + \left(\frac{\partial T}{\partial s}\right)_\rho \nabla^2 s\right]. \end{aligned} \tag{75}$$

Studying perturbation

$$\rho = \rho_0 + \rho' \exp[\jmath(\omega t - kz)] \qquad \text{and} \qquad s = s_0 + s' \exp[\jmath(\omega t - kz)] \tag{76}$$

one obtains by making the determinant of these perturbations in 75 that there are two waves with speeds V_1^2 and V_2^2 given by

$$V_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \qquad \text{and} \qquad V_2^2 = \frac{\rho_s}{\rho_n}(s - s_s)^2 \left(\frac{\partial T}{\partial s}\right)_\rho = \frac{\rho_s \rho_n}{\rho^2}(s_n - s_s)^2 \left(\frac{\partial T}{\partial s}\right)_\rho \tag{77}$$

The first one is the usual first-sound, density or pressure wave, and the second is the second sound namely, temperature or entropy wave. Note that the second sound velocity does not depend only on the entropy of the superfluid component, but also on the entropy of the normal component. From its wave speed one determine ρ_s/ρ_n, or equivalently ρ_s/ρ, indeed introducing the concentration $c_s = \frac{\rho_s}{\rho}$, from equation 77 we get:

$$c_s = \frac{\rho_s}{\rho} = \frac{1}{1 + \frac{V_2^2}{(s - s_n)^2}\left(\frac{\partial T}{\partial s}\right)_\rho}. \tag{78}$$

Finally we observe that in the last expression of equation 77, the term $(s_n - s_s)$ is analogous to the counterflow velocity V_{ns}, so it can be identified as an entropy counterflow. In fact, because the propagation of the second sound is linked to the heat flow, that is expressed in terms of $\mathbf{V}_{ns}$ by eq. 12, alongside the counterflow velocity there is a counterflow entropy, since each component (normal and superfluid) moves with its entropy.

3 Concluding remarks

In this paper we have presented a generalized two-fluid model for superfluid helium II in which the superfluid component is allowed to carry a small, but non-vanishing, amount of entropy. This extension relaxes one of the traditional postulates of the Landau–Tisza theory—namely, the assumption that the entropy of the superfluid fraction is identically zero for all temperatures below the lambda point—while remaining fully consistent with experimental evidence and the fundamental laws of thermodynamics.

Starting from the general framework of the rational theory of fluid mixtures, we have shown that helium II can be consistently described as a simple mixture, owing to the complete absence of dynamical interaction between the normal and superfluid components. By introducing a non-zero superfluid entropy that vanishes only in the zero-temperature limit, we have derived modified field equations and compared them with the classical Landau equations. The comparison shows that the Landau model is recovered as a limiting case of the present theory when the superfluid entropy is set to zero. In this sense, the standard two-fluid model appears as a particular, highly idealized approximation of a more general and thermodynamically consistent description.

An important physical consequence of the generalized model is the modification of the second sound velocity. Even a small superfluid entropy contribution leads to corrections in the propagation of temperature waves, which may become relevant in high-precision experiments. This result suggests that measurements of second sound and related thermo-mechanical effects could provide indirect evidence for, or constraints on, the entropy carried by the superfluid component.

In conclusion, the model developed here provides a coherent theoretical framework that bridges the traditional two-fluid picture and extended thermodynamic approaches. It preserves the successful features of the Landau–Tisza theory while offering a more flexible and potentially more accurate description of superfluid helium II.

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