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QUASI k-YAMABE GRADIENT SOLITONS: TRIVIALITY AND NONEXISTENCE*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

The purpose of our paper is to investigate the geometric behavior of complete noncompact and stochastically complete quasi k-Yamabe gradient solitons under appropriate conditions in order to obtain new triviality and nonexistence results. For this, we derive a suitable Bochner type formula and we apply it jointly with integrability criteria, Liouville type results and several maximum principles dealing, in particular, with the notion of convergence to zero at infinity and the concept of polynomial and exponential volume growth.

Keywords: complete and stochastically complete quasi k-Yamabe gradient solitons, σ_k-curvature, convergence to zero at infinity, polynomial and exponential volume growth.

MSC: 53C21, 53C24, 53C25.

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1 Introduction

Let (Σ^n, g) be an n-dimensional, $n \geq 3$, Riemannian manifold. We recall that the Schouten tensor $\mathcal{A}_g$ of (Σ^n, g) is given by

$$\mathcal{A}_g = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)}g\right),$$

where Ric and R denote, respectively, the Ricci tensor and the scalar curvature of (Σ^n, g). Moreover, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the symmetric endomorphism $g^{-1} \circ \mathcal{A}_g$, the σ_k-curvatures, $1 \leq k \leq n$, related to the metric g, are defined as the k-th elementary symmetric function of $\lambda_1, \ldots, \lambda_n$, that is,

$$\sigma_k(g) = \sum_{i_1 < \ldots < i_n} \lambda_{i_1} \ldots \lambda_{i_n}.$$

It is not difficult to see that $\sigma_1(g) = \dfrac{R}{2(n-1)}$ and, for this reason, the concept of σ_k-curvature is an appropriate generalization of the scalar curvature.

According to Tokura, Batista, Kai and Barboza [18], a Riemannian manifold (Σ^n, g), $n \geq 3$, is said to be a *quasi k-Yamabe gradient soliton*, denoted by (Σ^n, g, f), if there exists a smooth function $f \in \mathcal{C}^\infty(\Sigma^n)$, called *potential function*, and real constants $m, \rho \in \mathbb{R}$, with $m \neq 0$, such that

$$\nabla^2 f - \frac{1}{m}\nabla f \otimes \nabla f = (\sigma_k - \rho)g, \tag{1}$$

where σ_k, ∇f and $\nabla^2 f$ denote, respectively, the σ_k-curvature, the gradient and the hessian (related to the metric g) of the smooth function $f \in \mathcal{C}^\infty(\Sigma^n)$. For the case $k = 1$, the quasi 1-Yamabe gradient soliton is just the *quasi Yamabe gradient soliton* (for more details, we recommend [11]). We point out that, when the potential function is constant, the quasi k-Yamabe gradient soliton is said to be *trivial*. In this context, these authors have proved in [18] that any compact quasi k-Yamabe gradient soliton is trivial and, therefore, has constant σ_k-curvature.

Now, let us consider the positive smooth function $u = e^{-\frac{f}{m}} \in \mathcal{C}^\infty(\Sigma^n)$. Since $\nabla u = -\frac{u}{m}\nabla f$, with a straightforward computation, we obtain

$$\nabla^2 u = \frac{u}{m^2}\nabla f \otimes \nabla f - \frac{u}{m}\nabla^2 f. \tag{2}$$

Thus, in an equivalent way, from (2), we can rewrite (1) as follows

$$\nabla^2 u = -\frac{1}{m}(\sigma_k - \rho)ug. \tag{3}$$

In this setting, taking into account the relation between the smooth functions u and f, we highlight that u is a constant function if and only if f is a constant function. From now on, we will consider a quasi k-Yamabe gradient soliton (Σ^n, g, u) as a Riemmanian manifold (Σ^n, g) which satisfy (3), for some constants $m, \rho \in \mathbb{R}$, with $m \neq 0$.

In [16], Poddar, Sharma and Subramanian showed that every compact quasi Yamabe gradient soliton (Σ^n, g, u), with $m > 0$, has constant scalar curvature. More recently, in [4], the authors of the present manuscript obtained some triviality and nonexistence results to complete noncompact and stochastically complete quasi Yamabe gradient solitons by using suitable integrability conditions and several maximum principles dealing, in particular, with the notion of convergence to zero at infinity and polynomial and exponential volume growth.

Going a step further, here our purpose is to extend the techniques of [4] in order to establish new triviality and nonexistence results concerning complete noncompact and stochastically complete quasi k-Yamabe gradient solitons (Σ^n, g, u). With this aim, in Section 2 we derive a suitable Bochner type formula and we also quote another two auxiliary ones (see Proposition 1). Afterwards, in Section 3 we apply integrability criteria, Liouville type results and several maximum principles dealing, in particular, with the notion of convergence to zero at infinity and the concept of polynomial and exponential volume growth, to establish our main results (see Theorems 1 until 19).

2 Auxiliary formulas

Our goal in this section is to establish three auxiliary formulas, including a suitable Bochner type formula, related to a quasi k-Yamabe gradient soliton (Σ^n, g, u). These formulas will be essential for proving our main results in the next section and for this reason we will present them as the following proposition.

Proposition 1. *Let (Σ^n, g, u) be a quasi k-Yamabe gradient soliton. Then*

$$\Delta u = -\frac{n}{m}(\sigma_k - \rho)u, \tag{4}$$

$$|\nabla^2 u|^2 = \frac{n}{m^2}(\sigma_k - \rho)^2 u^2, \tag{5}$$

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 - \frac{1}{n-1}\mathrm{Ric}(\nabla u, \nabla u). \tag{6}$$

Proof: Taking the trace of (3), it is not difficult to see that we get (4). On the other hand, considering a (local) orthonormal frame $\{E_1, \cdots, E_n\}$ on Σ^n, from (3), we have

$$|\nabla^2 u|^2 = \sum_i |\nabla^2 u(E_i)|^2 = \sum_i \left|\frac{1}{m}(\sigma_k - \rho)uE_i\right|^2 = \frac{n}{m^2}(\sigma_k - \rho)^2 u^2.$$

Therefore, we also get (5).

In order to obtain (6), let us remember the classical Bochner formula (see for instance [5]):

$$\frac{1}{2}\Delta|\nabla u|^2 = \mathrm{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla\Delta u) + |\nabla^2 u|^2. \tag{7}$$

Moreover, choosing $\{E_1, \cdots, E_n\}$ to be a geodesic frame, we claim that

$$\mathrm{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla\Delta u) = \sum_i g(\nabla_{E_i}\nabla_{E_i}\nabla u, \nabla u). \tag{8}$$

Indeed, we have that

$$\begin{aligned}\mathrm{Ric}(\nabla u, \nabla u) &= \sum_i g(R(\nabla u, E_i)\nabla u, E_i) \\ &= \sum_i g(\nabla_{E_i}\nabla_{\nabla u}\nabla u - \nabla_{\nabla u}\nabla_{E_i}\nabla u + \nabla_{[\nabla u, E_i]}\nabla u, E_i).\end{aligned} \tag{9}$$

But, since $\{E_1, \cdots, E_n\}$ is a geodesic frame, we get

$$\sum_i g(\nabla_{\nabla u}\nabla_{E_i}\nabla u, E_i) = \sum_i \nabla u(g(\nabla_{E_i}\nabla u, E_i)) = \nabla u(\Delta u) = g(\nabla u, \nabla\Delta u). \tag{10}$$

Moreover, we also obtain

$$\begin{aligned}g(\nabla_{E_i}\nabla_{\nabla u}\nabla u + \nabla_{[\nabla u, E_i]}\nabla u, E_i) &= E_i(g(\nabla_{\nabla u}\nabla u, E_i)) - g(\nabla_{\nabla u}\nabla u, \nabla_{E_i}E_i) \\ &\quad - g(\nabla_{E_i}\nabla u, [E_i, \nabla u]) \\ &= E_i(g(\nabla_{E_i}\nabla u, \nabla u)) \\ &\quad - g(\nabla_{E_i}\nabla u, \nabla_{E_i}\nabla u - \nabla_{\nabla u}E_i) \\ &= g(\nabla_{E_i}\nabla_{E_i}\nabla u, \nabla u).\end{aligned} \tag{11}$$

Hence, inserting (10) and (11) into (9), we arrive at (8).

Thus, from (3) and (8), we get

$$\mathrm{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla\Delta u) = -\frac{1}{m}g(\nabla((\sigma_k - \rho)u), \nabla u). \tag{12}$$

But, from (4), we have

$$g(\nabla u, \nabla \Delta u) = -\frac{n}{m} g(\nabla((\sigma_k - \rho)u), \nabla u).$$

So, we conclude that

$$-\frac{1}{m} g(\nabla((\sigma_k - \rho)u), \nabla u) = -\frac{1}{(n-1)} \mathrm{Ric}(\nabla u, \nabla u). \tag{13}$$

Hence, from (7), (12) and (13), we reach our suitable Bochner type formula

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 - \frac{1}{n-1}\mathrm{Ric}(\nabla u, \nabla u).$$

□

3 Triviality and nonexistence results

This section is devoted to establish our triviality and nonexistence results concerning complete noncompact and stochastically complete quasi k-Yamabe gradient solitons. Our approach is based on the use of some integrability conditions, Liouville type theorems and several maximum principles in order to explore the formulas present in Proposition 1 to obtain our main results. Moreover, we point out that we will organize our results into subsections according to the main analytical tool which will be used to prove them.

3.1 Via integrability conditions

Here, we will explore some suitable integrability conditions and Liouville type theorems to get our first triviality results concerning complete noncompact quasi k-Yamabe gradient solitons (Σ^n, g, u). In this setting, to prove the next result, we will assume that the Ricci tensor is nonpositive in the direction of ∇u jointly with a suitable integrability condition on $|\nabla|\nabla u|^2|$.

Theorem 1. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\int_\Sigma \mathrm{Ric}(\nabla u, \nabla u) \leq 0$ and*

$$\int_\Sigma |\nabla|\nabla u|^2| = o(r), \tag{14}$$

then u is a harmonic function, $|\nabla u|$ is a constant function and $\sigma_k = \rho$ on (Σ^n, g).

Proof: From (4), we can verify that

$$\frac{n}{m^2}(\sigma_k-\rho)^2u^2=-\frac{u}{m}(\sigma_k-\rho)\Delta u.$$

Integrating this previous equality on a ball and using Stokes' Theorem, we obtain

$$\begin{aligned}\frac{n}{m^2}\int_{B_r}(\sigma_k-\rho)^2u^2&=-\int_{B_r}\frac{u}{m}(\sigma_k-\rho)\Delta u \qquad (15)\\&=\int_{B_r}\frac{1}{m}g(\nabla((\sigma_k-\rho)u),\nabla u)-\int_{\partial B_r}\frac{u}{m}(\sigma_k-\rho)\nabla_\nu u\\&=\frac{1}{n-1}\int_{B_r}\mathrm{Ric}(\nabla u,\nabla u)+\int_{\partial B_r}\frac{u}{m}(\rho-\sigma_k)\nabla_\nu u\\&\le\int_{\partial B_r}\frac{|u|}{|m|}|\sigma_k-\rho||\nabla u|,\end{aligned}$$

where we have used the hypothesis that $\int_\Sigma \mathrm{Ric}(\nabla u,\nabla u)\le 0$. On the other hand, from (3), we observe that

$$\begin{aligned}g(\nabla|\nabla u|^2,\nabla u)&=2g(\nabla_{\nabla u}\nabla u,\nabla u)=2\nabla^2u(\nabla u,\nabla u)\\&=-\frac{2}{m}(\sigma_k-\rho)ug(\nabla u,\nabla u),\end{aligned}$$

from Cauchy-Schwarz inequality, we get

$$\frac{2|u|}{|m|}|\sigma_k-\rho||\nabla u|\le|\nabla|\nabla u|^2|.$$

Thus, from (15), we have

$$\frac{n}{m^2}\int_{B_r}(\sigma_k-\rho)^2u^2\le\frac{1}{2}\int_{\partial B_r}|\nabla|\nabla u|^2|.$$

Therefore, from Fubini's Theorem and (14), we have

$$\int_{B_r}(\sigma_k-\rho)^2u^2\to 0$$

as $r\to+\infty$. In this picture, we conclude that

$$\int_\Sigma(\sigma_k-\rho)^2u^2=0.$$

Since $u > 0$, we have $\sigma_k = \rho$ on (Σ^n, g). Moreover, from (4), we get that u is a harmonic function and, from (5), we conclude that $|\nabla^2 u| = 0$. In this setting, from Kato's inequality, we also get

$$|\nabla|\nabla u|| \leq |\nabla^2 u| = 0,$$

which implies that $|\nabla u|$ is a constant function. □

Replacing $\int_\Sigma \mathrm{Ric}(\nabla u, \nabla u) \leq 0$ with $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, we obtain our first triviality result.

Theorem 2. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, and*

$$\int_\Sigma |\nabla|\nabla u|^2| = o(r), \tag{16}$$

then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).

Proof: Since we are assuming that $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, from (6), we get

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 - \frac{2}{n-1}\mathrm{Ric}(\nabla u, \nabla u) \geq \Gamma|\nabla u|^2,$$

where $\Gamma = \dfrac{2\alpha}{n-1} \in \mathbb{R}$ is a positive constant. Therefore, integrating the previous inequality on a ball and using Stokes' Theorem, we have

$$\begin{aligned} \Gamma \int_{B_r} |\nabla u|^2 &\leq \int_{B_r} \Delta|\nabla u|^2 \\ &= \int_{\partial B_r} \nabla_\nu |\nabla u|^2 \\ &\leq \int_{\partial B_r} |\nabla|\nabla u|^2|. \end{aligned}$$

From (16) and using Fubini's Theorem, we get

$$\int_{B_r} |\nabla u|^2 \to 0$$

as $r \to +\infty$. Thus, we conclude

$$\int_\Sigma |\nabla u|^2 = 0$$

and, hence, u is a constant function. In particular, from (4), we verify that $\sigma_k = \rho$ on (Σ^n, g). $\square$

We observe that, using Kato's inequality, we have

$$|\nabla|\nabla u|^2| = 2|\nabla u||\nabla|\nabla u|| \leq 2|\nabla u||\nabla^2 u|.$$

In this picture, if $|\nabla u| \in L^\infty(\Sigma^n)$ and $\displaystyle\int_\Sigma |\nabla^2 u| = o(r)$, we can rewrite Theorems 1 and 2 as follows.

Theorem 3. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\int_\Sigma \mathrm{Ric}(\nabla u, \nabla u) \leq 0$, $|\nabla u| \in L^\infty(\Sigma^n)$ and*

$$\int_\Sigma |\nabla^2 u| = o(r),$$

then u is a harmonic function, $|\nabla u|$ is a constant function and $\sigma_k = \rho$ on (Σ^n, g).

Theorem 4. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant, $|\nabla u| \in L^\infty(\Sigma^n)$ and*

$$\int_\Sigma |\nabla^2 u| = o(r),$$

then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).

In [19] Yau, generalizing a previous result due to Gaffney [8], established the following version of Stokes' Theorem on an n-dimensional complete noncompact Riemannian manifold Σ^n: *If $\omega \in \Omega^{n-1}(\Sigma^n)$ is an integrable $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that*

$$B_i \subset B_{i+1}, \quad \Sigma^n = \bigcup_{i\geq 1} B_i \quad \text{and} \quad \lim_{i\to+\infty} \int_{B_i} d\omega = 0.$$

Suppose Σ^n is oriented by the volume element $d\Sigma$, and let $L^q(\Sigma^n)$ be the space of Lebesgue q-integrable functions on Σ^n, that means

$$L^q(\Sigma^n) := \left\{ v : \Sigma^n \to \mathbb{R}; \int_\Sigma |v|^q d\Sigma < +\infty, \quad 1 \leq q < +\infty \right\}.$$

If $\omega = \iota_X d\Sigma$ is the contraction of $d\Sigma$ in the direction of a smooth vector field X on Σ^n, then Caminha obtained the following consequence of Yau's result (see [6, Proposition 2.1]).

Lemma 1. *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold (Σ^n, g), such that $\mathrm{div}_g X$ does not change sign on (Σ^n, g). If $|X| \in L^1(\Sigma^n)$, then $\mathrm{div}_g X = 0$.*

In this picture, we will use Lemma 1 to prove our next result.

Theorem 5. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\sigma_k - \rho$ does not change sign and $|\nabla u| \in L^1(\Sigma^n)$, then u is a harmonic function, $|\nabla u|$ is a constant function and $\sigma_k = \rho$ on (Σ^n, g).*

Proof: From (4), we have

$$\Delta u = -\frac{n}{m}(\sigma_k - \rho)u.$$

So, taking the smooth vector field $X = \nabla u \in \mathfrak{X}(\Sigma^n)$, we have that $|X| = |\nabla u| \in L^1(\Sigma^n)$ and $\mathrm{div}_g X = \Delta u$ does not change sign because we are assuming that $\sigma_k - \rho$ does not change sign. Therefore, applying Lemma 1, we obtain

$$0 = \Delta u = -\frac{n}{m}(\sigma_k - \rho)u,$$

which implies that $\sigma_k = \rho$ on (Σ^n, g). Moreover, from (5), we get $|\nabla^2 u| = 0$ and, from Kato's inequality, we verify

$$|\nabla |\nabla u|| \leq |\nabla^2 u| = 0.$$

Thus, $|\nabla u|$ is a constant function. □

Before we present our next result, let us remember the following lemma due to Yau [20].

Lemma 2. *Every complete noncompact Riemannian manifold with nonnegative Ricci curvature has infinity volume.*

In this picture, using Lemma 2, we can establish the following corollary of Theorem 5.

Corollary 1. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton with nonnegative Ricci curvature. If $\sigma_k - \rho$ does not change sign and $|\nabla u| \in L^1(\Sigma^n)$, then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).*

Proof: From Theorem 5, we have, in particular, that $|\nabla u|$ is a constant function and $\sigma_k = \rho$ on (Σ^n, g). On the other hand, taking into account that (Σ^n, g, u) is a complete noncompact Riemannian manifold with nonnegative

Ricci curvature, from Lemma 2, we conclude that (Σ^n, g, u) has infinity volume. In this setting, since we are assuming that $|\nabla u| \in L^1(\Sigma^n)$ and knowing that $|\nabla u|$ is a constant function, we conclude that $|\nabla u| = 0$ and, therefore, (Σ^n, g, u) is trivial. □

Now, we will use Lemma 1 jointly with the Bochner type formula (6) of Proposition 1 to get our next integrability result.

Theorem 6. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$ and $|\nabla|\nabla u|^2| \in L^1(\Sigma^n)$, then u is a harmonic function, $|\nabla u|$ is a constant function and $\sigma_k = \rho$ on (Σ^n, g).*

Proof: Let us consider the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$. Since we are assuming that $|\nabla|\nabla u|^2| \in L^1(\Sigma^n)$, we have that $|X| \in L^1(\Sigma^n)$. On the other hand, taking into account that $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$, from (6), we also have

$$\mathrm{div}_g X = \Delta|\nabla u|^2 = 2|\nabla^2 u|^2 - \frac{2}{n-1}\mathrm{Ric}(\nabla u, \nabla u) \geq 0.$$

So, applying Lemma 1, we conclude that $\mathrm{div}_g X = 0$. In particular, we get that $|\nabla^2 u|^2 = 0$ and, consequently, from (5), we have

$$\frac{n}{m^2}(\sigma_k - \rho)^2 u^2 = 0.$$

Therefore, since $u > 0$, we obtain that $\sigma_k = \rho$ on (Σ^n, g). Moreover, since $\sigma_k = \rho$, from (4), we get $\Delta u = 0$ and, therefore, u is a harmonic function. Lastly, from Kato's inequality, we have

$$|\nabla|\nabla u|| \leq |\nabla^2 u| = 0,$$

which implies that $|\nabla u|$ is a constant function. □

In order to get our next result, we will consider that $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$. In this setting, in a similar configuration of Theorem 6, we deduce the following theorem.

Theorem 7. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton whose Ricci tensor satisfies $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, and such that $|\nabla|\nabla u|^2| \in L^1(\Sigma^n)$, then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).*

Proof: Reasoning in a similar way of Theorem 6, let us consider the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$. It is easy to see that $|X| \in L^1(\Sigma^n)$ and

$$\mathrm{div}_g X = \Delta|\nabla u|^2 = 2|\nabla^2 u|^2 - \frac{2}{n-1}\mathrm{Ric}(\nabla u, \nabla u) \geq \Gamma|\nabla u|^2,$$

where $\Gamma = \dfrac{2\alpha}{n-1} \in \mathbb{R}$ is a positive constant.

Therefore, from Lemma 1, we have $\mathrm{div}_g X = 0$, which implies that $|\nabla u|^2 = 0$. In this setting, we conclude that u is a constant function and, hence, (Σ^n, g, u) is trivial. Moreover, from (4), we verify that $\sigma_k = \rho$ on (Σ^n, g). $\square$

Remembering that, from Kato's inequality, we get

$$|\nabla|\nabla u|^2| = 2|\nabla u||\nabla|\nabla u|| \leq 2|\nabla u||\nabla^2 u|,$$

assuming that $|\nabla u| \in L^\infty(\Sigma^n)$ and $|\nabla^2 u| \in L^1(\Sigma^n)$, we can rewrite Theorems 6 and 7 respectively as follows.

Theorem 8. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$, $|\nabla u| \in L^\infty(\Sigma^n)$ and $|\nabla^2 u| \in L^1(\Sigma^n)$, then u is a harmonic function, $|\nabla u|$ is a constant function and $\sigma_k = \rho$ on (Σ^n, g).*

Theorem 9. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton whose Ricci tensor satisfies $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, and such that $|\nabla u| \in L^\infty(\Sigma^n)$ and $|\nabla^2 u| \in L^1(\Sigma^n)$, then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).*

At this point, we recall that a smooth function $v \in \mathcal{C}^\infty(\Sigma^n)$ on a Riemannian manifold (Σ^n, g) is said to be a subharmonic function when $\Delta v \geq 0$ on (Σ^n, g). In this setting, we present the next lemma which is a Liouville type result due to Yau [20] (see also [15]).

Lemma 3. *Let v be a nonnegative smooth subharmonic function on a complete noncompact Riemannian manifold (Σ^n, g). If $v \in L^q(\Sigma^n)$, for some $q > 1$, then v is constant.*

Using Lemma 3, we get one more integrability result which is a kind of complementary version of Theorem 6.

Theorem 10. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$ and $|\nabla u|^2 \in L^q(\Sigma^n)$, $1 < q < +\infty$, then u is a harmonic function, $|\nabla u|$ is a constant function and $\sigma_k = \rho$ on (Σ^n, g).*

Proof: Since we are supposing that $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$, from (6), we obtain

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 - \frac{2}{n-1}\mathrm{Ric}(\nabla u, \nabla u) \geq 0.$$

In this picture, we oberseve that $|\nabla u|^2$ is a nonnegative subharmonic function such that $|\nabla u|^2 \in L^q(\Sigma^n)$, $1 < q < +\infty$. So, from Lemma 3, we get $|\nabla u|^2$ is a constant function and, from (5), we also get

$$0 = \Delta|\nabla u|^2 \geq |\nabla^2 u|^2 = \frac{n}{m^2}(\sigma_k - \rho)^2 u^2 \geq 0,$$

which implies that $\sigma_k = \rho$ on (Σ^n, g) and, from (4), we conclude that u is a harmonic function. □

One more time, we will replace the hypothesis $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$ with $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, in order to obtain a new version of Theorem 10, which is another triviality result.

Theorem 11. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, and $|\nabla u|^2 \in L^q(\Sigma^n)$, $1 < q < +\infty$, then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).*

Proof: Taking into account that $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, from (6), we have

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 - \frac{2}{n-1}\mathrm{Ric}(\nabla u, \nabla u) \geq \Gamma|\nabla u|^2,$$

where $\Gamma = \dfrac{2\alpha}{n-1} \in \mathbb{R}$ is a positive constant. In this setting, $|\nabla u|^2$ is a nonnegative subharmonic function such that $|\nabla u|^2 \in L^q(\Sigma^n)$, $1 < q < +\infty$. So, applying Lemma 3, we get $|\nabla u|^2$ is a constant function and, therefore,

$$0 = \Delta|\nabla u|^2 \geq \Gamma|\nabla u|^2 \geq 0,$$

which implies that $|\nabla u|^2 = 0$ and, hence, u is a constant function. Finally, from (4), we conclude that $\sigma_k = \rho$ on (Σ^n, g). □

We close this subsection by showing another triviality result involving integrability.

Theorem 12. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton such that $m(\sigma_k - \rho) \leq 0$ and $u \in L^q(\Sigma^n)$, $1 < q < +\infty$, then (Σ^n, g, u) is trivial and $R = \rho$ on Σ^n.*

Proof: Since $m \neq 0$ and $m(\sigma_k - \rho) \leq 0$, we can verify that $\dfrac{1}{m}(\sigma_k - \rho) \leq 0$. Therefore, from (4), we have

$$\Delta u = -\frac{n}{m}(\sigma_k - \rho)u \geq 0.$$

Thus, since we are assuming $u \in L^q(\Sigma^n)$, with $1 < q < +\infty$, Lemma 3 guarantees that u is constant and, hence, $\sigma_k = \rho$ on (Σ^n, g). □

3.2 Via volume growth

Before we present our next results, we will need some terminology on the concept of polynomial and exponential volume growth. In this setting, let us consider a (connected oriented) complete Riemannian manifold (Σ^n, g) and denote by $B(p,t)$ a geodesic ball centered at p and with radius t. Given a continuous function $\sigma : (0, +\infty) \to (0, +\infty)$ we say that Σ^n has volume growth like $\sigma(t)$ if there exists $p \in \Sigma^n$ such that

$$\mathrm{vol}(B(p,t)) = \mathcal{O}(\sigma(t))$$

as $t \to \infty$, where vol stands the volume, that is,

$$\mathrm{vol}(B(p,t)) = \int_{B(p,t)} d\Sigma.$$

With the above concept in mind, we can present the following lemma which corresponds to a particular case of a more general maximum principle due to Alías, Caminha and Nascimento (see [2, Theorem 2.1]).

Lemma 4. *Let (Σ^n, g) be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma^n)$ be a bounded smooth vector field on Σ^n, with $|X| \leq c$ for some positive constant $c \in \mathbb{R}$. Let $v \in \mathcal{C}^\infty(\Sigma)$ be a smooth function such that $g(\nabla v, X) \geq 0$ and $\mathrm{div}_g X \geq av$ on Σ^n, for some positive constant $a \in \mathbb{R}$.*

(i) If (Σ^n, g) has polynomial volume growth, then $v \leq 0$ on Σ^n.

(ii) If (Σ^n, g) has exponential volume growth, say like $e^{\beta t}$, then $v \leq \frac{c\beta}{a}$ on Σ^n.

Using this previous lemma, we get the following nonexistence result concerning complete noncompact quasi k-Yamabe gradient solitons with polynomial volume growth.

Theorem 13. *There is no complete noncompact quasi k-Yamabe gradient soliton (Σ^n, g, u) with polynomial volume growth such that $|\nabla u| \in L^\infty(\Sigma^n)$ and $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$.*

Proof: Suppose by contradiction the existence of such a complete noncompact quasi k-Yamabe gradient soliton (Σ^n, g, u). Taking the smooth vector field $X = \nabla u \in \mathfrak{X}(\Sigma^n)$, we have that $|X| = |\nabla u|$ is bounded and

$$g(\nabla u, X) = |\nabla u|^2 \geq 0.$$

Moreover, taking into account that $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$, we get

$$\mathrm{div}_g X = \Delta u = -\frac{n}{m}(\sigma_k - \rho)u \geq \frac{n\alpha}{m^2}u.$$

Hence, since (Σ^n, g) is complete noncompact and has polynomial volume growth, we can apply the item (i) of Lemma 4 to conclude that $u \leq 0$, which is absurd. $\square$

Keeping in mind the concept of polynomial volume growth, we present the first triviality result of this subsection as follows.

Theorem 14. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton whose Ricci tensor satisfies $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$. If (Σ^n, g) has polynomial volume growth and $|\nabla u|$, $|\nabla^2 u| \in L^\infty(\Sigma^n)$, then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).*

Proof: Initially, let us take the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$ and the smooth function $v = |\nabla u|^2 \in \mathcal{C}^\infty(\Sigma^n)$, by using Kato's inequality, we get

$$|X| = |\nabla|\nabla u|^2| = 2|\nabla u||\nabla|\nabla u|| \leq 2|\nabla u||\nabla^2 u| \in L^\infty(\Sigma^n),$$

because we are assuming that $|\nabla u|$, $|\nabla^2 u| \in L^\infty(\Sigma^n)$.

On the other hand, we also get

$$g(X, \nabla v) = |\nabla|\nabla u|^2|^2 \geq 0.$$

Furthermore, since $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, from (6), we verify that

$$\mathrm{div}_g X = \Delta|\nabla u|^2 \geq \Gamma|\nabla u|^2,$$

where $\Gamma = \dfrac{2\alpha}{n-1} \in \mathbb{R}$ is a positive constant.

In this picture, taking into account that (Σ^n, g) is complete noncompact and has polynomial volume growth, we can apply the item (i) of Lemma 4 to conclude that $|\nabla u|^2 = 0$ and, consequently, u must be a constant function. Thus, from (4), we get $\sigma_k = \rho$ on (Σ^n, g). $\square$

Next, dealing with the notion of exponential volume growth, we will get an estimate for $|u|_\infty$ through $|\nabla u|_\infty$, where u is the potential function of the quasi k-Yamabe gradient soliton (Σ^n, g, u).

Theorem 15. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton with exponential volume growth, say like $e^{\beta t}$. If $|\nabla u| \in L^\infty(\Sigma^n)$ and $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$, then*

$$|u|_\infty \leq \frac{m^2\beta}{n\alpha}|\nabla u|_\infty.$$

Proof: Taking the smooth vector field $X = \nabla u \in \mathfrak{X}(\Sigma^n)$ and following the same steps of the proof of Theorem 13, we get that X is a bounded smooth vector field, $g(\nabla u, X) \geq 0$ and $\operatorname{div}_g X \geq \dfrac{n\alpha}{m^2}u$, for some positive constant $\alpha \in \mathbb{R}$.

Hence, applying the item (ii) of Lemma 4, we obtain

$$|u| \leq \frac{m^2\beta}{n\alpha}|\nabla u|_\infty.$$

Therefore, we conclude that

$$|u|_\infty \leq \frac{m^2\beta}{n\alpha}|\nabla u|_\infty.$$

□

Inspired by Theorem 15, we present another kind of estimate result, where we get an estimate for $|\nabla u|_\infty$ through $|\nabla^2 u|_\infty$.

Theorem 16. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton with exponential volume growth, say like $e^{\beta t}$. If $|\nabla u|, |\nabla^2 u| \in L^\infty(\Sigma^n)$ and $\operatorname{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, then*

$$|\nabla u|_\infty \leq \frac{(n-1)\beta}{2\alpha}|\nabla^2 u|_\infty.$$

Proof: Taking the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$ and following the same steps of the proof of Theorem 14, we get that X is a bounded smooth vector field, $g(\nabla|\nabla u|^2, X) \geq 0$ and $\operatorname{div}_g X \geq \Gamma|\nabla u|^2$, where $\Gamma = \dfrac{2\alpha}{n-1} \in \mathbb{R}$ is a positive constant.

So, applying the item (ii) of Lemma 4, we obtain

$$|\nabla u|^2 \leq \frac{(n-1)\beta}{2\alpha}|\nabla u|_\infty|\nabla^2 u|_\infty.$$

Thus, we conclude that

$$|\nabla u|_\infty \leq \frac{(n-1)\beta}{2\alpha}|\nabla^2 u|_\infty.$$

□

3.3 Via stochastic completeness

We recall that a Riemannian manifold (Σ^n, g) is said to be *stochastically complete* if, for some (and, hence, for any) $(x, \tau) \in \Sigma^n \times (0, +\infty)$, the heat kernel $p(x, y, \tau)$ of the Laplace-Beltrami operator Δ satisfies the conservation property

$$\int_\Sigma p(x, y, \tau) d\mu(y) = 1. \tag{17}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have infinite life time. Furthermore, for the Brownian motion on a manifold, the conservation property (17) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [7, 9, 10, 17]).

On the other hand, following the terminology introduced by Pigola, Rigoli and Setti in [14], the Omori-Yau's maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold (Σ^n, g) if, for any smooth function $u \in \mathcal{C}^2(\Sigma^n)$ with $\sup_\Sigma u < +\infty$, there exists a sequence of points $(p_k) \subset \Sigma^n$ satisfying

$$\lim_k u(p_k) = \sup_\Sigma u, \quad \lim_k |\nabla u(p_k)| = 0 \quad \text{and} \quad \limsup_k \Delta u(p_k) \leq 0.$$

In this point of view, the classical result given by Omori and Yau in [12, 20] states that Omori-Yau's maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below.

But, as it was also observed by Pigola, Rigoli and Setti in [14], the validity of Omori-Yau's maximum principle on Σ^n does not depend on curvature bounds as would be expected. For instance, the Omori-Yau's maximum principle holds on every Riemannian manifolds which is properly immersed into a Riemannian space form with controlled mean curvature (see [14, Example 1.14]). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, following again the terminology introduced in [14], the weak Omori-Yau's maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold (Σ^n, g) if, for any smooth function $u \in \mathcal{C}^2(\Sigma^n)$ with $\sup_\Sigma u < +\infty$, there exists a sequence of points $(p_k) \subset \Sigma^n$ with the properties

$$\lim_k u(p_k) = \sup_\Sigma u \quad \text{and} \quad \limsup_k \Delta u(p_k) \leq 0.$$

In this setting, Pigola, Rigoli and Setti [13, 14] proved the following equivalence:

Lemma 5. *A Riemannian manifold* (Σ^n, g) *is stochastically complete if and only if the weak Omori-Yau's maximum principle holds on* (Σ^n, g).

Proceeding with our study, related to the notion of stochastic completeness, we present our next triviality result.

Theorem 17. *Let* (Σ^n, g, u) *be a stochastically complete quasi* k*-Yamabe gradient soliton whose Ricci tensor satisfies* $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$*, for some positive constant* $\alpha \in \mathbb{R}$*. If* $|\nabla u| \in L^\infty(\Sigma)$*, then* (Σ^n, g, u) *is trivial and, in particular,* $\sigma_k = \rho$ *on* (Σ^n, g).

Proof: From (6) and taking into account that $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$, we get

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 - \frac{2}{n-1}\mathrm{Ric}(\nabla u, \nabla u) \geq \Gamma|\nabla u|^2,$$

where $\Gamma = \dfrac{2\alpha}{n-1} \in \mathbb{R}$ is a positive constant. Moreover, since we are supposing that $|\nabla u| \in L^\infty(\Sigma^n)$, we can apply Lemma 5 to get a sequence $(q_k) \subset \Sigma^n$ such that

$$\lim_k |\nabla u|^2(q_k) = \sup_\Sigma |\nabla u|^2 \quad \text{and} \quad \limsup_k \Delta|\nabla u|^2(q_k) \leq 0.$$

Thus, we have

$$0 \geq \limsup_k \Delta|\nabla u|^2(q_k) \geq \Gamma \limsup_k |\nabla u|^2(q_k) = \Gamma \sup_\Sigma |\nabla u|^2 \geq 0,$$

which implies that $|\nabla u|^2 = 0$ and, therefore, u is a constant function. So, from (4), we get $\sigma_k = \rho$ on (Σ^n, g). □

We remind that a Riemannian manifold (Σ^n, g) (non necessarily complete) is called parabolic when the only subharmonic functions $v \in \mathcal{C}^\infty(\Sigma^n)$ which are bounded from above are the constant ones. Taking into account [10, Corollary 6.4], we have that every parabolic Riemannian manifold is stochastically complete. In this setting, we obtain the following consequence of Theorem 17.

Corollary 2. *Let* (Σ^n, g, u) *be a parabolic quasi* k*-Yamabe gradient soliton whose Ricci tensor satisfies* $\mathrm{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$*, for some positive constant* $\alpha \in \mathbb{R}$*. If* $|\nabla u| \in L^\infty(\Sigma)$*, then* (Σ^n, g, u) *is trivial and, in particular,* $\sigma_k = \rho$ *on* (Σ^n, g).

Considering [3] [Theorem 2.13], we can establish our second corollary of Theorem 17.

Corollary 3. *Let (Σ^n, g, u) be a complete quasi k-Yamabe gradient soliton whose Ricci tensor satisfies $\operatorname{Ric}(\nabla u, \nabla u) \leq -\alpha|\nabla u|^2$, for some positive constant $\alpha \in \mathbb{R}$ and $\operatorname{Ric} \geq -G(r)$, where r denotes the Riemannian distance function from a fixed origin in Σ^n and the function $G \in \mathcal{C}^1([0, +\infty))$ satisfies*

$$G(0) > 0, \quad G'(0) \geq 0 \quad \textit{and} \quad G^{-\frac{1}{2}} \notin L^1([0, +\infty)).$$

If $|\nabla u| \in L^\infty(\Sigma)$, then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).

Next, we will present a nonexistence result by assuming, in particular, that a stochastically complete quasi k-Yamabe gradient soliton (Σ^n, g, u) is such that $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$.

Theorem 18. *There is no stochastically complete quasi k-Yamabe gradient soliton (Σ^n, g, u) such that $u \in L^\infty(\Sigma^n)$ and $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$.*

Proof: Assume, by contradiction, that there exists such a quasi k-Yamabe gradient soliton. Since we are supposing that $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$, from (4), we get

$$\Delta u = -\frac{n}{m}(\sigma_k - \rho)u \geq \frac{n\alpha}{m^2}u.$$

Moreover, taking into account that $u \in L^\infty(\Sigma^n)$, Lemma 5 gives a sequence $(q_k) \subset \Sigma^n$ such that

$$\lim_k u(q_k) = \sup_\Sigma u \quad \text{and} \quad \limsup_k \Delta u(q_k) \leq 0.$$

Thus, we get

$$0 \geq \limsup_k \Delta u(q_k) \geq \frac{n\alpha}{m^2} \limsup_k u(q_k) = \frac{n\alpha}{m^2} \sup_\Sigma u > 0,$$

which is an absurd. □

Reasoning as in Corollaries 2 and 3, we obtain the following corollaries of Theorem 18.

Corollary 4. *There is no parabolic quasi k-Yamabe gradient soliton (Σ^n, g, u) such that $u \in L^\infty(\Sigma^n)$ and $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$.*

Corollary 5. *There is no complete quasi k-Yamabe gradient soliton (Σ^n, g, u) such that $u \in L^\infty(\Sigma^n)$, $m(\sigma_k - \rho) \leq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$, and $\operatorname{Ric} \geq -G(r)$, where r denotes the Riemannian distance function from a fixed origin in Σ^n and the function $G \in \mathcal{C}^1([0, +\infty))$ satisfies $G(0) > 0$, $G'(0) \geq 0$ and $G^{-\frac{1}{2}} \notin L^1([0, +\infty))$.*

3.4 Via convergence at infinity

In order to establish our last result, we need the concept of convergence to zero at infinity. Given a (connected) complete noncompact Riemannian manifold (Σ^n, g) and denoting by $d(\cdot, o) : \Sigma^n \to [0, +\infty)$ the Riemannian distance of Σ^n measured from a fixed point $o \in \Sigma^n$, a function $h \in \mathcal{C}^\infty(\Sigma^n)$ converges to zero at infinity when

$$\lim_{d(x,o)\to\infty} h(x) = 0.$$

The following lemma is due to Alías, Caminha and do Nascimento [1].

Lemma 6. *Let (Σ^n, g) be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma^n)$ be a smooth vector field on Σ^n. Assume that there exists a nonnegative, non-identically vanishing function $v \in \mathcal{C}^\infty(M)$ which converges to zero at infinity and such that $g(\nabla v, X) \geq 0$. If $\mathrm{div}_g X \geq 0$ on Σ^n, then $g(\nabla v, X) \equiv 0$ on Σ^n.*

To close this manuscript, we will use the notion of convergence to zero at infinity jointly with Lemma 6 to prove our last triviality result.

Theorem 19. *Let (Σ^n, g, u) be a complete noncompact quasi k-Yamabe gradient soliton whose Ricci tensor satisfies $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$. If $|\nabla u|$ converges to zero at infinity, then (Σ^n, g, u) is trivial and, in particular, $\sigma_k = \rho$ on (Σ^n, g).*

Proof: Once again, let us consider the smooth function $v = |\nabla u|^2 \in \mathcal{C}^\infty(\Sigma^n)$ and the smooth vector field $X = \nabla|\nabla u|^2 \in \mathfrak{X}(\Sigma^n)$. Since we are assuming that $\mathrm{Ric}(\nabla u, \nabla u) \leq 0$, from (6), we have

$$\mathrm{div}_g X = \Delta|\nabla u|^2 = 2|\nabla^2 u|^2 - \frac{2}{n-1}\mathrm{Ric}(\nabla u, \nabla u) \geq 0$$

and

$$g(X, \nabla v) = |\nabla|\nabla u|^2|^2 \geq 0.$$

Taking into account that $|\nabla u|^2$ converges to zero at infinity, from Lemma 6, we conclude that $|\nabla u|^2$ is a constant function. However, $|\nabla u|^2$ is a constant function that converges to zero at infinity and, therefore, $|\nabla u|^2 = 0$, which implies that u is a constant function. Moreover, from (4), we conclude that $\sigma_k = \rho$ on (Σ^n, g). $\square$

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