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ON FAT Ψ-SPACE CONSTRUCTIONS*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

A fat Ψ-space is a Tychonoff space obtained by replacing the isolated points of a Ψ-space with more structured topological components, commonly referred to as "building blocks". This paper surveys the principal known constructions of such spaces and introduces a new example developed within this framework.

Keywords: covering properties, Ψ-space, fat Ψ-space.
MSC: 54D20.

1 Introduction

Let ω be a countable discrete space, and let $\mathcal{A}$ be a maximal, with respect to inclusion, almost disjoint family of infinite subsets of ω. It is called *Isbell-Mrówka space* or Ψ*-space* ([25], see also [19]) the set $\Psi = \omega \cup \mathcal{A}$ topologized as follows: the points of ω are isolated and a basic neighborhood of the point $a \in \mathcal{A}$ takes the form $O_F(a) = \{a\} \cup (a \setminus F)$, where F is an arbitrary finite set (recall that a family of sets is almost disjoint if the intersection of every two distinct elements of the family is at most finite).

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A fat Ψ-space is constructed by replacing the countable discrete space ω with a space Y, formed as the union of countably or uncountably many clopen "building blocks" non trivial subsets. A closed discrete set Z is then added, such that certain sequences of subsets from the building blocks converge, in a specified way, to points in Z. This process defines a fat Ψ-space construction, and the resulting space $Y \cup Z$ is called a fat Ψ-space. Such spaces are usually, though not always, zero-dimensional and pseudocompact.

In this paper, we survey (rather by construction than by results) fat Ψ-space constructions used in the literature (i.e., which building blocks were used) and introduce a machine. The structure of this machine is a particular kind of a fat Ψ-space construction. Here we survey fat Ψ-space constructions found in the literature, focusing more on their construction, specifically the types of building blocks employed, than on the results they yield. We then introduce the construction of a machine, which represents a particular instance of a fat Ψ-space construction. All spaces are assumed to be Tychonoff unless otherwise stated.

2 Notations and basic definitions

In general, we follow the notation of [19]. The symbols $l(X)$, $d(X)$, and $nw(X)$ denote, respectively, the Lindelöf degree, the density, and the network weight of a topological space X. Recall that the network weight is the minimum cardinality of a network in the space (for further studies on these cardinal functions, see, for instance, [7,20]). A space is *countably compact* provided that every countable open cover has a finite subcover. For T_1 spaces, countable compactness is equivalent to the condition stating that every infinite set has a limit point. A space is *Lindelöf* provided that every open cover has a countable subcover. A space has *countable extent* provided that every closed discrete subspace is at most countable. A space X has the *Discrete Finite Chain Condition*, briefly *DFCC*, (*Discrete Countable Chain Condition*, briefly *DCCC*) provided that every discrete family of nonempty open sets is finite (respectively, at most countable). For regular spaces, DCCC property is equivalent to state "locally finite" instead of "discrete" (a synonim of this property is *pseudo-Lindelöf*). The countability of the extent and DCCC are well-known consequences of the Lindelöf property. A space is *pseudocompact* provided that every continuous real-valued function defined on X is bounded. Every countably compact space is DFCC and every DFCC space is pseudocompact. It is well-known that every normal pseudocompact space is countably compact [19]. The analog of this result for the Lindelöf-type properties says that every collectionwise normal DCCC spaces have countable extent [22]. A space is metaLindelöf if every open cover has a point-countable open refinement. A space X is *countably pracompact* if it has a dense subspace Y such that Y is countably compact in X (see [1]; a subspace $Y \subseteq X$ is *countably compact in* X if every infinite subset of Y has a limit point in X. Clearly, every countably compact is countably pracompact, and it is easy to see that every countably pracompact is DFCC (hence pseudocompact).

Let $\mathcal{U}$ be an open cover of a space X. For a set $A \subset X$, we write $st(A,\mathcal{U}) = st^1(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $st^{n+1}(A,\mathcal{U}) = st(st^n(A,\mathcal{U}),\mathcal{U})$, $n \in \mathbb{N}$. A spaces is *n-star-compact* (*n-star-Lindelöf*), $n \in \mathbb{N}$, provided for every open cover $\mathcal{U}$ there is a finite (respectively, countable) subset $F \subset X$ such that $St^n(F,\mathcal{U}) = X$. For $n = 1$ we say just *star-compact* (respectively, *star-Lindelöf*). A space is Lindelöf if and only if it is star-Lindelöf and meta-Lindelöf [6]. A space X is *$n\frac{1}{2}$-star-compact* (*$n\frac{1}{2}$-star-Lindelöf*) provided for every open cover $\mathcal{U}$ there is a finite (respectively, countable) subfamily $\mathcal{A} \subset \mathcal{U}$ such that $st^n(\bigcup\mathcal{A},\mathcal{U}) = X$ (see [18] where different terminology is used and [23]). Of course, every n-star-compact (n-star-Lindelöf) space is $n\frac{1}{2}$-star-compact ($n\frac{1}{2}$-star-Lindelöf), for every n. A space X is 1-*cl-star-compact* provided that for every open cover $\mathcal{U}$ there is a finite subset $F \subset X$ such that $st(F,\mathcal{U})$ is dense in X. Every star-compact space is 1-cl-star-compact, every 1-cl-star-compact is 2-star-compact and every 2-star-compact space is DFCC. Of course, countably compact $\Rightarrow$ star-compact $\Rightarrow$ $1\frac{1}{2}$-star-compact $\Rightarrow$ 2-star-compact $\Rightarrow$ $2\frac{1}{2}$-star-compact. In the class of regular spaces $2\frac{1}{2}$-star-compactness is equivalent to being DFCC, and also in the class of Tychonoff spaces these two previous properties are equivalent to pseudocompactness. The star covering properties have also been investigated in [9–11, 13].

A family $\mathcal{U}$ of nonempty sets is said to be *centered* (*n-linked*, where $n \geq 2$ is an integer) if any finite (respectively, n-element) subfamily of it has nonempty intersection; 2-linked is called just *linked*; $\mathcal{U}$ is a *CCC-family* if it does not contain uncountable cellular subfamiliesù; $\mathcal{U}$ is *σ-centered* (*σ-n-linked*) if it can be represented as a countable union of centered (respectively, n-linked) subfamilies.

A space is weakly separable [4] provided that it has a σ-centered π-base. A space X is centered-Lindelöf (linked-Lindelöf) if every open cover contains a σ-centered subcover (resp., σ-linked subcover) [6, 14, 23]. A space is CCC-Lindelöf if every open cover contains a CCC subcover.

Of course, Lindelöf $\Rightarrow$ star-Lindelöf $\Rightarrow$ centered-Lindelöf $\Rightarrow$ linked-Lindelöf $\Rightarrow$ $1\frac{1}{2}$-star-Lindelöf $\Rightarrow$ 2-star-Lindelöf $\Rightarrow$ $2\frac{1}{2}$-star-Lindelöf. In the class of regular spaces $2\frac{1}{2}$-star-Lindelöfness is equivalent to being DCCC. Also separable $\Rightarrow$ weakly separable $\Rightarrow$ centered-Lindelöf.

3 Some fat Ψ-space constructions

A Ψ-space $\Psi = \omega \cup \mathcal{A}$ is a Hausdorff locally compact zero-dimensional (hence Tychonoff) space. Since ω is countably compact in Ψ, Ψ is countably pracompact. Then, by [23, Theorem 15], Ψ is 1-cl-star-compact and hence pseudocompact. Note that ω is an open discrete subspace of Ψ, and $\mathcal{A}$ is a closed discrete subspace of Ψ. Hence, we do not consider the trivial case $|\mathcal{A}| < \omega$. Then it is easy to see that $\omega_1 \leq |\mathcal{A}| \leq \mathfrak{c}$ and it is easy to construct a maximal almost disjoint family of cardinality $\mathfrak{c}$ (for more information on possible values of $|\mathcal{A}|$ see [17]). Since $\mathcal{A}$ is infinite closed and discrete in Ψ, Ψ is not countably compact; moreover, it is not even $1\frac{1}{2}$-star-compact by [23, Theorem 28]. Further Ψ is a separable, hence star-Lindelöf, space. Finally, Ψ is an example of a space having countable network weight and arbitrarily large Lindelöf number (see [2, 8, 12] for spaces having countable network weight).

In the following, we survey (see also [23]) some type of fat Ψ-space constructions used in the literature. In particular, we will describe which building blocks were used and for which purpose.

In [3] Berner used $\mathfrak{c}^+$ many building blocks, each of them homeomorphic to the Cantor set D^ω to construct the first example of a pseudocompact space which is not countably pracompact. In [24] Matveev and Uspenskii used the same building block to construct, for every cardinal τ, a pseudocompact space of cardinality $> \tau$ having a G_δ-diagonal and proved that, assuming CH, this space can be made 2-star-compact.

In [26], Scott used countably many building blocks each of them is the countable (Tychonoff product) power of the lexicographical square of the Cantor set, to construct (assuming CH) a pseudocompact metaLindelöf space which is not compact. In [18, Example 2.3.2], also assuming CH, a modification of Scott's construction is used to construct a pseudocompact metaLindelöf first countable Tychonoff space which is not 2-star-compact.

Since only countably many building blocks are used in Scott's construction, the example from [18] has a dense σ-compact subspace. In [15] a modification of the cited Example from [18] is used to obtain, under the assumption $2^{\omega_1} = 2^\omega$, a Tychonoff space with property $(*)$, i.e. such that every locally countable family of nonempty open sets is countable, which is not 2-star-Lindelöf. The difference from [18] is that now the authors take ω_1

many "building blocks" instead of countably many, and use the convergence on the type of the one point Lindelöfication of the discrete space of cardinality ω_1 instead of the usual convergence sequences. The cost of this is that this space is only DCCC, but not pseudocompact. Another difference from the example from [18] is that now the authors use the assumption $2^{\omega_1} = 2^{\omega}$ instead of CH.

In [6, Example 2.4] the author constructs an example of a Hausdorff, weakly separable (hence centered-Lindelöf) space which is not star-Lindelöf; the building block for this construction is the "$\mathbf{c}$-discretization" of the original topology of Cantor set D^{ω}, i.e. D^{ω} with the new topology the basic open sets of which are open sets of the usual topology with arbitrary $< \mathbf{c}$ points removed. Note that the space of the previous example has good properties: it is a feebly compact (i.e., every locally finite family of non-empty open sets is finite; see, for example [27]), a ccc space with a countable pseudobase (see [21]) and under CH, it is meta-Lindelöf.

In [5] a Tychonoff, centered-Lindelöf space X which is not star-Lindelöf is constructed using countably many building blocks homeomorphic to ω_1. Note that $st\text{-}l(X) = d(X) = \omega_1$ and that the space X is zero-dimensional and scattered (recall that $st\text{-}l(X) = \min\{\kappa : \forall$ open cover $\mathcal{U}$ of X, $\exists F \subset X$ such that $|F| \leq \kappa$ and $St(F,\mathcal{U}) = X\}$; moreover a space Y is *scattered* provided every closed subset $A \subset Y$ contains a point which is isolated in A).

In [23] the author present an example of a zero dimensional 2-star-compact space which is not absolutely 2-star-compact (a space X is *absolutely* 2-*star-compact* if for every open cover $\mathcal{U}$ and every dense subset $Y \subset X$, there exists a finite subset $A \subset Y$ such that $st^2(A,\mathcal{U}) = X$ [23]). In fact, it is a modification of [18, Example 2.3.2] (of, under CH, a first countable pseudocompact non 2-star-compact space). Now the author use nearly the same enumeration technique (and therefore CH can not be omitted) but another building block for the fat Ψ-space construction; in particular, the author uses $\beta\omega \setminus \omega$ as a building block.

In [16], the authors construct a machine producing a non star-Lindelöf space whenever a non separable space is put in. In particular, with the aid of this machine, they obtain an example of locally compact, ccc, non-star-Lindelöf space and a consistent example of first countable, locally compact, ccc, non-star-Lindelöf space. The construction of the machine is a particular kind of fat Ψ-space.

A machine producing a non star-Lindelöf space $\tilde{X}$ whenever a non separable space X is given as input, is constructed in [16]. The machine preserves many properties of X: if X is a first countable space, or a Baire space, or a subparacompact space, or a Moore space, then so is $\tilde{X}$. This machine provides an example of locally compact zero-dimensional ccc space which is not star-Lindelöf.

4 A particular kind of fat Ψ-space construction

Now we introduce a special construction that is a particular kind of a fat Ψ-space construction. Also, we present some results and pose some open questions.

The machine. Let X be a zero-dimensional space and $\mathcal{B}$ a family of nonempty clopen subsets of X such that $|\mathcal{B}| \leq \mathbf{c}$. Fix an almost disjoint family $\mathcal{A}$ of infinite subsets of ω such that $|\mathcal{A}| = |\mathcal{B}|$; let f be a bijection of $\mathcal{B}$ onto $\mathcal{A}$. Put $\mathcal{M} = \mathcal{M}(X, \mathcal{B}, \mathcal{A}) = (X \times \omega) \cup \mathcal{R}$ where $\mathcal{R} = \{r_B : B \in \mathcal{B}\}$ and the points r_B are not in $X \times \omega$ and distinct for distinct $B \in \mathcal{B}$. We topologize $\mathcal{M}$ as follows: $X \times \omega$ is open in $\mathcal{M}$; a basic neighborhood of the point r_B takes the form $O_F(r_B) = \{r_B\} \cup (B \times (f(B) \setminus F))$, where F is a finite subset of ω. $\mathcal{M}$ is the output of the machine. Usually, the choice of $\mathcal{A}$ will not be crucial, so we write $\mathcal{M}(X, \mathcal{B})$ or even just $\mathcal{M}$ instead of $\mathcal{M}(X, \mathcal{B}, \mathcal{A})$.

We have:

- $X \times \omega$ is dense in $\mathcal{M}$;
- the sets $O_F(r_B)$ are clopen in $\mathcal{M}$. Indeed, they are open by construction. Assume, by contradiction, there exist $B \in \mathcal{B}$ and $x \in \overline{O_F(r_B)} \setminus O_F(r_B)$ with $O_F(r_B) = \{r_B\} \cup (B \times (f(B) \setminus F))$. Since B is clopen in X, we have that $B \times \{n\}$ is clopen in $\tilde{X}$; then $x \notin X \times \{n\}$ for every $n \in \omega$. Also, $x \neq r_B$ for any $B \in \mathcal{B}$. Indeed, if there exists $B' \in \mathcal{B}$ such that $x = r_{B'}$, since $\mathcal{A}$ is an almost disjoint family, $F' = f(B) \cap f(B')$ is a finite set the neighbourhood $O_{F'}(r_{B'}) = \{r_{B'}\} \cup (B' \times (f(B') \setminus F'))$ is a neighbourhood of x disjoint from $O_F(r_B)$;
- for every $B, B' \in \mathcal{B}$ there exist finite subsets F, F' of ω, we have that $O_F(r_B) \cap O_{F'}(r_{B'}) = \emptyset$ (it follows by the previous item);
- $\mathcal{M}$ is zero-dimensional, hence Tychonoff.

Notice the following obvious property of the machine.

Proposition 1. If $\mathcal{B}_0 \subset \mathcal{B}$ and for every $B \in \mathcal{B}_0$, F_B is chosen arbitrarily, then $\bigcap\{O_{F_B}(r_B) : B \in \mathcal{B}_0\} \neq \emptyset \Rightarrow \bigcap\{O_\emptyset(r_B) : B \in \mathcal{B}_0\} \neq \emptyset \Rightarrow \bigcap \mathcal{B}_0 \neq \emptyset$.

Question 1. What properties are preserved by the machine?

The machine preserves first countability, the Fréchet property, and sequentiality as shown by the following two propositions.

Proposition 2. Let X be a zero-dimensional space and let $\mathcal{B}$ be as defined in the construction of the machine. If X is first countable, then $\mathcal{M}(X, \mathcal{B})$ is first countable.

Proof. Since $X \times \omega$ is open and first countable in $\mathcal{M}(X, \mathcal{B})$, we have that $\chi(a, \mathcal{M}(X, \mathcal{B})) = \aleph_0$, for every $a \in X \times \omega$. Additionally, $\chi(r_B, \mathcal{M}(X, \mathcal{B})) = \aleph_0$, for every $r_B \in \mathcal{R}$, as a consequence of the construction of the machine. □

Proposition 3. Let X be a zero-dimensional space and let $\mathcal{B}$ be as defined in the construction of the machine. If X has the Fréchet property (resp., is sequential), then $\mathcal{M}(X, \mathcal{B})$ has the Fréchet property (resp., is sequential).

Proof. The structure of the machine guarantees that both sequentiality and the Fréchet property are preserved. □

Note that the machine does not preserve sequential compactness, as demonstrated by the following example. The authors are unsure whether it is possible to modify the construction of the machine to also preserve sequential compactness.

Example 1. Sequential compactness is not preserved by the machine.

Suppose X is an infinite sequentially compact space. Let $S = \{x_n : n \in \omega\}$ be a convergent sequence in X. Fix a clopen subset $B \in \mathcal{B}$ containing all but finitely many points of S. Set $f(B) \setminus F = \{y_n : n \in \omega\}$, where F is a finite subset of ω. Consider the sequence $\mathcal{S} = \{(x_n, y_n) : n \in \omega\} \subseteq X \times \omega$. In this case, $\mathcal{M}$ is not sequentially compact because the sequence $\mathcal{S}$ does not converge in $\mathcal{M}$.

In [14], a variety of examples of centered-Lindelöf spaces which are not star-Lindelöf were constructed. In the same paper the authors prove [14, Proposition 3] that every Hausdorff locally compact centered-Lindelöf space is star-Lindelöf and show that a Hausdorff locally compact $1\frac{1}{2}$-star-Lindelöf need not be star-Lindelöf (even CCC-Lindelöf).

Then, they posed the following question:

Question 2. [14, Question 1] Is every Hausdorff locally compact linked-Lindelöf space star-Lindelöf?

Question 3. Can the machine be used to give a (partial) answer to the previous question?

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