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APPLICATIONS OF HEAT CONDUCTION WITH INTERNAL VARIABLES IN METALS (Ag, Au, Cu, Pb)*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

In this contribution, we revisit the physical mathematical framework introduced in a previous paper, where stress-strain relations for viscoanelastic isotropic media were derived within irreversible thermodynamics with internal variables. We present the fundamental equations and briefly discuss their conceptual implications. In subsequent paper quantitative applications of the model for the analysis of physical phenomena in heat propagation can also be extended to the more general case of an anisotropic medium. On this occasion, the model was used to investigate heat propagation phenomena, with results benchmarked against the Fourier and Cattaneo–Vernotte frameworks.

Keywords: viscoanelasticity, irreversible thermodynamics, internal variables, heat equation.

MSC: 74D05, 80A17, 35Q79.

Biographical Note

Professor Liliana Restuccia has played a pivotal role in the advancement of mathematical physics, inspiring generations of students and colleagues

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through her vision and dedication. Her career, extending over several decades, has been distinguished by a rare and exemplary balance between rigorous mathematical analysis and profound physical insight. The present contribution seeks to highlight the continuity of her scientific interests with recent developments in continuum mechanics and thermodynamics. Her research activity has embraced a wide spectrum of themes in mathematical physics, often conducted in close and fruitful collaboration with colleagues. During my academic career, first as her university professor and later as her collaborator, I had the privilege of working alongside her on fundamental problems in the thermodynamics of nonlinear irreversible processes, within the framework of the theory of internal variables developed jointly by Professor G. A. Kluitenberg and myself [5–13]

1 Introduction

In [14], we show that thermoconduction phenomena in visco-anelastic media may be studied by using the systematic procedure of (CIT-IV). For the study of the behavior of rheological media, in presence of viscous and anelastic phenomena, in addition to determining the stress-strain relationships it is important to obtain the heat propagation equation. For this purpose the tensor $\varepsilon^{(1)}_{\alpha\beta}$ (inelastic strain) and the vector $\boldsymbol{\xi}$ are introduced as internal variables which charaterize both stress-strain relationships and which generalize the Fourier heat equation (parabolic type equation).

In the context of irreversible processes an important role is played by the flow of heat which, classicaly, is not considered to be a state variable. Therefore we will suppose that the specific entropy s, depends not only on the the specific internal energy u, the total strain $\varepsilon_{\alpha\beta}$ and the tensor $\varepsilon^{(1)}_{\alpha\beta}$ describing the inelastic strain, but also on a vectorial dynamic variable, $\boldsymbol{\xi}$, that is odd function of microscopic particles velocities that have influence on the propagation phenomena which occur in the medium.

In Sect. 2, introducing a vectorial dynamical variable $\boldsymbol{j}$ and a stress field $\tau^{(eq)}_{\alpha\beta}$ which is of a thermoelastic nature, an explicit form for the entropy production is derived.

In Sects 3-4, by virtue of the dynamical variable, influencing the thermal transport phenomena, the heat equation is obtained. In particular, in the isotropic case, when the medium has symmetry properties that, under orthogonal transformations, are invariant with respect to all rotations and inversions of the frame of axes, it is obtained that the heat flux can be split in two parts: a first contribution $\boldsymbol{J}^{(0)}$, governed by Fourier law and a second contribution $\boldsymbol{J}^{(1)}$, obeying Mawell-Cattaneo-Vernotte equation (MCV) ([3], [15], [16], [17]) in which a relaxation time is present.

In Sect. 5 we obtain a general temperature equation which generalizes the analogous equations of Fourier and MCV.

Finally, in Sect. 6 we applied the equation of heat conduction with internal variables in metals (Ag, Au, Cu, Pb).

2 The balance equation of entropy

In the context of irreversible processes an important role is played by the heat flux which, classicaly, is not considered to be a state variable. Therefore, we will suppose that the specific entropy s, depends not only on the specific internal energy u, the total strain $\varepsilon_{\alpha\beta}$, and the tensor $\varepsilon^{(1)}_{\alpha\beta}$ describing the inelastic strain but also on a vectorial dynamic variable, $\boldsymbol{\xi}$, that is odd function of microscopic particles velocities that have influence on the propagation phenomena which occur in the medium (see [14])

$$s = s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{1}$$

where ξ_α $(\alpha = 1, 2, 3)$ is the α-component of the vector $\boldsymbol{\xi}$.
We define the *absolute temperature*

$$T^{-1} \stackrel{\text{def}}{=} \frac{\partial}{\partial u} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{2}$$

the equilibrium-stress tensor

$$\tau^{(eq)}_{\alpha\beta} \stackrel{\text{def}}{=} -\varrho T \frac{\partial}{\partial \varepsilon_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{3}$$

the *affinity-stress* conjugate to $\varepsilon^{(1)}_{\alpha\beta}$

$$\tau^{(1)}_{\alpha\beta} \stackrel{\text{def}}{=} \varrho T \frac{\partial}{\partial \varepsilon^{(1)}_{\alpha\beta}} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{4}$$

and the vector $\boldsymbol{j}$ conjugate to the internal vector variable $\boldsymbol{\xi}$

$$j_\alpha \stackrel{\text{def}}{=} \varrho T \frac{\partial}{\partial \xi_\alpha} s(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \xi_\alpha)\,, \tag{5}$$

where $\varrho \stackrel{\text{def}}{=} \nu^{-1}$ is the mass density.

By using (1) we can obtain the following Gibbs relation

$$T ds = du - \nu\, \tau^{(eq)}_{\alpha\beta} d\varepsilon_{\alpha\beta} + \nu\, \tau^{(1)}_{\alpha\beta}\, d\varepsilon^{(1)}_{\alpha\beta} + \nu\, j_\alpha\, d\xi_\alpha\,, \tag{6}$$

where ν is the specific volume.
From (6) we have:

$$\varrho T \frac{ds}{dt} = \varrho \frac{du}{dt} - \tau^{(eq)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} + \tau^{(1)}_{\alpha\beta} \frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt}\,, \tag{7}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \tag{8}$$

is the substantial derivative respect to time, $\boldsymbol{v}$ is the velocity field and ∇ is the gradient.

To analyze phenomena due to viscous flows (analogus to those which occurs during flows in ordinary viscous liquids and gases) we introduce the following viscous stress tensor

$$\tau^{(vi)}_{\alpha\beta} \stackrel{\text{def}}{=} \tau_{\alpha\beta} - \tau^{(eq)}_{\alpha\beta}\,, \tag{9}$$

where $\tau_{\alpha\beta}$ is the mechanical stress tensor which occurs in the equation of motion

$$\varrho \frac{dv_\alpha}{dt} = \varrho\, F_\alpha + \frac{\partial \tau_{\alpha\beta}}{\partial x_\beta} \tag{10}$$

and in the first law of thermodynamics

$$\varrho \frac{du}{dt} = -\nabla \cdot \boldsymbol{J}^{(q)} + \tau_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt}. \tag{11}$$

In (10) the force F_α is the volume force per unit of mass and in (11) the vector $\boldsymbol{J}^{(q)}$ is the heat flux.
The equation (11), by virtue of (9) becomes

$$\varrho \frac{du}{dt} = -\nabla \cdot \boldsymbol{J}^{(q)} + \left(\tau^{(eq)}_{\alpha\beta} + \tau^{(vi)}_{\alpha\beta}\right) \frac{d\varepsilon_{\alpha\beta}}{dt}, \tag{12}$$

which substituted in the equation (7) gives

$$\varrho\, \frac{ds}{dt} = T^{-1} \left(-\nabla \cdot \boldsymbol{J}^{(q)} + \tau^{(vi)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} + \tau^{(1)}_{\alpha\beta} \frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} + j_\alpha \frac{d\xi_\alpha}{dt}\right) \tag{13}$$

and we have

$$\varrho \frac{ds}{dt} = -\text{div}\left(\frac{\boldsymbol{J}^{(q)}}{T}\right) + \sigma^{(s)} \tag{14}$$

and the following entropy production

$$\sigma^{(s)} = T^{-1}\Big[\boldsymbol{J}^{(q)} \cdot \Big(-T^{-1}\mathrm{grad}\, T\Big) + \tau^{(vi)}_{\alpha\beta}\,\frac{d\,\varepsilon_{\alpha\beta}}{dt} \;+\; \tau^{(1)}_{\alpha\beta}\,\frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} \;+\; j_\alpha\,\frac{d\xi_\alpha}{dt}\Big] \geq 0. \tag{15}$$

3 Phenomenological equations

According to the usual procedure of non-equilibrium thermodynamics, by virtue of the form (15) for the entropy production, we have for anisotropic media the following phenomenological equations (see [14]):

$$J^{(q)}_\alpha = L^{(q)(q)}_{\alpha\beta}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + L^{(q)(0)}_{\alpha(\mu\nu)}\,\frac{d\varepsilon_{\mu\nu}}{dt} \;+ L^{(q)(1)}_{\alpha(\mu\nu)}\tau^{(1)}_{\mu\nu} \;+\; L^{(q)(\xi)}_{\alpha\mu}\,\frac{d\xi_\mu}{dt}\,, \tag{16}$$

$$\tau^{(vi)}_{\alpha\beta} = L^{(0)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + L^{(0)(0)}_{(\alpha\beta)(\mu\nu)}\,\frac{d\varepsilon_{\mu\nu}}{dt} \;+ L^{(0)(1)}_{(\alpha\beta)(\mu\nu)}\tau^{(1)}_{\mu\nu} + L^{(0)(\xi)}_{(\alpha\beta)\mu}\,\frac{d\xi_\mu}{dt}\,, \tag{17}$$

$$\frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} = L^{(1)(q)}_{(\alpha\beta)\mu}\left(-T^{-1}\frac{\partial T}{\partial x^\mu}\right) + L^{(1)(0)}_{(\alpha\beta)(\mu\nu)}\,\frac{d\varepsilon_{\mu\nu}}{dt} \;+ L^{(1)(1)}_{(\alpha\beta)(\mu\nu)}\tau^{(1)}_{\mu\nu} + L^{(1)(\xi)}_{(\alpha\beta)\mu}\,\frac{d\xi_\mu}{dt}\,, \tag{18}$$

$$j_\alpha = L^{(\xi)(q)}_{\alpha\beta}\left(-T^{-1}\frac{\partial T}{\partial x^\beta}\right) + L^{(\xi)(0)}_{\alpha(\mu\nu)}\,\frac{d\varepsilon_{\mu\nu}}{dt} \;+ L^{(\xi)(1)}_{\alpha(\mu\nu)}\,\tau^{(\xi)(1)}_{\mu\nu} \;+ L^{(\xi)(\xi)}_{\alpha\beta}\,\frac{d\xi_\beta}{dt}\,, \tag{19}$$

where the quantities $L^{(q)(q)}_{\alpha\beta}$, $L^{(q)(0)}_{\alpha(\mu\nu)}$ $L^{(q)(1)}_{\alpha(\mu\nu)}$, ..., are called phenomenological thensors.

By using (16)-(19), from equation (15) one has (see [14]):

$$\begin{aligned} T\sigma^{(s)} =& L^{(q)(q)}_{\alpha\beta}\Big(T^{-2}\frac{\partial T}{\partial x^\alpha}\frac{\partial T}{\partial x^\beta}\Big) + \Big[L^{(\xi)(q)}_{\alpha\beta} + L^{(q)(\xi)}_{\alpha\beta}\Big]\frac{d\xi_\alpha}{dt}\Big(-T^{-1}\frac{\partial T}{\partial x^\beta}\Big) \\ &+ L^{(0)(0)}_{(\alpha\beta)(\mu\nu)}\frac{d\varepsilon_{\alpha\beta}}{dt}\,\frac{d\varepsilon_{\mu\nu}}{dt} + \Big[L^{(1)(0)}_{(\alpha\beta)(\mu\nu)} + L^{(0)(1)}_{(\mu\nu)(\alpha\beta)}\Big]\,\tau^{(1)}_{\alpha\beta}\,\frac{d\varepsilon_{\mu\nu}}{dt} \\ &+ L^{(1)(1)}_{(\alpha\beta)(\mu\nu)}\tau^{(1)}_{\alpha\beta}\,\tau^{(1)}_{\mu\nu} + L^{(\xi)(\xi)}_{\alpha\beta}\frac{d\xi_\alpha}{dt}\,\frac{d\xi_\beta}{dt}\,, \end{aligned} \tag{20}$$

where the entropy production $\sigma^{(s)} \geq 0$. If the media are isotropic the equations (16)-(19) become:

$$J^{(q)}_\alpha = -\lambda^{(q,q)}\frac{\partial T}{\partial x^\alpha} + \lambda^{(q,\xi)}\,\frac{d\xi_\alpha}{dt}\,, \tag{21}$$

$$\tau^{(vi)}_{\alpha\beta} = \eta^{(0,0)}_s\frac{d\varepsilon_{\alpha\beta}}{dt} + (\eta^{(0,0)}_v - \eta^{(0,0)}_s)\,\frac{d\varepsilon}{dt}\,\delta_{\alpha\beta} + \eta^{(0,1)}_s\tau^{(1)}_{\alpha\beta} + (\eta^{(0,1)}_v - \eta^{(0,1)}_s)\,\tau^{(1)}\,\delta_{\alpha\beta}\,, \tag{22}$$

$$\frac{d\varepsilon^{(1)}_{\alpha\beta}}{dt} = \eta_s^{(1,0)}\frac{d\varepsilon_{\alpha\beta}}{dt} + (\eta_v^{(1,0)} - \eta_s^{(1,0)})\,\frac{d\varepsilon}{dt}\,\delta_{\alpha\beta} + \eta_s^{(1,1)}\tau^{(1)}_{\alpha\beta} + (\eta_v^{(1,1)} - \eta_s^{(1,1)})\,\tau^{(1)}\,\delta_{\alpha\beta}\,, \tag{23}$$

$$j_\alpha = -\lambda^{(\xi,q)}\,\frac{\partial T}{\partial x^\alpha} + \lambda^{(\xi,\xi)}\,\frac{d\xi_\alpha}{dt}\,, \tag{24}$$

where the scalars quantities $\lambda^{(q,q)}$, $\lambda^{(q,\xi)}$, $\eta_s^{(0,0)}$, $\eta_v^{(0,0)}$ ... are phenomenological coefficients and in particular $\lambda^{(q,q)}$, $\eta_s^{(0,0)}$ and $\eta_v^{(0,0)}$ may be called the heat conductivity coefficient, the shear viscosity and the volume viscosity, respectively.

4 Linear equations of state

Let f the specific free energy of the medium

$$f = u - T\,s. \tag{25}$$

With the aid of Gibbs relation (6) we have:

$$df = \nu\,\tau^{(eq)}_{\alpha\beta}d\varepsilon_{\alpha\beta} - \nu\tau^{(1)}_{\alpha\beta}d\varepsilon^{(1)}_{\alpha\beta} - \nu\,j_\alpha\;d\xi_\alpha - sdT\,, \tag{26}$$

where

$$\nu\,\tau^{(eq)}_{\alpha\beta} = \frac{\partial f}{\partial\varepsilon_{\alpha\beta}}, \quad \nu\,\tau^{(1)}_{\alpha\beta} = -\frac{\partial f}{\partial\varepsilon^{(1)}_{\alpha\beta}}, \quad \nu\,j_\alpha = -\frac{\partial f}{\partial\xi_\alpha}, \quad s = -\frac{\partial f}{\partial T}. \tag{27}$$

Let us choose a configuration $\Sigma^{(0)}$ with uniform temperature T_0 and in which s_0 is the specific entropy, ν_0 is the specific volume and $(\tau^{(eq)}_{\alpha\beta})_0$, $(\tau^{(1)}_{\alpha\beta})_0$, $(\tau_{\alpha\beta})_0$, $(\varepsilon_{\alpha\beta})_0, (\varepsilon^{(1)}_{\alpha\beta})_0$ and $(j_\alpha)_0$ are zero.
Let us suppose that the deviation from $\Sigma^{(0)}$ are sufficiently small (with $\varrho \approx \varrho_0$) and the free energy for anisotropic medium can be written in the following form (see [14])

$$\begin{aligned} f = \nu_0\Big\{&\frac{1}{2}\Big[a^{(0)(0)}_{\alpha\beta\mu\nu}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\mu\nu} + a^{(1)(1)}_{\alpha\beta\mu\nu}\,\varepsilon^{(1)}_{\alpha\beta\nu}\,\varepsilon^{(1)}_{\mu\nu} + a^{(\xi)(\xi)}_{\alpha\beta}\xi_\alpha\,\xi_\beta\Big] \\ &+ a^{(0)(1)}_{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta}\,\varepsilon^{(1)}_{\mu\nu} + a^{(0)(\xi)}_{\alpha\beta\mu}\varepsilon_{\alpha\beta}\,\xi_\mu + a^{(1)(\xi)}_{\alpha\beta\mu}\varepsilon^{(1)}_{\alpha\beta}\,\xi_\mu \\ &+ (T - T_0)\Big[a^{(0)(T)}_{\alpha\beta}\varepsilon_{\alpha\beta} + a^{(1)(T)}_{\alpha\beta}\,\varepsilon^{(1)}_{\alpha\beta} + a^{(\xi)(T)}_{\alpha}\xi_\alpha\Big]\Big\} - \Psi(T)\,, \end{aligned} \tag{28}$$

where Ψ is a function of the temperature and the phenomenological tensors "a" are constants, i.e. they do not depend on temperature and the strains and are determined by the physical properties of the medium in the reference state.

We have

$$\Psi \;=\; c_{(\varepsilon)}\, T \log\left(\frac{T}{T_0}\right) \;+\; s_0\, T - c_{(\varepsilon)}(T - T_0) - u_0\,, \tag{29}$$

where $c_{(\varepsilon)}$ is constant and represnts the specific heat at constant deformation, and s_0 and u_0 are the specific entropy and the specific energy in the reference state, respectively.

In the isotropic case we have [14]):

$$\begin{aligned}\tau^{(eq)}_{\alpha\beta} = {} & a^{(0,0)}\, \varepsilon_{\alpha\beta} \,+ (b^{(0,0)} - a^{(0,0)})\, \varepsilon\, \delta_{\alpha\beta} \,+ a^{(0,1)} \varepsilon^{(1)}_{\alpha\beta} \\ & + (b^{(0,1)} - a^{(0,1)})\, \varepsilon^{(1)} \delta_{\alpha\beta} + \, a^{(0,T)}\, (T - T_0)\, \delta_{\alpha\beta}\,,\end{aligned} \tag{30}$$

$$\begin{aligned}\tau^{(1)}_{\alpha\beta} = {} & a^{(1,1)}\, \varepsilon^{(1)}_{\alpha\beta} \,+ (b^{(1,1)} - a^{(1,1)})\, \varepsilon^{(1)}\, \delta_{\alpha\beta} \,+ a^{(0,1)} \varepsilon_{\alpha\beta} \\ & + (b^{(0,1)} - a^{(0,1)})\, \varepsilon\, \delta_{\alpha\beta} \,+\, a^{(1,T)}\, (T - T_0)\, \delta_{\alpha\beta}\,,\end{aligned} \tag{31}$$

$$j_\alpha \;=\; -\, a^{(\xi,\xi)}\, \xi_\alpha\,. \tag{32}$$

5 Heat flows in visco-anelastic processes.

From (32) and (24) we have:

$$\boldsymbol{J}^{(q)} = \boldsymbol{J}^{(0)} \,+\, \boldsymbol{J}^{(1)}\,, \tag{33}$$

where $\boldsymbol{J}^{(0)}$ satisfies the Fourier's law and $\boldsymbol{J}^{(1)}$ obeys the Maxwell-Cattaneo-Vernotte law, i.e.

$$\boldsymbol{J}^{(0)} \;=\; -\lambda^{(0)}\, \nabla T\,, \tag{34}$$

with

$$\lambda^{(0)} \;=\; \lambda^{(q,q)} \;-\; \frac{\lambda^{(q,\xi)}\, \lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}}\,, \tag{35}$$

and

$$\boldsymbol{J}^{(1)} = -\frac{\lambda^{(q,\xi)}\, a^{(\xi,\xi)}}{\lambda^{(\xi,\xi)}}\, \boldsymbol{\xi}\,, \tag{36}$$

where $\boldsymbol{J}^{(1)}$ satisfies the following equation:

$$t_1\, \frac{d\boldsymbol{J}^{(1)}}{dt} + \boldsymbol{J}^{(1)} \;=\; -\frac{\lambda^{(\xi,q)}\, \lambda^{(q,\xi)}}{\lambda^{(\xi,\xi)}}\, \nabla T\,, \tag{37}$$

with t_1 the relaxation time:

$$t_1 \;=\; \frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}\,. \tag{38}$$

We consider a constant strain, i.e.

$$\frac{d}{dt} = \frac{\partial}{\partial t} \tag{39}$$

and we assume that the specific internal energy u is related to the temperature by

$$du = c_{(v)} dT\,, \tag{40}$$

being $c_{(v)}$ the heat capacity per unit of mass at constant volume. By virtue of (39) and (40) the first law of thermostatic (11) becomes

$$\varrho\, c_{(v)}\,\frac{\partial T}{\partial t} \;+\; \nabla\cdot\left(\boldsymbol{J}^{(0)}+\boldsymbol{J}^{(1)}\right) = 0. \tag{41}$$

By using (34) and (37), we have the following temperature equation

$$t_1\,\frac{\partial^2 T}{\partial t^2} \;+\; \frac{\partial T}{\partial t} \;=\; \alpha'\,\triangle T \;+\; t_2\,\frac{\partial \triangle T}{\partial t}, \tag{42}$$

where

$$\alpha' \;=\; \frac{\lambda^{(q,q)}}{\varrho\, c_{(v)}}, \tag{43}$$

$$t_1 = \frac{\lambda^{(\xi,\xi)}}{a^{(\xi,\xi)}}, \tag{44}$$

$$t_2 = \frac{\lambda^{(0)}}{\lambda^{(q,q)}}\, t_1\,, \tag{45}$$

where t_1, α' and t_2 are parametres expressed through the phenomenological and state coefficients. Equation (42) generalizes the temperature equation with analoguos equations of Fourier and Maxwell-Cattaneo-Vernotte.

Special cases:

- If the relaxation time t_1 (38) can be neglected with respect to the heat propagation time t ($t_1 \ll t$ and hence also t_2 can be neglected respect to t by virtue of (45)) the equation (42) becomes

$$\frac{\partial T}{\partial t} = \alpha' \triangle T\,, \tag{46}$$

  i.e. the Fourier equation.

- If the phenomenological coefficients satisfy the following condition from (35) one has: $\lambda^{(0)} = 0$

$$\lambda^{(q,q)} = \frac{\lambda^{(q,\xi)}, \lambda^{(\xi,q)}}{\lambda^{(\xi,\xi)}} \,, \tag{47}$$

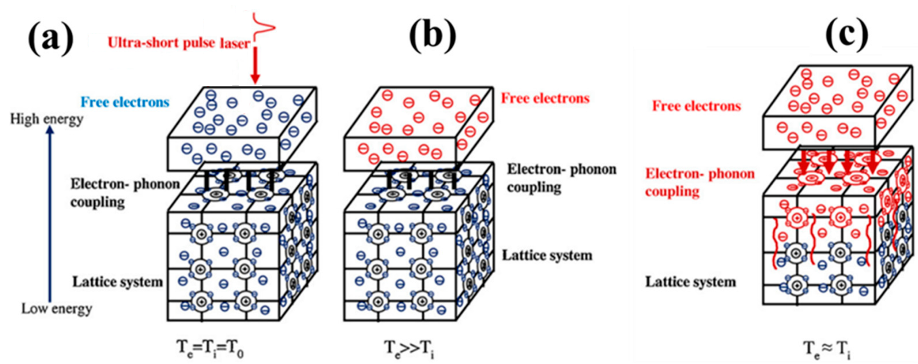
and the relation (45) becomes

$$t_2 = 0 \,, \tag{48}$$

and the equation (42) takes the form of the Maxwell-Cattaneo-Vernotte equation

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \alpha' \triangle T = 0 \,. \tag{49}$$

6 Local physical phenomenous applications to ultra short one dimensional processes



In the case of an constant strain medium, the generalized heat equation allows to characterize the temperature distribution in a metal film under the action of an ultrafast laser pulse. Infact α' is the effective thermal diffusivity in the phonon-electron or pure-phonon system and t_2 and t_1 are two intrinsic time scales characterizing the fast-transient response on microscales. Equation (42) is the generalized energy equation accounting for the effect of microstructural interaction (microscale response in space) in the fast-transient (microscale response in time) process. Table 1 shows the physical significance of the phenomenological and state coefficients with the corresponding units of measurement.

Table 2 shows the parameter, α', t_1 and t_2 values for some metals.

Table 1: Physical significance: $\lambda^{(\xi,\xi)}$, $\lambda^{(q,q)}$, $\lambda^{(0)}$, $a^{(\xi,\xi)}$.

Coefficient	***Description***	***Measurement unit***
$\lambda^{(\xi,\xi)}$, $\lambda^{(q,q)}$, $\lambda^{(0)}$ (35)	Thermal conductivity	$W/(m\ ^oK)$
$a^{(\xi,\xi)}$ (32)	Thermal conductivity rate	$W/(m\ ^oK\ s)$

Table 2: Parameters values for some metals.

Metal	$\boldsymbol{\alpha'}$ (43) $(10^{-4} \times m^2/s)$	$\boldsymbol{t_1}$ (44) $(picoseconds)$	$\boldsymbol{t_2}$ (45) $(picoseconds)$
Ag	1.6620	0.7438	89.286
Cu	1.1283	0.4348	70.883
Au	1.2495	0.7438	89.286
Pb	0.2301	0.1670	12.097

Considering a one-dimensional medium (in space), see Figure 1, it is sufficient to study the time-history of heat propagation. In this case, the equation (42) reduces to:

$$t_1 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha' \frac{\partial^2 T}{\partial x^2} + t_2 \frac{\partial}{\partial t} \left(\frac{\partial^2 T}{\partial x^2} \right) . \tag{50}$$

Two initial conditions are needed due to the presence of the wave term in equation (50):
$T = T_0$ and $\dfrac{\partial T}{\partial t} = \Lambda_0$ at $t = 0$.
By imposing a suddenly raised temperature at the boundary $T = T_w$ at $x = 0$, we study the way in which heat propagates through the solid medium. At a distance far away from the boundary, the thermal disturbance vanishes:

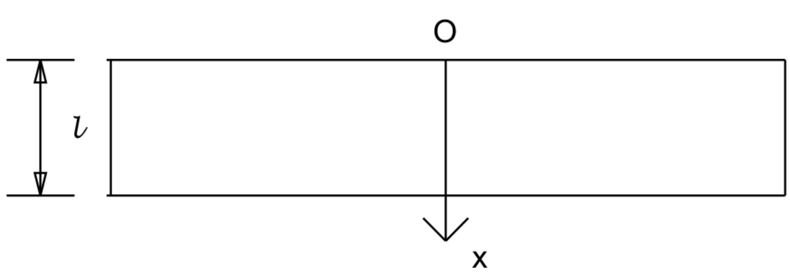


Figure 1: Thickness of metal sheet: $l = 1.0\text{E}^{-6}$ meter.

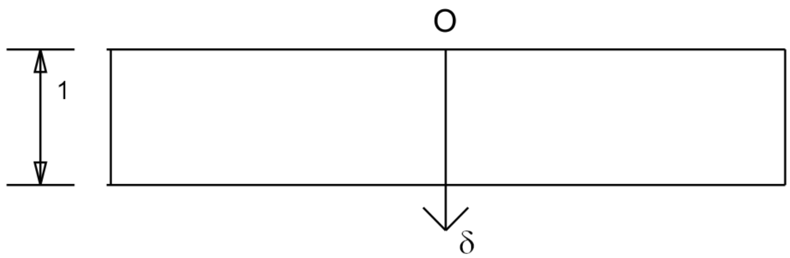


Figure 2: Geometry of dimensionaless problem

$\dfrac{\partial T}{\partial x} = 0$ at $x \longrightarrow l\,.$
Introducing the following dimensionless variables:

$$\theta = \frac{T - T_0}{T_w - T_0},$$

$$\delta = \frac{x}{l},$$

$$\beta = \frac{t}{l^2/\alpha'},$$

temperature, space and time, respectively, we obtain the dimensionaless equation:

$$z_1 \frac{\partial^2 \theta}{\partial \beta^2} + \frac{\partial \theta}{\partial \beta} = \frac{\partial^2 \theta}{\partial \delta^2} + z_2 \frac{\partial}{\partial \beta} \frac{\partial^2 \theta}{\partial \delta^2}\,, \tag{51}$$

with:

$$z_1 = \frac{t_1}{(l^2/\alpha')} \;\; ; \;\; z_2 = \frac{t_2}{(l^2/\alpha')}.$$

The initial conditions, and the boundary conditions thus become:
$\theta = 0 \quad \text{and} \quad \dfrac{\partial \theta}{\partial \beta} = \Gamma_0 \quad \text{at} \quad \beta = 0,$
$\theta = 1 \quad \text{at} \quad \delta = 0 \quad \text{and} \quad \dfrac{\partial \theta}{\partial \delta} = 0 \quad \text{at} \quad \delta = 1,$
with

$$\Gamma_0 = \frac{\Lambda_0\,(l^2/\alpha')}{T_w - T_0}.$$

The parameters z_1 and z_2 weigh the relative importance of the lagging times (in both the temperature gradient and the heat flux vector) to the diffusion time. They are the physical parameters governing the transition of the physical mechanisms.

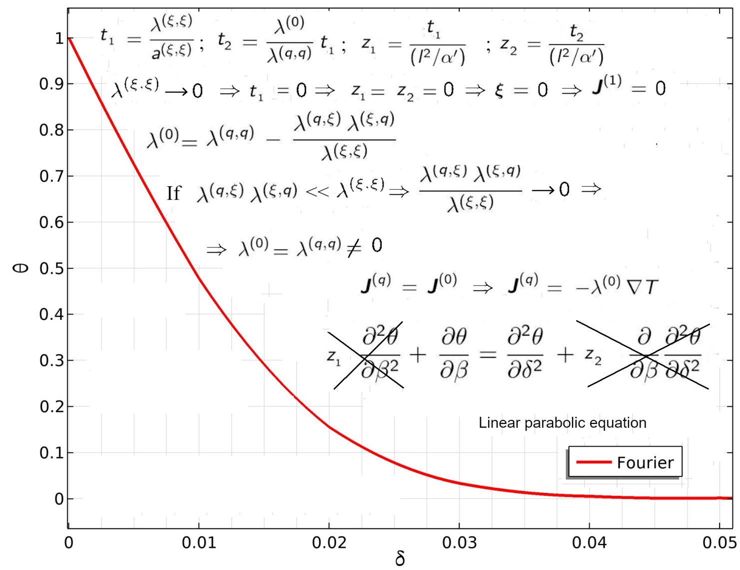


Figure 3: Normalized temperature trend with respect to dimensionless space according to the Fourier's equation.

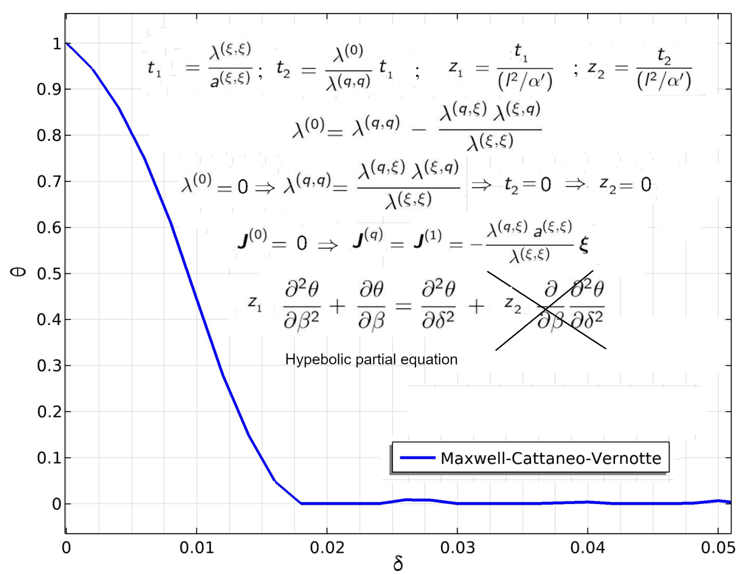


Figure 4: Normalized temperature trend with respect to dimensionless space according to the Maxwell-Cattaneo-Vernotte equation.

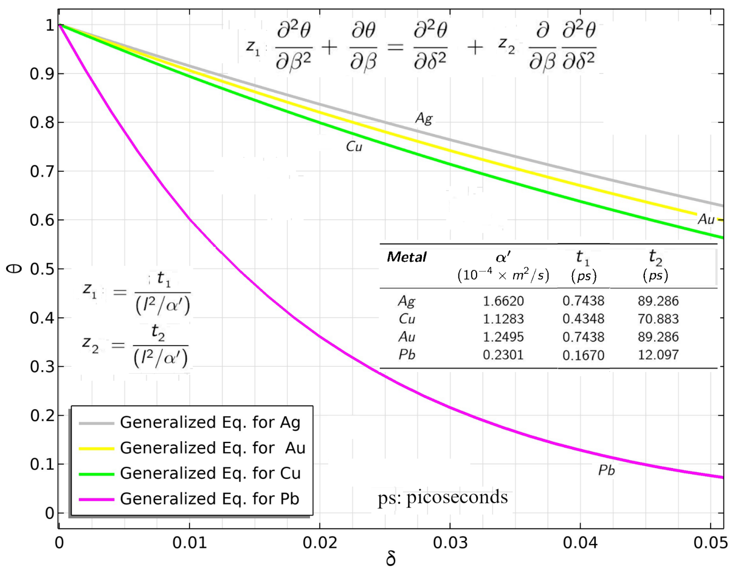


Figure 5: Normalized temperature trend with respect to dimensionless space according to the V. Ciancio's equation.

The equations were solved numerically using the finite element method (FEM) by engineer Bruno Felice Filippo Flora with the COMSOL software [1,2,4]. The simulation results are shown in the Figures 3, 4, 5.

7 Concluding remarks

These equations illustrate the interplay between mechanics and thermodynamics in complex materials. While originally presented in the context of viscoanelasticity, they provide a framework adaptable to various non-equilibrium systems.

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