Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 17, No. 3/2025

SOME BACKGROUNDS OF TEMPERATURE, SECOND LAW AND MATTER TENSOR*

Wolfgang Muschik†

Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

DOI 10.56082/annalsarscimath.2025.3.269

Abstract

Some backgrounds of temperature, entropy production and matter tensor are sketchy discussed.

Keywords: Schottky system, Clausius inequality, non-equilibrium entropy, contact temperature, state space, higher derivatives, balances on state space, axiom of no-reversible directional derivatives, non-symmetric matter tensor, Belinfante-Rosenfeld procedure.

MSC: 80A05, 83C05.

1 Introduction

This report is not a tutorial contribution [3], some knowledge of recent theoretical thermodynamics [11] and of Einstein's field equation is needed. Why write and read this brief document? Sometimes a subject is discussed without being interested in its background. A prominent example may be temperature, if equilibrium- and non-equilibrium temperatures are not properly separated. The practitioner knows what temperature is: temperature is what the thermometer indicates, but what does it indicate and what are the properties of the thermometer? The guess may be, that is the non-equilibrium contact temperature sketched in section 2.

---

[*] Accepted for publication on November 02, 2025

[†] `muschik@physik.tu-berlin.de`, Institut für Physik und Astronomie Technische Universität Berlin, Hardenbergstr. 36, D-10623 BERLIN, Germany

Another example for an interesting background is the fact that the entropy production turns out to be independent of the directional derivatives, or in another wording, that the entropy production is a state function. The background of this statement is the axiom of no-reversible directional derivatives which is sketched in section 3.

An example for a missing background is the symmetry of the matter tensor of Einstein's field equation in General Relativity Theory (GRT), because there are materials with non-symmetric energy-momentum tensor like liquid crystals. Of course, nobody is interested to describe liquid crystals general-relativistically. But the problem is vice-versa: Is GRT able to describe as a general theory also materials of non-symmetric matter tensors ? The Belinfante-Rosenfeld procedure for generating symmetric tensors from non-symmetric ones cannot help as sketched in section 4.

2 Contact temperature

We consider a closed Schottky system[1] [8,12] interacting with its environment by net heat exchange $\overset{\bullet}{Q}$ through an inert partition[2]. Presupposing that the environment is in equilibrium, its thermostatic temperature is T. Then Clausius inequality

$$\oint \frac{\overset{\bullet}{Q}}{T} dt \ \leq \ 0 \tag{1}$$

is valid[3].

Presupposing that also in non-equilibrium an entropy balance equation of the Schottky system is valid [6]

$$\overset{\bullet}{S} = \ \Phi + \Sigma, \qquad \Sigma \ \geq \ 0 \tag{2}$$

(entropy exchange Φ and production Σ). Clear is, that Φ cannot be defined by use of the environment's temperature T because S is a state function of the Schottky system and T is not a quantity of the system

$$\Phi \ \neq \ \frac{\overset{\bullet}{Q}}{T}. \tag{3}$$

---

[1] A Schottky system is a discrete system, that means a "box" which is in interaction with its environment.

[2] An inert partition does not absorb or emit heat, power or material.

[3] For performing the integral, a state space belonging to the Schottky system must be introduced [7].

Consequently, we define the entropy exchange by using an up to now unknown non-equilibrium temperatur Θ belonging to the Schottky system [1,2]

$$\Phi \;:=\; \frac{\overset{\bullet}{Q}}{\Theta} \;\longrightarrow\; 0 \;=\; \oint \overset{\bullet}{S}\,dt \;=\; \oint \frac{\overset{\bullet}{Q}}{\Theta}dt + \oint \Sigma dt \;\geq\; \oint \frac{\overset{\bullet}{Q}}{\Theta}dt \tag{4}$$

according to (2), resulting in a Clausius-like inequality. From (4) follows

$$\oint \frac{\overset{\bullet}{Q}}{\Theta}dt \;\leq\; 0, \tag{5}$$

which is different from (1).

Comparing (1) with (5) the question arises, which integrand is greater than the other? We set[4] provisional with regard to an interpretation below

$$\left(\frac{\overset{\bullet}{Q}}{\Theta} - \frac{\overset{\bullet}{Q}}{T}\right) \;=\; \left(\frac{1}{\Theta} - \frac{1}{T}\right)\overset{\bullet}{Q} \overset{\#}{\geq} \;0. \tag{6}$$

From this inequality follows

$$\Theta \overset{>}{<} T \;\rightarrow \overset{\bullet}{Q} \overset{<}{>} 0. \tag{7}$$

The question arises: what are the temperatures, if the net heat exchange is zero ? The assumption of contradiction is that in this case the temperatures are different

$$\overset{\bullet}{Q} \;=\; 0 \;\xrightarrow{?}\; \Theta - T \overset{>}{<} \rightarrow \overset{\bullet}{Q} \overset{<}{>} 0. \tag{8}$$

The assumption of different temperatures at vanishing heat exchange results in the contradiction (8). Consequently,

$$\overset{\bullet}{Q} \;=\; 0 \;\rightarrow\; \Theta \;=\; T \tag{9}$$

is valid. This relation allows to define Θ, a non-equilibrium temperature called *Contact Temperature* [4].

A Schottky system in a special non-equilibrium state is contacted with equilibrium environments of different thermostatic temperatures, net heat exchanges occur. There is one equilibrium environment of temperature T for which this net heat exchange vanishes. According to (9), the value of the system's non-equilibrium temperature Θ is defined by the well known thermostatic temperature T of the environment.

---

[4] $\overset{\#}{\square} \longrightarrow \square$ is a set quantity.

Definition of the Contact Temperature [6]: The system's contact temperature is that thermostatic temperature of the system's equilibrium environment for which the net heat exchange between the system and its environment through an inert partition vanished by change of sign.

3 Second law and material

For describing materials, a *state space* $\mathbf{Z} \ni \mathbf{z}$ is necessary on which –after inserting the constitutive equations into the balance equations– the linear differential operators of the balance equations (d_t, ∇) act by applying the chain rule, thus generating so-called *higher derivatives* $\boldsymbol{y}^\top := (d_t\mathbf{z}, \nabla\mathbf{z})$ [10]. Thus, time and position derivatives of the state space variables appear in the balance equations which for discrimination are called *balances on state space*

$$\mathbf{A}(\mathbf{z})\boldsymbol{y} = \boldsymbol{C}(\mathbf{z}), \qquad \boldsymbol{B}(\mathbf{z})\boldsymbol{y} = D(\mathbf{z}) \geq 0, \tag{10}$$

and which are linear in the higher derivatives which can be interpreted as *directional derivatives* in state space. Here $(10)_2$ is the entropy balance equation and $(10)_1$ are all other balances in matrix formulation.

Because there are less balance equations than higher derivatives, these ones are not determined by the balances on state space, rather higher derivatives exist forming a kernel of the balances on state space

$$\mathbf{A}(\mathbf{z})\boldsymbol{y}_{ker} = 0. \tag{11}$$

Due to their linearity, higher derivatives can be combined linearly representing different directional derivatives. According to the dissipation inequality, there are three kinds of directional derivatives: irreversible, reversible and not-satisfying the inequality ones.

Concerning the directional derivatives, some axiomatic statements can be formulated:
i) *A reversible directional derivative cannot be generated by non-reversible ones.*
This statement results in the *Coleman-Noll-Mizel formulation* of the 2^{nd} law:
ii) *All solutions of the balance equations on state space have to satisfy the dissipation inequality.*
A second axiomatical statement is
iii) *Directional derivatives which are in the kernel of the balance equations on state space do not enter the entropy production* σ

$$\sigma(\boldsymbol{y} + \boldsymbol{y}_{ker}) \;=\; \sigma(\boldsymbol{y}). \tag{12}$$

This results in the

Axiom of No-reversible Directional Derivatives

iv) *Except in equilibria, reversible directional derivatives in state space do not exist,*
that is, reversible directional derivatives appear only in equilibrium and the entropy production in non-equilibrium does not depend on them, it is a state space function. Consequently, reversible directional derivatives do not exist in non-equilibrium.

Another result of (12) is of interest [13]: the *Liu Relations*

$$\boldsymbol{B}(\boldsymbol{z}) \;=\; \boldsymbol{\lambda}(\boldsymbol{z})\mathbf{A}(\boldsymbol{z}) \quad \text{and} \quad \boldsymbol{\lambda}(\boldsymbol{z})\boldsymbol{C}(\boldsymbol{z}) \;\geq\; D(\boldsymbol{z}), \tag{13}$$

which connect the balance equations with the dissipation inequality and which is unexpected independent of the directional derivatives.

4 Matter tensor

In general-covariant continuum physics (GCCP) is presupposed that the energy-momentum tensor T^{ab} and the current of spin density S^{kab} [5] satisfy the following balance equations

$$T^{ab}{}_{;a} \;=\; k^b, \qquad S^{kab}{}_{;k} \;=\; m^{ab}, \tag{14}$$

$$S^{kab} \;=\; -S^{kba}, \quad m^{ab} \;=\; -m^{ba}. \tag{15}$$

In non-isolated systems, $k^b \neq 0$ denotes an external force density, and $m^{ab} \neq 0$ is an external momentum density. The energy-momentum tensor may be not symmetric.

The question arises, how to make these general-covariant balances (14) compatible with Einstein's field equations of General Relativity Theory (GRT) [5]

$$R^{ab} - \frac{1}{2}g^{ab}R \;=\; \kappa\Theta^{ab} \quad \Longrightarrow \quad \Theta^{ab} \;=\; \Theta^{ba}, \quad \Theta^{ab}{}_{;b} \;=\; 0. \tag{16}$$

This question seems to be solved by:

Proposition 1. (see [5]) *The energy-momentum balance (14)$_1$ is only compatible with Einstein's equations (16), if the inhomogeneity of the energy-momentum balance (14)$_1$ has the form*

$$k^b \;=\; T^{ab}{}_{;\,a} \;=\; -\frac{1}{2}R^b_{klm}S^{klm} \tag{17}$$

---

[5] often shortly denoted as spin tensor

(R^b_{klm} is the curvature tensor), and additionally beyond that, the anti-symmetric part of the energy-momentum tensor is given by the divergence of the spin tensor

$$2T^{[ab]} = S^{kab}{}_{;k} = m^{ab}. \tag{18}$$

The equations (17) and (18) represent the basic balances of energy-momentum and spin in GCCP and which enforce the compatibility of them with Einstein's equations (16). The way to generate these equations is called *Belinfante-Rosenfeld procedure* which starts by using the ansatzes

$$B^{ab} := T^{ab} - \frac{1}{2}\Sigma^{kab}{}_{;k} \tag{19}$$

$$\Sigma^{kab} := S^{kab} + S^{abk} + S^{bak}, \quad \Sigma^{kab} = -\Sigma^{akb}, \tag{20}$$

$$\Sigma^{k[ab]}{}_{;k} = S^{kab}{}_{;k} = m^{ab}, \quad \Sigma^{k(ab)}{}_{;k} = 2S^{(ab)k}{}_{;k} \tag{21}$$

which are composed of the energy-momentum tensor and of divergences of the spin tensor.

Demanding

$$B^{[ab]} \doteq 0, \quad B^{ab}{}_{;a} \doteq 0, \tag{22}$$

results in (17) and (18) [9][6]. The decisive step of the Belinfante-Rosenfeld procedure is the setting

$$B^{(ab)} = T^{(ab)} - S^{(ab)k}{}_{;k} \doteq \Theta^{ab} = \frac{1}{\kappa}\Big(R^{ab} - \frac{1}{2}g^{ab}R\Big) =: \frac{1}{\kappa}G^{ab}, \tag{23}$$

that the symmetric part of the procedure generating tensor B^{ab} (19) equals the gravitation generating energy-momentum tensor Θ^{ab} of matter in Einstein's equations (16) which couple matter with geometry G^{ab}.

The result (23) of the Belinfante-Rosenfeld procedure seems strange in some respect: According to $(22)_1$ and $(21)_1$

$$B^{[ab]} = T^{[ab]} - \frac{1}{2}m^{ab} = 0 \tag{24}$$

$T^{[ab]}$ and m^{ab} have no influence on the gravitation according to $(23)_2$. The external momentum density m^{ab} generates the material quantity $T^{[ab]}$, thus coupling spin with energy-momentum. According to (17), the external forces vanish with the curvature, a fact which is not true in Special Relativity

---

[6] A setting is marked by $\doteq$, a definition by A:=B, A is defined by B

Theory (SRT). According to (23)$_3$ also the matter tensor Θ vanishes with the curvature and in this case $T^{(ab)}$ is determined by $S^{(ab)k}{}_{;k}$ different from matter theory.

The result is: The Belinfante-Rosenfeld procedure has to be modified with ansatzes different from (19) and (20), if SRT is presupposed.

References

[1] W. Muschik and G. Brunk, A concept of non-equilibrium temperature, *Int. J. Eng. Sc.* 15 (1977), 377-389.

[2] W. Muschik, Empirical foundation and axiomatic treatment of non-equilibrium temperature, *Arch. Rat. Mech. Anal.* 66 (1977), 379-401.

[3] W. Muschik, Survey of some branches of thermodynamics, *J. Non-Equilib. Thermodyn.* 33 (2008), 165-198.

[4] W. Muschik, Contact quantities and non-equilibrium entropy of discrete systems, *J. Non-Equilib. Thermodyn.* 34 (2009), 75-92.

[5] W. Muschik and H.-H. Borzeszkowski, Mathisson-Papapetrou equations as conditions for the compatibility of general relativity and continuum physics, *Gen. Relat. Gravit.* 46 (2014), 1677.

[6] W. Muschik, Second law and non-equilibrium entropy of Schottky systems: doubts and verification, *Entropy* 20 (2018), 740.

[7] W. Muschik, Discrete systems in thermal physics and engineering: a glance from non-equilibrium thermodynamics, *Continuum Mech. Thermodyn.* 33 (2021), 2411-2430.

[8] W. Muschik, Thermodynamical foundations of closed discrete non-equilibrium systems, *J. Non-Equilib. Thermodyn.* 47 (2022), 227-231.

[9] W. Muschik, A modified Belinfante/Rosenfeld procedure for testing the compatibility of general-covariant continuum physics and general relativity theory, *Atti Accademia Peloritana dei Pericolanti* 101 (2023), 1-15.

[10] W. Muschik, Second law and Liu relations: the non-reversible direction axiom: revisited, *Meccanica* 59 (2024), 1643-1654.

[11] C. Papenfuss, *Continuum Thermodynamics and Constitutive Theory*, Springer Nature Switzerland, AG 2020.

[12] W. Schottky, *Thermodynamik*, Erster Teil §1, Springer, Berlin 1929.

[13] V. Triani, C. Papenfuss, V.A. Cimmelli and W. Muschik, Exploitation of the second law: Coleman-Noll and Liu procedure in comparison, *J. Non-Equilib. Thermodyn.* 33 (2008), 47-60.