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THE INFLUENCE OF PARAMETERS ON STABILIZATION FOR HOMOGENEOUS POLYNOMIAL DYNAMICAL SYSTEMS IN THE PLANE*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

The problem of stabilization of dynamical systems is very important, as part of the control systems field. The theory of positive polynomials in control has the seeds in the 1980's, based on the work of Naum Zuselevich. They can be used to solve a variety of problems in robust control, non-linear control and also in non-convex optimization. The present paper approaches the problem of finding a stabilizing feedback for homogeneous polynomial systems in the plane. It is known that the polynomial systems in the plane have a lot of special properties which can be easier approached thanks to the dimension 2. The case of systems arising from excitable media is taken into account, and the results will be used to deduce properties for further detailed analysis.

Keywords: kinematics of mixing, control, Lyapunov stability, polynomial system, feedback stabilization.

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1 Introduction and mathematical framework

There are many domains where the complex dynamical systems are modelled as polynomial differential equations. Dynamics of population growth in ecology, prices and business cycles in economics, spread of epidemics in network science, and motion of electromechanical systems in control engineering, are only some examples where some slight deviations from the "assumptions of linearity" imply working with vector fields. On the other hand, polynomial vector fields have always been at the heart of diverse branches of mathematics. Among different qualitative questions we can ask, the stability problem is the most important. Almost universally, the approach of stability of dynamical systems leads to Lyapunov second method and its variants. Between the appliances, in the last decades the Lyapunov theory has been refreshed by the advances in the theory of convex optimization and in particular semidefinite programming (SDP). But despite the rich research in this field, many elusive problems remained. Let us remind the following fundamental question, still open: given a polynomial vector field, is there an algorithm that can decide whether the origin is a (locally or globally) asymptotically stable equilibrium point? A well-known conjecture of Arnold states that there cannot be such an algorithm; i.e., that the problem is undecidable [1] . This problem of decidability is part of the field of "P versus NP problems", a major unsolved conjecture in the computer science. It is important to mention the distinctions between establishing results for stability, versus answering questions on existence of Lyapunov functions. The connections between the two are always important and must be noted. More details on this point can be found in [12]. In the present paper it is approached the stabilization problem for homogeneous polynomial dynamical systems in the plane, and tested how the parameter influence works in this analysis type. Let us consider the affine control system in the plane:

$$\dot{x} = f(x) + u \cdot g(x), \tag{1}$$

where the state $x \in \mathbb{R}^n$ is a smooth vector field, $u \in \mathbb{R}, f(0) = 0$ and f and g are also smooth vector fields. In [2] it is proved that if the system (1) admits a Lyapunov function, then the system is stabilizable at the origin by a nonlinear feedback law, which is smooth for $x \neq 0$. The main idea is to construct homogeneous feedback laws that preserve the homogeneity of the resulting closed loop system. In [11,15] more details can be found.

Let us consider the homogeneous dynamical systems in the plane of the form [11]

$$\begin{cases} \dot{x}_1 = & P_1(x_1,x_2) + u \cdot Q_1(x_1,x_2) \\ \dot{x}_2 = & P_2(x_1,x_2) + u \cdot Q_2(x_1,x_2), \end{cases} \tag{2}$$

where $(x_1,x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}$, P_1, P_2 are homogeneous polynomials of degree $2k+1$, Q_1, Q_2 are homogeneous polynomials of degree q, and k, q are integers. The problem is to find a feedback $(x_1,x_2) \to u(x_1,x_2)$ which is homogeneous of degree $2k+1-q$ and asymptotically stabilizes the control system (2). If such a function exists, then the system is globally asymptotically stabilizable (GAS) at the origin. In [11] there are given some methods for constructing a homogeneous feedback law. There are a lot of cases taken into account there, concerning the polynomials P_1, P_2, Q_1, Q_2, all based essentially on a theorem given by [4] in which the author gives necessary and sufficient conditions for stability of homogeneous systems in the plane. We follow the technique from [11] and call here only the essential notations and results. Let us recall the following definition.

Definition 1. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial function. We say that P is *homogeneous of degree* $d \in \mathbb{N}$ if $P(\lambda x) = \lambda^d P(x), \quad \forall \lambda \in \mathbb{R}, \quad \forall x \in \mathbb{R}^2$. In [4] it is called the Hahn theorem which gives necessary and sufficient conditions for a two-dimensional system like the following:

$$\begin{cases} \dot{x}_1 = & X_1(x_1,x_2) \\ \dot{x}_2 = & X_2(x_1,x_2), \end{cases} \tag{3}$$

with $X_1(0,0) = 0, X_2(0,0) = 0$, and X_1, X_2 Lipschitz, continuous and homogeneous of degree p, to be asymptotically stable.

Now consider the polynomial system (2) with the above statements. The problem is to find a homogeneous feedback of degree $2k+1-q$ which stabilizes the system (2).

We need to define the following polynomial functions, which will be essential in the calculus [11]:

$$\Phi(x_1,x_2) = \det \begin{pmatrix} X_1(x_1,x_2) & x_1 \\ X_2(x_1,x_2) & x_2 \end{pmatrix} \tag{4}$$

$$G(x_1,x_2) = \det \begin{pmatrix} Q_1(x_1,x_2) & x_1 \\ Q_2(x_1,x_2) & x_2 \end{pmatrix} = x_2 Q_1(x_1,x_2) - x_1 Q_2(x_1,x_2) \tag{5}$$

$$H(x_1,x_2) = \det \begin{pmatrix} P_1(x_1,x_2) & x_1 \\ P_2(x_1,x_2) & x_2 \end{pmatrix} = x_2 P_1(x_1,x_2) - x_1 P_2(x_1,x_2)$$

$$F(x_1,x_2) = \det\begin{pmatrix} P_1(x_1,x_2) & Q_1(x_1,x_2) \\ P_2(x_1,x_2) & Q_2(x_1,x_2) \end{pmatrix}$$

$$= Q_2(x_1,x_2)\,P_1(x_1,x_2) - Q_1(x_1,x_2)\,P_2(x_1,x_2)\,.$$

It is easy to observe that the degrees of the homogeneous polynomials G, H and F are $q+1$, $2k+2$ and $2k+q+1$ respectively.

Let $u(x_1,x_2)$ be an homogeneous feedback of degree $2k+1-q$. The closed loop system (2) by the feedback u can be written as

$$\begin{cases} \dot{x_1} = P_1(x_1,x_2) + uQ_1(x_1,x_2) = X_1(x_1,x_2) \\ \dot{x_2} = P_2(x_1,x_2) + uQ_2(x_1,x_2) = X_2(x_1,x_2)\,. \end{cases} \tag{6}$$

If we recall the function $\Phi(x_1,x_2) = x_2X_1(x_1,x_2) - x_1X_2(x_1,x_2)$, then it is clear that

$$\phi(x_1,x_2) = H(x_1,x_2) + u\ G(x_1,x_2)\,. \tag{7}$$

Since $u(x_1,x_2)$ is homogeneous of degree $2k+1-q$, the function $\Phi(x_1,x_2)$ is homogeneous of degree $2k+2$. This function plays a very important role in the study of the stability of the closed loop system (6). Moreover, it can be constructed the function $\Phi(x_1,x_2)$ satisfying the following conditions [11]:

(I) The function Φ is C^∞ in $R^2\backslash\{(0,0)\}$ and homogeneous of degree $2k+2$;

(II) The functions $(c_ix_1 - \widetilde{c_i}x_2)^{\eta_i}$ divide $\Phi(x_1,x_2) - H(x_1,x_2)$ for all $i \in I_1$; Here $c_i, \widetilde{c_i}$ are between the zeros of the function G, η_{i} is the multiplicity order and I_1 is a set of index;

(III) If the set of points $\xi \in R^2\backslash(0,0)$ such that $\Phi(\xi) = 0$ is non-empty and the point $\xi = (\xi_1,\xi_2)$ satisfies $\Phi(\xi) = 0$, then

$$\left\langle (X_1(\xi), X_2(\xi))^T \middle| (\xi)^T \right\rangle < \ 0.$$

Here the relation $\left\langle x^T \middle| y^T \right\rangle = \sum_{i=1}^{2} x_iy_i$ denotes the Euclidean inner product.

In order to construct the function Φ, we have to use the numerical data P_1, P_2, $Q_1$and Q_2 to compute the functions $G(x_1,x_2)$, $H(x_1,x_2)$, $F(x_1,x_2)$ and their zeros. We take into account here only the case when $G(x_1,x_2)$ is definite, i.e. it has no zeros on the unit sphere. Then the following result holds.

Theorem 1 [11]. *If there exists a point M, $M = (m_1, m_2)$ such that $G(M) \cdot F(M) > 0$, then the function Φ defined by*

$$\Phi(x_1, x_2) = (m_2 x_1 - m_1 x_2)^{2k+2} \tag{8}$$

satisfies the conditions (I)-(III) and the feedback

$$u(x_1, x_2) = \frac{\Phi(x_1, x_2) - H(x_1, x_2)}{G(x_1, x_2)} \tag{9}$$

is C^∞ on $R^2 \backslash \{(0,0)\}$, homogeneous of degree $2k + 1 - q$ and stabilizes the system (2).

2 Recent results on kinematics of mixing in 2d case

The paradigm of mixing of fluids is substantially relied to the development of various branches of science and technology. Mixing is connected with turbulence, natural sciences, but equally to various branches of engineering. Despite its universality, it has been quite difficult to construct a general framework for analysis. Therefore, it is very unlikely that a single explanation can capture all the features of the problem.

Mixing is intimately related to stretching and folding of material surfaces (or lines, in two dimensions) and the theory can be synthetized in terms of a kinematical description. Even though the kinematical foundations were available for a long time (starting with Truesdell in 1954, Truesdell & Toupin in 1960) a standpoint like this was established rather with the works of [13]. In some cases, the existing kinematical foundations are sufficient and it is possible to manage the exact calculation for the material elements' stretching, for example in steady curvilinear flows and slowly varying flows. But even these cases can lead to poor mixing unless special precautions are taken [3].

The mathematical formula of a flow is:

$$x = \Phi_t(X), \quad X = \Phi_{t=0}(X). \tag{10}$$

Here Φ, the *flow*, is a diffeomorphism of class C^k and it must satisfy the equation

$$J = \det(D(\Phi_t(X))) = \det\left(\frac{\partial x_i}{\partial X_j}\right). \tag{11}$$

The equation (11) implies two particles, X_1 and X_2, which occupy the same position $\mathbf{x}$ at a moment, and D means the differentiation with respect to the reference configuration.

Chaotic motions have a natural way for increasing the mixing efficiency of a flow. The basic idea is that the Eulerian velocity field:

$$\left(\frac{d\mathbf{x}}{dt}\right)_{\mathbf{x}} = \mathbf{v}\left(\mathbf{x}, t\right), \quad \text{with } \nabla\mathbf{v} = 0 \tag{12}$$

admits chaotic particle trajectories for simple forms of $\mathbf{v}(\mathbf{x} = \mathbf{X}\ at\ t = 0)$.

Chaotic mixing is tightly related to dynamical systems field. This can be easily observed from relation (11). However, the dynamical systems appliances can be over-estimated. For example, it is not possible to predict a-priori the "degree of chaos", even in simple two-dimensional experiments, without doing the experiment first. In fact there are two very important aspects that distinguish the chaotic mixing. First, it is related to *rate* a process, that means rapid mixing, rather than long-time behaviour. The second aspect is that the perturbations from integrability are *large*, because it happens when the best mixing occurs. However, few experiments, analyses, computations, revealed some of the basic mechanisms acting in simple chaotic flows, and thus some "windows" for some complicated situations are opened.

The basic measure in the kinematics of mixing is the *deformation gradient* **F**, with the formula:

$$\mathbf{F} = (\nabla_X \Phi_t(\mathbf{X}))^T, \quad F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right). \tag{13}$$

In order to evaluate the mixing process, we need to evaluate the *stretching* and *folding* of material elements. If the gradient **F** is non-singular, the deformation measures are defined: the *length deformation* λ and *surface deformation* η, with the following relations:

$$\lambda = (\mathbf{C} : \mathbf{MM})^{1/2}, \quad \eta = (\det(\mathbf{F}) \cdot \left(\mathbf{C}^{-1} : \mathbf{NN}\right)^{1/2} \tag{14}$$

with $\mathbf{C}$ $(= \mathbf{F}^T \cdot \mathbf{F})$ the *Cauchy-Green deformation tensor,* and the vectors **M**, **N** the orientation versors in length and surface respectively.

λ and η are used for studying the basic qualitative quantity in the kinematics of mixing: the deformation efficiency. It can be naturally quantified, namely, the following formulas are defined [14]:

$$e_\lambda = \frac{D(\ln\lambda)/Dt}{(\mathbf{D} : \mathbf{D})^{1/2}} \leq 1 \tag{15}$$

the *deformation efficiency in length*, and

$$e_\eta = \frac{D(\ln \eta)/Dt}{(\mathbf{D} : \mathbf{D})^{1/2}} \leq 1 \tag{16}$$

the *deformation efficiency in surface* (for the case of an isochoric flow)

The above quantities are very important in the study of the mixing efficiency; various forms of them are used in practice for simplifying the calculus. More details can be found in [14].

When approaching the mixing model as a dynamical system, it is convenient to start from the widespread basic 2d mathematical form [14]:

$$\begin{cases} \dot{x}_1 = & G \cdot x_2 \\ \dot{x}_2 = & KG \cdot x_1, \quad -1 < K < 1, \quad G \in \mathbb{R}. \end{cases} \tag{17}$$

Although this system is a linear one, when associating the corresponding initial conditions

$$x_1(0) = x_1(t = 0) = X_1; \quad x_2(0) = x_2(t = 0) = X_2 \tag{18}$$

one obtains the trajectory of the Cauchy problem (17)-(18) with a nonlinear behaviour.

In fact the solution is quite complex and has an interesting geometric standpoint. The streamlines of the model satisfy the relation

$$x_2^2 - K \cdot x_1^2 = \text{const},$$

which corresponds to some ellipses with the axes rate $\left(\frac{1}{|K|}\right)^{\frac{1}{2}}$ if K is negative, and some hyperbolas with the angle $\beta = \arctan\left(\frac{1}{|K|}\right)^{\frac{1}{2}}$ between the extension axes and x_2, if K is positive [14].

This isochoric flow has a 3d version, quite easy to construct, namely associating for the third component, the movement velocity of the system [7]:

$$\begin{cases} \dot{x}_1 &= G \cdot x_2 \\ \dot{x}_2 &= K \cdot G \cdot x_1, \ -1 < K < 1, c = \text{const.} \\ \dot{x}_3 &= c \end{cases} \tag{19}$$

For the basic 2d mixing flow dynamical system, few analyses were done, in order to create a panel for its behaviour. The stability of the system was analysed, in the original and perturbed form. One form with important features was the following slightly perturbed form [8]:

$$\begin{cases} \dot{x_1} = Gx_2 + x_1 \\ \dot{x_2} = KGx_1 - x_2, \end{cases} \quad -1 < K < 1, G \in R. \tag{20}$$

For this model, the feedback linearization like in [10] was performed, then a SOS Lyapunov function was found *both* for the initial and for the feedback linearized model [8].

It is known that the Lyapunov computational analysis is a powerful tool for stability, since it has appliances for studying how *small variations in the initial conditions* will introduce small variations in the phenomenon evolution. A slightly perturbed form like (20) of the dynamical system is very good also for analysing the parameter influence on the stability analysis.

In a next stage, another important form of the mixing flow dynamical system was taken into account, the model perturbed with a logistic type term [9]:

$$\begin{cases} \dot{x_1} = Gx_2 \\ \dot{x_2} = KGx_1 + G(x_2 - x_1), \end{cases} \quad -1 < K < 1, G \in R. \tag{21}$$

Since it is a nonlinear model, finding a Lyapunov function for it is quite difficult, therefore the "controlled form" of the model was used. In this context, an eigenvalue criterion for stability was straightforward to apply, and there were found good results, the first stability criterion works in feasible conditions for the parameters [9].

3 Finding a stabilizing feedback for the mixing dynamical system in a perturbed form

Although a nonlinear model, the mixing flow dynamical system for the kinematics of mixing can reach the stability. This was checked in recent works. Taking into account the previous perturbations of the model, a question arise in an obvious way. *Up to what point a model like the mixing model can be perturbed, such as it is possible to apply a feasible stabilisation technique?* A possible answer to this question can be done by the homogeneous polynomial systems' analysis in the plane. In what follows we focus on the mixing model 2d dynamical system in the slightly perturbed form (20), a form which allows using a stabilization technique for finding a polynomial feedback in plane. We choose a simple variant for the values set of the integers k and q, in order to make easier testing the feasibility of the technique. Namely, the set $k = 0, q = 2$ was chosen. Thus, the following variant for the perturbed dynamical system is taken into account:

$$\begin{cases} \dot{x}_1 = & Gx_2 - x_1 + u \cdot (134x_1^2 - 58x_1x_2 + 75x_2^2) \\ \dot{x}_2 = & KGx_1 - x_2 + u \cdot (70x_1^2 - 30x_1x_2 + 125x_2^2), \end{cases} \tag{22}$$

where $-1 < K < 1$, $G \in \mathbb{R}$.

It is easy to observe that $deg(P_1) = deg(P_2) = 1, deg(Q_1) = deg(Q_2) = 2$ and the polynomials are:

$$P_1(x_1, x_2) = Gx_2 - x_1 \tag{23}$$

$$P_2(x_1, x_2) = KGx_1 - x_2 \tag{24}$$

$$Q_1(x_1, x_2) = (134x_1^2 - 58x_1x_2 + 75x_2^2) \tag{25}$$

$$Q_2(x_1, x_2) = (70x_1^2 - 30x_1x_2 + 125x_2^2). \tag{26}$$

Furthermore, we note that

1. The origin is a solution for the system (22);
2. The polynomials P_1, P_2, and Q_1, Q_2 are homogeneous of degree 1 and 2 respectively.

Thus we can apply the Theorem 1 and check if it is possible to find a polynomial feedback $u(x_1, x_2)$ for the system (22). Let us first construct the functions F, G, H. Following the relations (5), we find the functions as follows:

$$\begin{aligned} F(x_1, x_2) \quad = \quad & (-70 - 134KG)x_1^3 + (125G + 75)x_2^3 \\ & + (164 + 70G + 58KG)x_1^2x_2 \\ & - (183 + 30G + 75KG)x_1x_2^2 \end{aligned} \tag{27}$$

$$G(x_1, x_2) = 75x_2^3 - 70x_1^3 + 164x_1^2x_2 - 183x_1x_2^2 \tag{28}$$

$$H(x_1, x_2) = Gx_2^2 - KGx_1^2. \tag{29}$$

It is obvious from the above relations that the functions depend on the parameters K and G, $-1 < K < 1, \quad G \in \mathbb{R}$. Therefore it is more convenient to test the Theorem 1 for some basic non zero values for a point $M(m_1, m_2)$.

Let us choose the symmetric values $(1, -1)$ and $(-1, 1)$. Thus, we have two cases:

- a) $(m_1, m_2) = (1, -1)$;
- b) $(m_1, m_2) = (-1, 1)$.

In the case a), the calculus gives

$$F(1,-1) = -100G - 133KG, \quad G(1,-1) = -492.$$

The condition $F(M) \cdot G(M) > 0$ gives the inequality

$$492G(100 + 133K) > 0. \tag{30}$$

Combining the cases $G > 0$ and $G < 0$ (both feasible) and $-1 < K < 1$ we obtain the variant

$$G > 0, \quad K > -0.75 \tag{31}$$

which is feasible. In the case b) the calculus gives

$$F(-1,1) = 492 + 267KG + 225G, \quad G(-1,1) = 617 > 0.$$

The condition $F(M) \cdot G(M) > 0$ gives the inequality

$$G(225 + 267K) > -492. \tag{32}$$

This inequality is not feasible, because K must be in the strip $-1 < K < 1$. Thus, the first case, the case $(m_1, m_2) = (1, -1)$ remains a feasible case, in which the Theorem 1 works. Furthermore, we can construct for this case the function $\Phi(x_1, x_2)$ and then the polynomial feedback $u(x_1, x_2)$ following the formulas (8) and (9) respectively. We obtain $\Phi(x_1, x_2) = (-x_1 - x_2)^2$ and the calculus according to (9) gives the feedback law u as follows:

$$\Phi(x_1, x_2) = \frac{(1 + KG)x_1^2 + (1 - G)x_2^2 + 2x_1x_2}{75x_2^3 - 70x_1^3 + 164x_1^2x_2 - 308x_1x_2^2}, -1 < K < 1, G \in \mathbb{R}. \tag{33}$$

It is important to mention that G, KG are the mixing model's parameters, and the function $G(x_1, x_2)$ is a polynomial function, thus G has different significance in the two notations.

4 Conclusions

In this paper the stabilization of 2d dynamical systems in a polynomial form is taken into account. The 2d mixing model perturbed with homogeneous polynomials is considered. It is found a point $M = (1, -1)$ and corresponding a polynomial feedback law $u(x_1, x_2)$ in the form (33), which stabilizes the system (22) in some feasible conditions for the parameters. The conclusion is that the 2d mixing model in a convenient perturbed form can be stabilized. The feedback law (33) has an interesting form and depends on the parameters, therefore a next aim is to analyse the influence of parameters G and KG on the behaviour of $u(x_1, x_2)$. Also, testing some other polynomials Q_1, Q_2 on one hand, and analysing the existence of more points $M(m_1, m_2)$ for which the system can be stabilized, on the other hand, will bring useful information concerning the qualitative analysis of the systems arising from excitable media.

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