Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 17, No. 3/2025

# ALGEBRAIC AND GEOMETRIC ASPECTS OF BIPARTITE PLANAR GRAPHS*

Maurizio Imbesi† Monica La Barbiera‡

*Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary*

DOI 10.56082/annalsarscimath.2025.3.197

**Abstract**

Let $B_{2t}$ be a bipartite planar graph with an even number of regions. We are able to find bounds for the graded Betti numbers and the projective dimension of the quotient ring associated to the graph. We will also investigate the minimal vertex covers and the maximum matchings related to such a graph.

**Keywords:** planar graphs, standard algebraic invariants, vertex covers.
**MSC:** 05C99, 13D02, 13F20.

## Introduction

Let $G$ be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and $R = K[X_1, \dots, X_n]$ be the polynomial ring over a field $K$, with variables $X_i$ associated to vertices $v_i$ of $G$. The monomial ideal $I_G = (X_iX_j \mid \{v_i, v_j\}$ is an edge of $G) \subset R$ is said the edge ideal of $G$. In this paper we are interested to extract specific information about some invariants linked to the minimal graded resolution of $R/I_G$ when $G$ is the bipartite planar graph $B_{2t}$, $t \geq 1$ integer, with $2t$ the

*Accepted for publication on September 26, 2025

†maurizio.imbesi@unime.it, Department of Mathematical and Computer Sciences, Physical and Earth Sciences, University of Messina, Italy

‡monica.labarbiera@unict.it, Department of Electrical, Electronic and Computer Engineering, University of Catania, Italy

number of its regions. In [1], the $K$-algebra $K[B_{2t}]$ related to the graph $B_{2t}$ was studied using its geometry and the Hilbert function of $K[B_{2t}]$ of it was computed.

The paper is structured as follows. In Section 1 some preliminary notions about the planar graphs $B_{2t}$ are given. In Section 2 we study all the graded Betti numbers that appear in the minimal graded resolution of $R/\mathcal{I}$, where $\mathcal{I}$ is the edge ideal of $B_{2t}$, using some geometric properties of $B_{2t}$. The graded Betti numbers determine the rank of the free modules in the minimal graded resolution of $R/\mathcal{I}$ and in general it is not possible to give a generic formula to compute them. We are able to give bounds for them in terms of the number of the regions of $B_{2t}$. As a particular case, we study the second Betti number of degree 3 of $R/\mathcal{I}$ linked to the number of the regions of these planar graphs and give an explicit formula to compute it. The last two sections of the paper are devoted to consider important sets associated to a bipartite planar graph $G$: the minimal vertex covers and the maximum matchings of $G$. The study of the vertex covering of a graph is intensively examined among others in [6–9, 11]. It consists in finding a vertex cover of minimum cardinality, that is a minimal subset $\mathcal{A}$ of the vertex set of $G$ such that each edge of $G$ is incident with one vertex in $\mathcal{A}$. More precisely, we describe the minimal vertex covers of the bipartite planar graphs $B_{2t}$ and connect to the vertex covers of $B_{2t}$ some algebraic aspects such as dimension and height. There is a correspondence among the minimal vertex covers and the minimal primes of the edge ideal. If all minimal vertex covers have the same size, then the graph is unmixed. The unmixed bipartite graphs were characterized in [14], and some generalizations of them were given in [5]. We will verify that the planar graphs $B_{2t}$ are not unmixed. Furthermore, as algebraic topic, we will compute the dimension of $R/\mathcal{I}$ and establish bounds for the projective dimension of $R/\mathcal{I}$ by connecting the geometry of $B_{2t}$ with graph-theoretical properties. The problem of finding maximum matchings for the bipartite graphs $B_{2t}$ is to associate the geometry of $B_{2t}$ with the minimal vertex covers. We prove that the graphs $B_{2t}$, for $t$ odd, have perfect matchings of König type ( [12]), say a collection $e_1, \ldots, e_g$ of pairwise disjoint edges such that the union of the vertices in which $e_1, \ldots, e_g$ are incident is the vertex set of the graph and $g$ is the height of its edge ideal. Finally we give a complete description of these matchings.

# 1 Preliminary notions

Let $G$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$ which consists of pairs $\{v_i, v_j\}$ of adjacent vertices, for some $v_i, v_j \in V(G)$. A graph $G$ on vertices $v_1, \ldots, v_n$ is *complete* if there exists an edge for all pair $\{v_i, v_j\}$ of vertices of $G$. It is denoted by $K_n$. A graph $G$ is *bipartite* if its vertex set $V(G)$ can be partitioned into disjoint subsets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_m\}$, and any edge joins a vertex of $V_1$ to a vertex of $V_2$. A bipartite graph $G$ is said to be *complete* if every vertex of $V_1$ is adjacent to all the vertices of $V_2$. It is denoted by $K_{n,m}$.

**Definition 1.** *A graph $G$ is said* planar *if it has an embedding in the plane such that each pair of edges is intersected only in the common vertices.*

A planar graph is subdivided by its edges into plane regions.

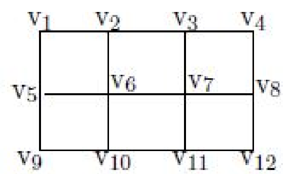
**Theorem 1** (see [4], Theorem 11.13)**.** *A graph is planar if and only if it has no subgraphs containing $K_5$ and $K_{3,3}$.* □

The complete graphs $K_5$ and $K_{3,3}$ are the minimal, not planar, graphs. In fact, it is not possible to represent these graphs in the plane so that the edges are not intersected only in the vertices.

Now, we consider the class of bipartite planar graphs $B_{2t}$ introduced in [1]. Let $B_{2t}$ be the planar graph with $r = 2t$ regions, $t \geq 1$ an integer, on vertex set $V(B_{2t}) = \{v_1, \ldots, v_{3t+3}\}$ and edge set $E(B_{2t}) = \{\{v_i, v_{i+1}\} | 1 \leq i \leq 3t+2, \quad i \neq t+1, 2t+2\} \cup \{\{v_i, v_{i+t+1}\} | 1 \leq i \leq 2t+2\}$. By Theorem 1, $B_{2t}$ is a planar graph, for all $t \geq 1$. Moreover, $B_{2t}$ is a bipartite planar graph: in fact, the vertex set of $B_{2t}$ can be partitioned into disjoint subsets $V_1$ and $V_2$, with $N = |V_1| + |V_2| = 3t+3$ and $|V_i|$ denotes the number of vertices of $V_i$ for $i = 1, 2$. Two cases occur:
1) If $t$ is even and $N = 3t+3$ is odd one has $V_1 = \{v_i \mid i \text{ odd}, 1 \leq i \leq 3t+3\}$ with $|V_1| = \frac{3t+4}{2}$ and $V_2 = \{v_i \mid i \text{ even}, 1 \leq i \leq 3t+3\}$ with $|V_2| = \frac{3t+2}{2}$.
2) If $t$ is odd and $N = 3t+3$ is even one has $V_1 = \{v_1, v_3, \ldots, v_t\} \cup \{v_{2+(t+1)}, v_{4+(t+1)}, \ldots, v_{t+1+(t+1)}\} \cup \{v_{1+(2t+2)}, v_{3+(2t+2)}, \ldots, v_{t+(2t+2)}\}$ and $V_2 = \{v_2, v_4, \ldots, v_{t+1}\} \cup \{v_{1+(t+1)}, v_{3+(t+1)}, \ldots, v_{t+(t+1)}\} \cup \{v_{2+(2t+2)}, v_{4+(2t+2)}, \ldots, v_{t+1+(2t+2)}\}$.

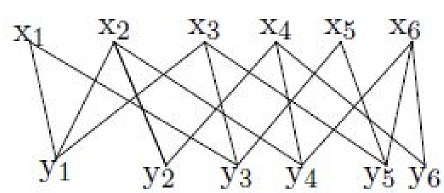
Note that $|\{v_1, v_3, v_5, \ldots, v_t\}| = |\{v_2, v_4, v_6, \ldots, v_{t+1}\}| = \frac{t+1}{2}$, hence one has $|V_1| = |V_2| = \frac{3t+3}{2}$. Then the graph $B_{2t}$ has vertex set $V(B_{2t}) = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, such that its edges join the vertices of $V_1$ to vertices of $V_2$ only, as follows from the definition of $E(B_{2t})$.

**Example 1.** *$G = B_6$, with $V(B_6) = \{v_1, \ldots, v_{12}\}$ and $E(B_6) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_8\}, \{v_9, v_{10}\}, \{v_{10}, v_{11}\}, \{v_{11}, v_{12}\}, \{v_1, v_5\}, \{v_2, v_6\}, \{v_3, v_7\}, \{v_4, v_8\}, \{v_5, v_9\}, \{v_6, v_{10}\}, \{v_7, v_{11}\}, \{v_8, v_{12}\}\}$.*




*$V(B_6)$ can be partitioned into disjoint subsets:*
*$V(B_6) = \{v_1, v_3, v_6, v_8, v_9, v_{11}\} \cup \{v_2, v_4, v_5, v_7, v_{10}, v_{12}\} = V_1 \cup V_2$.*
*If we rename $\{x_1, \ldots, x_6\}$ the vertices of $V_1$ and $\{y_1, \ldots, y_6\}$ the vertices of $V_2$, then the edge set can be written as:*

*$E(B_6) = \{\{x_1, y_1\}, \{x_2, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_4, y_4\}, \{x_5, y_5\}, \{x_6, y_5\}, \{x_6, y_6\}, \{x_1, y_3\}, \{x_3, y_1\}, \{x_2, y_4\}, \{x_4, y_2\}, \{x_5, y_3\}, \{x_3, y_5\}, \{x_6, y_4\}, \{x_4, y_6\}, \{x_3, y_4\}\}$.*




*The two pictures represent the same planar graph $B_6$.*

Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field $K$, with one variable $X_i$ for each vertex $v_i$ of $G$.

**Definition 2.** *We call* edge ideal *associated to a graph $G$ the ideal of $R$, denoted by $I_G$, which is generated by monomials of degree two, $X_iX_j$, on the variables $X_1, \ldots, X_n$, such that $\{v_i, v_j\} \in E(G)$, for $1 \leq i, j \leq n$.*

Bipartite graphs determine monomial ideals in the polynomial ring in two sets of variables $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$, where $n$ is the number of the vertices $x_1, \ldots, x_n$ in $V_1$ and $m$ is the number of the vertices $y_1, \ldots, y_m$ in $V_2$.

The edge ideal $I_G$ associated to a bipartite graph $G$ is the ideal of $R$ which is generated by the monomials of degree two, $X_iY_j$, on two disjoint sets of variables $X_1, \dots, X_n; Y_1, \dots, Y_m$, such that $\{x_i, y_j\} \in E(G)$, for $1 \le i \le n$ and $1 \le j \le m$.

In the following, we put $I_G = \mathcal{I}$ when $G = B_{2t}$.

## 2 Graded Betti numbers associated to $B_{2t}$

We are interested in finding bounds for the graded Betti numbers that appear in the minimal graded resolution of the edge ideal of $B_{2t}$, in particular we give upper bounds for them in terms of the number of the regions of $B_{2t}$.

**Definition 3.** *Let $G$ be a graph on vertex set $V(G)$. We call* induced subgraph *of $G$ the graph $H \subseteq G$ which has an edge between any two vertices of it if and only if there is an edge between them in $G$.*

**Proposition 1** (see [10], 4.1.1 Proposition)**.** *Let $G$ be a graph. If $H$ is an induced subgraph of $G$ on a subset of the vertices of $G$, then*

$$b_{i_j}(H) \le b_{i_j}(G), \quad \forall i, j\,,$$

*where $b_{i_j}(H)$ (resp. $b_{i_j}(G)$) are the graded Betti numbers associated to $H$ (resp. $G$).* □

**Proposition 2.** *Let $B_{2t}$ be the bipartite planar graph, $r = 2t$ be the number of its regions and $\mathcal{I}$ be its edge ideal. Let $b_{i_j}(B_{2t})$ be the graded Betti numbers in the minimal graded resolution of $R/\mathcal{I}$. Then:*

1) $b_{i_j}(B_{2t}) \le \sum_{\substack{k+l=i+1 \\ k,l \ge 1}} \binom{\frac{3r+12}{4}}{k}\binom{\frac{3r+4}{4}}{l}$, *if $t$ is even;*

2) $b_{i_j}(B_{2t}) \le \sum_{\substack{k+l=i+1 \\ k,l \ge 1}} \binom{\frac{3r+10}{4}}{k}\binom{\frac{3r+6}{4}}{l}$, *if $t$ is odd.*

*Proof.* $B_{2t}$ is a bipartite planar graph on two disjoint vertex sets $V_1 = \{x_1, \dots, x_n\}$ and $V_2 = \{y_1, \dots, y_m\}$, but it is not complete. Moreover, it is an induced subgraph of the complete bipartite graph on vertex sets $\overline{V_1} = \{x_1, \dots, x_{n+1}\}$ and $V_2 = \{y_1, \dots, y_m\}$.

1) If $t$ is even we have $|V_1| = n = \frac{3t+4}{2}$ and $|V_2| = m = \frac{3t+2}{2}$. $B_{2t}$ is an induced subgraph of the complete bipartite graph $K_{n+1,m}$, where $n + 1 = \frac{3t+6}{2}$ and $m = \frac{3t+2}{2}$, that is $V(B_{2t}) \subset V(K_{n+1,m})$ and $|E(B_{2t})| < |E(K_{n+1,m})|$. Then, by Proposition 1, we have: $b_{i_j}(B_{2t}) \le b_{i_j}(K_{n+1,m})$, where $b_{i_j}(K_{n+1,m})$ are the graded Betti numbers of $R/I(K_{n+1,m})$. By [10],

5.2.4 Theorem, we have: $b_{i_j}(K_{n+1,m}) = \sum_{\substack{k+l=i+1 \\ k,l\geq 1}} \binom{n+1}{k}\binom{m}{l}$.

It follows:

$$b_{i_j}(B_{2t}) \leq \sum_{\substack{k+l=i+1 \\ k,l\geq 1}} \binom{\frac{3t+6}{2}}{k}\binom{\frac{3t+2}{2}}{l}, \quad t = \frac{r}{2}\ .$$

2) If $t$ is odd we have $|V_1| = |V_2| = \frac{3t+3}{2}$. $B_{2t}$ is an induced subgraph of the complete bipartite graph $K_{n+1,m}$, where $n+1 = \frac{3t+5}{2}$ and $m = \frac{3t+3}{2}$, that is $V(B_{2t}) \subset V(K_{n+1,m})$ and $|E(B_{2t})| < |E(K_{n+1,m})|$. As before we obtain

$$b_{i_j}(B_{2t}) \leq \sum_{\substack{k+l=i+1 \\ k,l\geq 1}} \binom{\frac{3t+5}{2}}{k}\binom{\frac{3t+3}{2}}{l}, \quad t = \frac{r}{2}\ .$$

□

It is possible to express the second graded Betti number in degree 3 of $R/I_G$ in terms of graph theoretical properties for any graph $G$.

For a simple graph $G$ there exists the so-called *edge graph* $L(G)$ of $G$ (see [13]). It has vertex set equal to the edge set of $G$ and two vertices of $L(G)$ are adjacent whenever the corresponding edges of $G$ have one common vertex:

$$V(L(G)) = E(G) = \{f_1, \dots, f_q\}$$

$$E(L(G)) = \{(f_i, f_j) \mid f_i = \{v_i, v_j\}, \quad f_j = \{v_j, v_k\}, \quad i \neq j, \quad j \neq k\}.$$

If $G$ is a simple graph on vertices $v_1, \dots, v_n$ , then the number of edges of $L(G)$ is given by

$$|E(L(G))| = -|E(G)| + \sum_{i=1}^{n} \frac{\deg^2(v_i)}{2},$$

where $\deg(v_i)$ is the number of edges incident with $v_i$.

**Theorem 2** (see [2])**.** *Let $G$ be a graph and $I_G$ be its edge ideal. If*

$$\dots \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to R \to R/I_G \to 0$$

*is the minimal graded resolution of $R/I_G$ and $L(G)$ is the edge graph of $G$, then*

$$b = |E(L(G))| - N_3,$$

*where $N_3$ is the number of the triangles of $G$.* □

In particular, for $G = B_{2t}$ we can establish the following

**Theorem 3.** *Let $B_{2t}$ be the bipartite planar graph, $r = 2t$ be the number of its regions and $\mathcal{I}$ be its edge ideal. If*

$$\ldots \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to R \to R/\mathcal{I} \to 0$$

*is the minimal graded resolution of $R/\mathcal{I}$, then:*

1) $q = \frac{5}{2}r + 2$;

2) $b = 6r - 2$.

*Proof.* 1) $q = |E(B_{2t})| = |\{\{v_i, v_{i+1}\} : \ 1 \leq i \leq 3t+2, \ i \neq t+1, t+2\}| + |\{\{v_i, v_{i+t+1}\} : \ 1 \leq i \leq 2t+2\}| = (3t+2-2) + (2t+2) = 5t+2 = \frac{5}{2}r + 2$.
2) By Theorem 2, $b = |E(L(B_{2t}))| - N_3$, where $N_3 = 0$ because the graph is bipartite. One has $|E(L(B_{2t}))| = -|E(B_{2t})| + \sum_{i=1}^{N} \frac{\deg^2(v_i)}{2}$, where $N = 3t+3$. We observe that $B_{2t}$ has $N = 3(t+1)$ vertices representable in the plane on three horizontal lines and on each line there are $t+1$ vertices. In fact, the representation in the plane of $B_{2t}$ is a sequence of squares without chords arranged in 2 rows and $t$ columns. It follows that

$$\sum_{i=1}^{3t+3} \frac{\deg^2(v_i)}{2} = 4\left(\frac{2^2}{2}\right) + 2t\left(\frac{3^2}{2}\right) + (t-1)\left(\frac{4^2}{2}\right) = 17t = \frac{17}{2}r\,,$$

where $\deg(v_1) = \deg(v_{t+1}) = \deg(v_{2t+3}) = \deg(v_{3t+3}) = 2$, $\deg(v_i) = 3$ for $2 \leq i \leq t$, $i = t+2, 2t+2$ and $2t+4 \leq i \leq 3t+2$, $\deg(v_i) = 4$ for $t+3 \leq i \leq 2t+1$. Then $b = |E(L(B_{2t}))| = -\left(\frac{5}{2}r + 2\right) + \frac{17}{2}r = 6r - 2\,.$ □

## 3 Algebraic topics on minimal vertex covers of $B_{2t}$

**Definition 4.** *Let $G$ be a graph with vertex set $V(G)$. A subset $\mathcal{A} \subset V(G)$ is called a* minimal vertex cover *for $G$ if:*

(1) *each edge of $G$ is incident with one vertex in $\mathcal{A}$;*

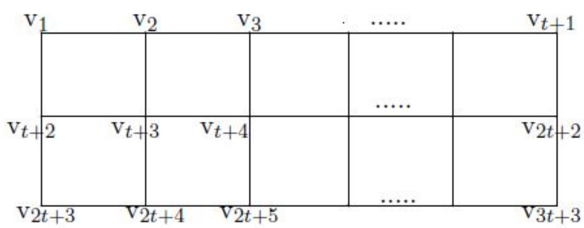
(2) *there is no proper subset of $\mathcal{A}$ with this property.*

If $\mathcal{A}$ satisfies condition (1) only, then $\mathcal{A}$ is called a *vertex cover* of $G$ and $\mathcal{A}$ is said to cover all the edges of $G$. The smallest number of vertices in any minimal vertex cover of $G$ is said *vertex covering number*. We denote it by $\alpha_0(G)$.

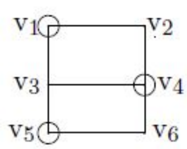
**Proposition 3.** *Let $B_{2t}$ be the bipartite planar graph with $r = 2t$ regions and $t \geq 1$. Then:*

$$\alpha_0(B_{2t}) = \begin{cases} \frac{3}{4}r + \frac{3}{2} & \textit{if } t \textit{ odd}, \\ \frac{3}{4}r + 1 & \textit{if } t \textit{ even}. \end{cases}$$

*Proof.* Let $V(B_{2t}) = \{v_1, \ldots, v_{3t+3}\}$ and $E(B_{2t}) = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq 3t+2,\ i \neq t+1, 2t+2, 3t+3\} \cup \{\{v_i, v_{i+t+1}\} | 1 \leq i \leq 2t+2\}$. Hence the representation of $B_{2t}$ in the plane is a sequence of squares without chords arranged in 2 rows and $t$ columns.




For $t = 1$ and $\alpha_0(B_2) = 3$, it is




and $\mathcal{A}(B_2) = \{v_1, v_4, v_5\}$, $\mathcal{A}'(B_2) = \{v_2, v_3, v_6\}$ are minimal vertex covers of $B_2$.

For $t > 1$ a minimal vertex cover of $\alpha_0(B_{2t})$ is given by adding to the minimal vertex cover of $B_2$ one vertex for each even column and two vertices for each odd column. Hence, if $t$ is odd

$$\alpha_0(B_{2t}) = \alpha_0(B_2) + 1\left(\frac{t-1}{2}\right) + 2\left(\frac{t-1}{2}\right) = \frac{3}{2}t + \frac{3}{2} = \frac{3}{4}r + \frac{3}{2},$$

where $\frac{t-1}{2}$ is the number of the even or odd columns in the graph for $t > 1$; if $t$ is even

$$\alpha_0(B_{2t}) = \alpha_0(B_2) + 1\left(\frac{t}{2}\right) + 2\left(\frac{t}{2} - 1\right) = \frac{3}{2}t + 1 = \frac{3}{4}r + 1,$$

where $\frac{t}{2}$ is the number of the even columns and $\frac{t}{2} - 1$ is the number of the odd columns in the graph for $t > 1$. □

**Proposition 4.** *Let $B_{2t}$ be the bipartite planar graph with $r = 2t$ regions and $t \geq 1$. Then the minimal vertex covers with cardinality $\alpha_0(B_{2t})$ are:*

$$\mathcal{A}(B_{2t}) = \begin{cases} V_1, V_2 & \text{if } t \text{ odd },\\ V_2 & \text{if } t \text{ even }.\end{cases}$$

*Proof.* If $t$ is odd, $\alpha_0(B_{2t}) = \frac{3t+3}{2}$, that is the cardinality of the vertex sets $V_1$ and $V_2$, where $V_1 = \{v_1, v_3, \ldots, v_t\} \cup \{v_{2+(t+1)}, v_{4+(t+1)}, \ldots, v_{t+1+(t+1)}\} \cup \{v_{1+(2t+2)}, v_{3+(2t+2)}, \ldots, v_{t+(2t+2)}\}$ and $V_2 = \{v_2, v_4, \ldots, v_{t+1}\} \cup \{v_{1+(t+1)}, v_{3+(t+1)}, \ldots, v_{t+(t+1)}\} \cup \{v_{2+(2t+2)}, v_{4+(2t+2)}, \ldots, v_{t+1+(2t+2)}\}$ are two minimal sets of vertices that cover all the edges of $B_{2t}$. If $t$ is even, $\alpha_0(B_{2t}) = \frac{3t+2}{2}$, that is the cardinality of the vertex set $V_2 = \{v_i \mid i \text{ even}, 1 \leq i \leq 3t{+}3\}$. $V_2$ is the only subset of vertices with cardinality $\alpha_0(B_{2t})$ that covers all the edges of $B_{2t}$. □

Now we consider some algebraic aspects linked to the minimal vertex covers.

An immediate consequence of Proposition 3 is the following

**Corollary 1.** *Let $\mathcal{I}$ be the edge ideal of $B_{2t}$ with $r = 2t$ regions. Then:*

$$ht(\mathcal{I}) = \begin{cases} \frac{3}{4}r + \frac{3}{2} & \text{if } t \text{ odd },\\ \frac{3}{4}r + 1 & \text{if } t \text{ even }.\end{cases}$$

*Proof.* It is known that the vertex covering number $\alpha_0(G)$ of a graph $G$ is equal to the height of the edge ideal of it, $\mathrm{ht}(I_G)$ (see [13], 6.1.18). Hence, the assertion follows from Proposition 3. □

**Proposition 5.** *Let $B_{2t}$ be the bipartite planar graph with $r = 2t$ regions, $t \geq 1$, and $\mathcal{I}$ be its edge ideal. Then:*

$$dim(R/\mathcal{I}) = \begin{cases} \frac{3}{4}r + \frac{3}{2} & \text{if } t \text{ odd },\\ \frac{3}{4}r + 2 & \text{if } t \text{ even }.\end{cases}$$

*Proof.* Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ and $\mathcal{I} \subset R$ be the edge ideal of $B_{2t}$ with $|V(B_{2t})| = n + m = 3t + 3$. By [13], 2.1.7, we have $\dim(R/\mathcal{I}) = \dim(R) - \mathrm{ht}(\mathcal{I})$ and, by [13], 6.1.18, $\mathrm{ht}(\mathcal{I}) = \alpha_0(B_{2t})$. Hence $\dim(R/\mathcal{I}) = (n + m) - \alpha_0(B_{2t}) = 3t + 3 - \alpha_0(B_{2t})$. Then, by Proposition 3, it follows:

1) $\dim(R/\mathcal{I}) = \frac{3}{2}r + 3 - (\frac{3}{4}r + \frac{3}{2}) = \frac{3}{4}r + \frac{3}{2}$, if $t$ is odd,

2) $\dim(R/\mathcal{I}) = \frac{3}{2}r + 3 - (\frac{3}{4}r + 1) = \frac{3}{4}r + 2$, if $t$ is even. □

**Proposition 6.** *Let $B_{2t}$ be the bipartite planar graph with $r = 2t$ regions and $\mathcal{I}$ be its edge ideal. Then, for the projective dimension of $R/\mathcal{I}$, we have:*
1) $\frac{3}{4}r + \frac{3}{2} < pd_R(R/\mathcal{I}) \leq \frac{3}{2}r + 3$, *if $t$ is odd;*
2) $\frac{3}{4}r + 1 < pd_R(R/\mathcal{I}) \leq \frac{3}{2}r + 3$, *if $t$ is even.*

*Proof.* For the lower bounds, by [13], Theorem 2.5.14, one has $\mathrm{pd}_R(R/\mathcal{I}) > ht(\mathcal{I})$. Hence, by [13], Proposition 6.1.18, it follows that $\mathrm{pd}_R(\mathcal{I}) > \alpha_0(B_{2t})$. Then the thesis follows from Proposition 3.
For the upper bounds, we observe that $B_{2t}$ is an induced subgraph of the complete bipartite graph $K_{n+1,m}$, where $n = \frac{3t+4}{2}$ and $m = \frac{3t+2}{2}$ as in Proposition 2. The projective dimension of a graph is affected by some simple transformations such as deleting some edges. So, as a consequence of [10], Proposition 4.1.3, we have $\mathrm{pd}_R(R/\mathcal{I}) \leq \mathrm{pd}_R(R/I(K_{n+1,m})) = n+1+m-1 = n + m$ (see [10], Proposition 4.2.9), with $n + m = 3t + 3 = \frac{3}{2}r + 3$. □

Finally, we recall the one to one correspondence among the minimal vertex covers of $G$ and minimal primes of $I_G$. In fact, $\wp$ is a minimal prime ideal of $I_G$ if and only if $\wp = (\mathcal{A})$ for some minimal vertex cover $\mathcal{A}$ of $G$ (see [13], 6.1.16). Thus the primary decomposition of the edge ideal of $G$ is given by $I_G = (\mathcal{A}_1) \cap \cdots \cap (\mathcal{A}_p)$, where $\mathcal{A}_1, \ldots, \mathcal{A}_p$ are the minimal vertex covers of $G$. If all the minimal vertex covers of $G$ have the same cardinality, $G$ is said an *unmixed* graph. In order to study the unmixedness of $B_{2t}$ we recall the following result.

**Proposition 7** (see [14]). *Let $G$ be a bipartite graph with no isolated vertices. Then $G$ is unmixed if and only if there is a bipartition $V_1 = \{x_1, \ldots, x_m\}$, $V_2 = \{y_1, \ldots, y_m\}$ of $G$ such that:*

1) $\{x_i, y_i\} \in E(G)$ *for all $i$;*
2) *if* $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$, *then* $\{x_i, y_k\} \in E(G)$, *$i, j, k$ distinct.* □

**Theorem 4.** *$B_{2t}$ is not unmixed, for all $t > 0$.*

*Proof.* If $t$ is odd, using the characterization of unmixed bipartite graphs as in the Proposition 7, it is enough to verify that, if $\{x_i, y_j\}, \{x_j, y_k\} \in E(B_{2t})$, then $\{x_i, y_k\} \notin E(B_{2t})$. Let

$$V_1 = \{v_1, v_3, \ldots, v_t\} \cup \{v_{2+(t+1)}, v_{4+(t+1)}, \ldots, v_{t+1+(t+1)}\}$$
$$\cup \{v_{1+(2t+2)}, v_{3+(2t+2)}, \ldots, v_{t+(2t+2)}\}$$

and

$$V_2 = \{v_2, v_4, \ldots, v_{t+1}\} \cup \{v_{1+(t+1)}, v_{3+(t+1)}, \ldots, v_{t+(t+1)}\}$$

$$\cup\{v_{2+(2t+2)}, v_{4+(2t+2)}, \dots, v_{t+1+(2t+2)}\}$$

the two disjoint vertex sets of $B_{2t}$. Replacing with $\{x_1, \dots, x_{\frac{3t+3}{2}}\}$ the vertices of $V_1$ and with $\{y_1, \dots, y_{\frac{3t+3}{2}}\}$ the vertices of $V_2$, we have $v_1 = x_1$, $v_{1+(t+1)} = y_{\frac{t+1}{2}+1}$, $v_{2+(t+1)} = x_{\frac{t+1}{2}+1}$, $v_{3+(t+1)} = y_{\frac{t+1}{2}+2}$. Then

$$\{x_1, y_{\frac{t+1}{2}+1}\}, \{x_{\frac{t+1}{2}+1}, y_{\frac{t+1}{2}+2}\} \in E(B_{2t}),$$

but $\{x_1, y_{\frac{t+1}{2}+2}\} \notin E(B_{2t})$. If $t$ is even, it is sufficient to observe that $V_1$ and $V_2$ are two minimal vertex covers with $|V_1| > |V_2|$. Hence, $B_{2t}$ is not unmixed. □

## 4 Perfect matchings of $B_{2t}$

Let $G$ be a graph. A minimal vertex cover $\mathcal{A}$ of $G$ is linked to the *set of the independent edges*. The edges of $G$ that have no common vertices are called *independent edges*. The *independence number* of a graph $G$, denoted by $\beta_1(G)$, is the maximum number of its independent edges.

**Definition 5.** *A* matching *of $G$ is a set $\mathcal{M}$ of independent edges.*

**Definition 6.** *$G$ has a* perfect matching *if it has an even number of vertices and there is a set of independent edges covering all the vertices.*

This means that there is a pairing off of all the vertices of $G$.

**Definition 7.** *A* maximum matching *of $G$ is a matching $\mathcal{M}$ such that every other matching $\mathcal{M}'$ satisfies $|\mathcal{M}'| < |\mathcal{M}|$. In this case $|\mathcal{M}| = \beta_1(G)$.*

**Remark 1.** *Let $\mathcal{M}$ be a maximum matching and $\mathcal{A}$ a minimal cover of a graph $G$. Note that each edge of $\mathcal{M}$ must be covered by at least one vertex of $\mathcal{A}$ and each vertex of $\mathcal{A}$ can cover at most one edge of $\mathcal{M}$. It follows: $\beta_1(G) \leq \alpha_0(G)$.*

**Definition 8.** *A* perfect matching of König type *of $G$ is a collection $e_1, \dots, e_g$ of pairwise disjoint edges such that the union of the vertices in which $e_1, \dots, e_g$ are incident is the vertex set of $G$ and $g$ is equal to the height of $I_G$.*

**Remark 2.** *A graph $G$ satisfies the König property if the maximum number of independent edges of $G$ equals the height of $I_G$. Hence a graph with a perfect matching of König type has the König property. In [3] it is proved that the converse is true for unmixed graphs.*

We are interested in analyzing bipartite matching problem, namely in finding a matching with the maximum number of edges. Clearly, the size of any matching is at most the size of any vertex cover. This follows from the fact that, given any matching $\mathcal{M}$, a vertex cover $\mathcal{A}$ must contain at least one of the vertices of each edge in $\mathcal{M}$. The maximum size of a matching is at most the minimal cardinality of a vertex cover.

**Proposition 8** (see [13], Proposition 6.1.7)**.** *For any bipartite graph $G$, the size of a maximum matching is equal to the size of a minimal vertex cover, that is $\beta_1(G) = \alpha_0(G)$.* □

**Theorem 5.** *Let $B_{2t}$ be the bipartite planar graph with $r = 2t$ regions, $t$ odd. Each maximum matching is a perfect matching of cardinality $\frac{3}{4}r + \frac{3}{2}$.*

*Proof.* $B_{2t}$ is a bipartite graph, then, by Proposition 8, $\beta_1(B_{2t}) = \alpha_0(B_{2t})$. Hence any vertex of the minimal vertex cover is incident upon independent edges. Then $B_{2t}$ has maximum matching with cardinality $\beta_1(B_{2t}) = |\mathcal{M}(B_{2t})| = |V_1| = |V_2| = \frac{3}{4}r + \frac{3}{2}$, $r = 2t$. Moreover $B_{2t}$ has an even number of vertices and $|V_1| = |V_2|$, this means that there is a pairing off of all the vertices of $B_{2t}$. It follows that each maximum matching is a perfect matching. □

**Theorem 6.** *Let $B_{2t}$ be the bipartite planar graph with $r = 2t$ regions, $t$ odd. $B_{2t}$ has perfect matching of König type.*

*Proof.*

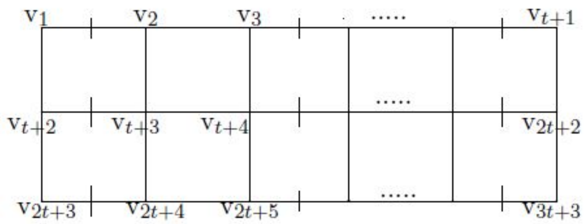
$$V_1 = \{v_1, v_3, \ldots, v_t\} \cup \{v_{2+(t+1)}, v_{4+(t+1)}, \ldots, v_{t+1+(t+1)}\} \\ \cup \{v_{1+(2t+2)}, v_{3+(2t+2)}, \ldots, v_{t+(2t+2)}\}$$

and

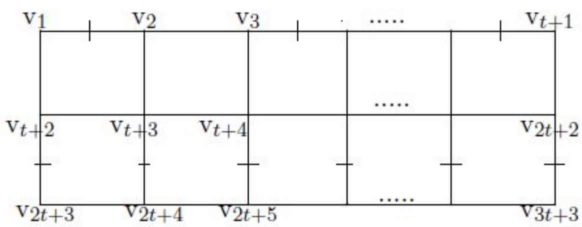
$$V_2 = \{v_2, v_4, \ldots, v_{t+1}\} \cup \{v_{1+(t+1)}, v_{3+(t+1)}, \ldots, v_{t+(t+1)}\} \\ \cup \{v_{2+(2t+2)}, v_{4+(2t+2)}, \ldots, v_{t+1+(2t+2)}\}$$

are minimal vertex covers of $B_{2t}$ with cardinality $\alpha_0(B_{2t}) = \frac{3}{4}r + \frac{3}{2}$. Note that $\beta_1(B_{2t}) = \alpha_0(B_{2t}) = \frac{3}{4}r + \frac{3}{2}$ and any vertex of the minimal vertex cover is incident upon independent edges. Hence, by the geometry of the planar graph $B_{2t}$, we obtain the following maximum matchings:

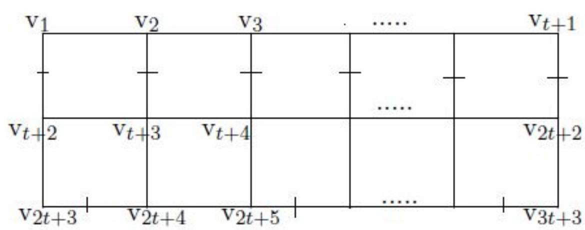
- $\mathcal{M} = \{\{v_{i-1}, v_i\} \mid i \text{ even},\ 2 \le i \le 3t + 3\}$.




- $\mathcal{M} = \{\{v_{i-1}, v_i\} \mid i \text{ even}, \;\; 2 \leq i \leq t+1\} \cup \{\{v_{i+t}, v_{i+2t+1}\} \mid 2 \leq i \leq t+2\}$.




- $\mathcal{M} = \{\{v_i, v_{i+t+1}\} \mid 1 \leq i \leq t+1\} \cup \{\{v_{i-1}, v_i\} \mid i \text{ even}, 2t+4 \leq i \leq 3t+3\}$.




The other perfect matchings of König type are obtained by the previous schemes through different combinations of the columns in the representation of the graph. In all the cases $\mathcal{M}$ is a matching such that $|\mathcal{M}| = \alpha_0(B_{2t}) = ht(I(B_{2t}))$ and the union of the vertices in which the edges of $\mathcal{M}$ are incident coincides with the vertex set of $B_{2t}$. Hence $B_{2t}$ has perfect matchings of König type. $\square$

**Remark 3.** *Each maximum matching $\mathcal{M}(B_{2t})$ is a complete matching from $V_2$ to $V_1$ (being $|V_2| < |V_1|$). This means that $\mathcal{M}(B_{2t})$ covers each vertex of $V_2$, but not all the vertices of $V_1$; in fact, $|\mathcal{M}(B_{2t})| = \beta_1(B_{2t}) = |V_2| = \frac{3}{4}r+1$.*

**Acknowledgements.** The research that led to the paper was partially supported by a grant of the group GNSAGA of INdAM, Italy.
Special thanks to the late unforgettable Prof. Tania Restuccia for the helpful discussions during the drafting of the paper.

## References

[1] L.R. Doering and T. Gunston, Algebras arising from bipartite planar graphs, *Commun. Algebra* 24 (1996), 3589-3598.

[2] S. Eliahou, and R.H. Villarreal, The second Betti number of an edge ideal, *Aportaciones Matematicas, Ser. Comunicaciones* 25 (1999), 101-106.

[3] I. Gitler, E. Reyes and R.H. Villarreal, Blowup algebras of ideals of vertex covers of bipartite graphs, *Contemp. Math.* 376 (2000), 273-279.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1972.

[5] M. Imbesi and M. La Barbiera, Algebraic properties of non-squarefree graph ideals, *Math. Rep.* 15 (2013), 107-113.

[6] M. Imbesi and M. La Barbiera, Monomial ideals of graphs with loops, *Anal. Ştiinţ. Univ. Al. I Cuza Iaşi (S.N.) Mat.* 60 (2014), 321-336.

[7] M. Imbesi and M. La Barbiera, On a class of vertex cover ideals, *Anal. Ştiinţ. Univ. Al. I Cuza Iaşi (S.N.) Mat.* 61 (2015), 319-326.

[8] M. Imbesi and M. La Barbiera, Vertex covers in graphs with loops, *Turk. J. Math.* 42 (2018), 1192-1202.

[9] M. Imbesi, M. La Barbiera and P.L. Staglianò, On generalized graph ideals of complete bipartite graphs, *J. Ramanujan Math. Soc.* 32 (2017), 121-133.

[10] S. Jacques, *Betti Numbers of Graph Ideals*, PhD Thesis, Univ. Sheffield, 2004.

[11] M. La Barbiera, A note on unmixed ideals of Veronese bi-type, *Turk. J. Math.* 37 (2013), 1-7.

[12] S. Morey, E. Reyes and R.H. Villarreal, Cohen-Macaulay, shellable and unmixed clutters with a perfect matching of König type, *J. Pure Appl. Algebra* 212 (2008), 1770-1786.

[13] R.H. Villarreal, *Monomial Algebras, $2^{nd}$ edit.*, Chapman and Hall/CRC, New York, 2015.

[14] R.H. Villarreal, Unmixed bipartite graphs, *Revista Colombiana de Matematicas* 41 (2007), 393-395.