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SIMILARITY SOLUTIONS AND NONLINEAR WAVE PROPAGATION IN A MULTI-TEMPERATURE GAS MIXTURE*

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

A set of field equations for a multi-temperature gas mixture in spherical symmetry is considered within the framework of Rational Extended Thermodynamics. The invariance with respect to stretching group of transformations is investigated and the associated canonical variables are introduced. This makes it possible to write the system in autonomous form. Special similarity solutions are determined and the propagation of weak discontinuities in a non-constant state of the original system is studied. Finally, the evolution of the discontinuity is illustrated for a mixture of Helium and Argon.

Keywords: acceleration waves, rational extended thermodynamics, similarity solution, rarefied gas mixture.

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1 Introduction

Rational Extended Thermodynamics (RET) [28, 31, 32] is a non-equilibrium theory that extends the set of field variables beyond the classical ones mass density, velocity, and temperature by including additional non-equilibrium quantities such as the stress tensor, dynamic pressure, heat flux, and other non-equilibrium fields. The field equations are balance laws, closed by local and instantaneous constitutive relations. The constitutive functions are determined by means of universal physical principles, such as the entropy principle (existence of an entropy inequality and concavity of the entropy density) and the principle of relativity. The exploitation of the entropy principle is made by use of the Lagrange multipliers (main fields). It has been shown that in terms of these main fields the equation system is symmetric hyperbolic. Hyperbolicity guarantees finite speeds of propagation, while symmetric hyperbolic systems imply well-posedness of Cauchy problems (i.e. existence, uniqueness, and continuous dependence on the data).

RET was first applied to monoatomic gases [28], yielding many interesting results. Recently, it has been generalized to dense and rarefied polyatomic gases both in the classical [31, 32] and in the relativistic framework [29, 2]. Rational Extended Thermodynamics yielded significant results also for gas mixture, see the recent books [31, 32]. In particular, it is remarkable that RET is able to describe the thermal-diffusion effect, as shown in [4] and in the reference therein. In the last decade, different materials are described inside Rational Extended Thermodynamic theory: metal electrons [7], quantum systems [34], graphene [35], biological models [6, 17, 18], nanofluids [9], gas bubbles [14, 15] and many others.

In the case of Mixture, RET moves in parallel with the kinetic theory with similar results. In 1967, Goldman and Sirovich [23] introduced a 13-moment model of gas mixtures with the multi-temperature (MT) assumption, which considers a different temperature field for each mixture constituent. After then, many authors worked almost independently on the same model for Maxwell molecules. We recall among the others the paper by Tsendin [36], the one by Goebel, Harris and Johnson [22], the model derivation by Kremer [25], the most recent work by Bisi, Groppi and Spiga [11]. Recently, the model for MT-gas mixture obtained in [22] has been studied in [3] showing some interesting mathematical and physical properties.

In this paper, the equations of RET applied to a binary multi-temperature mixture [22] are used to describe the propagation of acceleration waves in a non-constant state. The study of propagation of acceleration waves or weak discontinuities is a classical problem in literature, see for example

[37, 12, 13]. It describes a disturbance that propagates in an unperturbed state through which all field variables are continuous, while the first derivatives have a jump. The aim is to identify a critical wave amplitude below which the acceleration wave does not degenerate into a shock. From a mathematical point of view, this problem found its natural application to system of hyperbolic equations, like those obtained in the context of Rational Extended Thermodynamics. Recent studies of accelerations waves in models described by RET can be found in chemotaxis [6], gas-bubbles [14], mixtures [3], granular gasses [8], pit corrosion [30] and many others.

In 2000, the field equations of Rational Extended Thermodynamics for a rarefied gas were first formulated in spherical symmetry [20], and later also in cylindrical symmetry [27]. The study of how the solution depends on radial symmetry has provided many interesting results, prompting various studies in this direction. The heat transfer problem has also been investigated under different conditions and symmetries (see for example [5, 10] and the references therein). The aim of this paper is to study the propagation of acceleration waves in a non-constant state in a binary multi-temperature mixture described by Rational Extended Thermodynamics in spherical symmetry. The numerical solution is shown and the results are similar to the planar case presented in [3].

The structure of this article is as follows: in Section 2, the field equations are recalled and the dimensionless form are introduced. In Section 3, we determine the conditions for the existence of similarity solutions. Finally, in Section 4, the propagation of acceleration waves is studied, and a particular solution is presented.

2 Field equations

In this paper, we consider the field equations for a binary gas-mixture of classical ideal monatomic gases derived in [22] using the 13-moment Grad distribution function. These equations are written in spherical coordinates and the fields are assumed to depend only on the radial coordinate r and on the time t. With these assumptions, the set of the ten field equations for the ten fields, that are the densities ρ_α, velocities v_α, pressures p_α, deviatoric part of the stress tensors σ_α and heat fluxes q_α of both components ($\alpha =$

1, 2), assumes the form

$$
\begin{aligned}
&\frac{\partial \rho_\alpha}{\partial t} + \rho_\alpha \frac{\partial v_\alpha}{\partial r} + v_\alpha \frac{\partial \rho_\alpha}{\partial r} + \frac{2}{r}\rho_\alpha v_\alpha = 0,\\
&\rho_\alpha \frac{\partial v_\alpha}{\partial t} + \rho_\alpha v_\alpha \frac{\partial v_\alpha}{\partial r} + \frac{\partial \sigma_\alpha}{\partial r} + \frac{\partial p_\alpha}{\partial r} + \frac{3}{r}\sigma_\alpha = \Sigma_\alpha,\\
&\frac{\partial p_\alpha}{\partial t} + v_\alpha \frac{\partial p_\alpha}{\partial r} + \frac{2}{3}\frac{\partial q_\alpha}{\partial r} + \frac{5}{3}p_\alpha \frac{\partial v_\alpha}{\partial r} + \frac{2}{3}\sigma_\alpha \frac{\partial v_\alpha}{\partial r} +\\
&\qquad + \frac{10}{3}\frac{p_\alpha v_\alpha}{r} + \frac{4}{3}\frac{q_\alpha}{r} - \frac{2}{3}\frac{\sigma_\alpha v_\alpha}{r} = \frac{U_\alpha}{3},\\
&\frac{\partial \sigma_\alpha}{\partial t} + v_\alpha \frac{\partial \sigma_\alpha}{\partial r} + \frac{8}{15}\frac{\partial q_\alpha}{\partial r} + \frac{4}{3}p_\alpha \frac{\partial v_\alpha}{\partial r} + \frac{7}{3}\sigma_\alpha \frac{\partial v_\alpha}{\partial r} +\\
&\qquad - \frac{8}{15}\frac{q_\alpha}{r} + \frac{8}{3}\sigma_\alpha \frac{v_\alpha}{r} - \frac{4}{3}p_\alpha \frac{v_\alpha}{r} = Z_\alpha,\\
&\frac{\partial q_\alpha}{\partial t} + v_\alpha \frac{\partial q_\alpha}{\partial r} + \frac{16}{5}q_\alpha \frac{\partial v_\alpha}{\partial r} + \frac{p_\alpha - \sigma_\alpha}{\rho_\alpha}\frac{\partial \sigma_\alpha}{\partial r} + \frac{5}{2}\frac{p_\alpha + \sigma_\alpha}{\rho_\alpha}\frac{\partial p_\alpha}{\partial r} +\\
&\qquad - \frac{p_\alpha}{\rho_\alpha^2}\left(\frac{7\sigma_\alpha}{2} + \frac{5p_\alpha}{2}\right)\frac{\partial \rho_\alpha}{\partial r} + \frac{3}{r}\frac{p_\alpha - \sigma_\alpha}{\rho_\alpha}\sigma_\alpha + \frac{14}{5}\frac{q_\alpha}{r}v_\alpha = W_\alpha.
\end{aligned}
\tag{1}
$$

Equation $(1)_1$ represents the conservation law of mass for each constituent. The production term vanishes since mixtures of monatomic gases don't react chemically.

Equations $(1)_{2-5}$ follow from the balance laws for momentum, energy, stress tensor and heat flux for the α-constituent. In this paper, we assume that their production terms, Σ_α, U_α, Z_α and W_α, are the product of some functions of the two temperatures T_α and the Maxwellian productions, obtained in [22]. So they read (for $\beta \neq \alpha$)

$$
\begin{aligned}
&\Sigma_\alpha = f_1(T_1, T_2)\, C\,(v_\beta - v_\alpha),\\
&U_\alpha = f_2(T_1, T_2)\, D\left[3k_B(T_\beta - T_\alpha) + m_\beta (v_2 - v_1)^2\right],\\
&Z_\alpha = f_3(T_1, T_2)\left[-E_\alpha \sigma_\alpha + G_\alpha \sigma_\beta + \frac{2}{3}\rho_\beta G_\alpha (v_2 - v_1)^2\right],\\
&W_\alpha = f_4(T_1, T_2)\left\{-H_\alpha q_\alpha + I_\alpha q_\beta + \right.\\
&\left. + \rho_\beta \frac{v_\beta - v_\alpha}{2} I_\alpha \left[\frac{5k_B(T_\beta - T_\alpha)}{m_\beta} + (v_2 - v_1)^2\right] + (v_\beta - v_\alpha)(L_\alpha \sigma_\alpha + I_\alpha \sigma_\beta)\right\},
\end{aligned}
\tag{2}
$$

where k_B represents the Boltzman constant, m_α the molecular mass and we have

$$\begin{aligned}
&C=\tfrac{\nu_1^{12}\rho_1\rho_2}{m_1+m_2}, && D=\tfrac{2C}{m_1+m_2},\\
&T_\alpha=\tfrac{p_\alpha}{\frac{k_B}{m_\alpha}\rho_\alpha}, && E_\alpha=\tfrac{3\nu_2^{\alpha\alpha}\rho_\alpha}{4m_\alpha}+\tfrac{\left(4m_\alpha\nu_1^{12}+3m_\beta\nu_2^{12}\right)\rho_\beta}{2(m_1+m_2)^2},\\
&G_\alpha=\tfrac{m_\beta\left(4\nu_1^{12}-3\nu_2^{12}\right)\rho_\alpha}{2(m_1+m_2)^2}, && H_\alpha=\tfrac{\nu_2^{\alpha\alpha}\rho_\alpha}{2m_\alpha}+\tfrac{\left(\left(3m_\alpha^2+m_\beta^2\right)\nu_1^{12}+2m_1m_2\nu_2^{12}\right)\rho_\beta}{(m_1+m_2)^3},\\
&I_\alpha=\tfrac{2m_\beta^2\left(2\nu_1^{12}-\nu_2^{12}\right)\rho_\alpha}{(m_1+m_2)^3}, && L_\alpha=\tfrac{-8m_\alpha m_\beta\nu_1^{12}+\left(m_\alpha-3m_\beta\right)m_\beta\nu_2^{12}}{2(m_1+m_2)^3}\rho_\beta.
\end{aligned} \tag{3}$$

The coefficients $\nu_1^{\alpha\gamma}$ and $\nu_2^{\alpha\gamma}$ are characteristic of the Maxwellian molecules. They are given by

$$\nu_1^{\alpha\gamma}=2\pi\mathcal{A}_1(5)\sqrt{\chi_{\alpha\gamma}}\sqrt{\tfrac{m_\alpha+m_\gamma}{m_\alpha m_\gamma}},\quad \nu_2^{\alpha\gamma}=\tfrac{\mathcal{A}_2(5)}{\mathcal{A}_1(5)}\nu_1^{\alpha\gamma}, \tag{4}$$

where $\mathcal{A}_1(5)=0.422$, $\mathcal{A}_2(5)=0.436$ (see [16]), while $\chi_{\alpha\gamma}$ denote the Maxwell force constants. The response functions $f_i\,(i=1,\dots,4)$ are related to the relaxation times and, in turn, to viscosity and heat conductivity coefficients which we assume dependent on both temperatures of the mixture. See [22, 3] for more details.

Next, for later convenience, let us introduce the following dimensionless quantities and parameters:

$$\begin{aligned}
&\hat r=\tfrac{r}{L}, && \hat t=\tfrac{t}{\Upsilon}, && \hat\rho_\alpha=\tfrac{\rho_\alpha}{p_\alpha^0}\tfrac{k_B}{m_\alpha}T_\alpha^0,\\
&\hat T_\alpha=\tfrac{T_\alpha}{T_\alpha^0}, && \hat v_\alpha=\tfrac{v_\alpha}{c_0}, && \hat q_\alpha=\tfrac{q_\alpha}{p_\alpha^0c_0},\\
&\varkappa_\alpha=\tfrac{k_B}{m_\alpha}\tfrac{T_\alpha^0}{c_0^2}, && \hat p_\alpha=\tfrac{p_\alpha}{p_\alpha^0}, && \hat\sigma_\alpha=\tfrac{\sigma_\alpha}{p_\alpha^0},
\end{aligned} \tag{5}$$

where ρ_α^0, T_α^0, $p_\alpha^0=k_B/m_\alpha T_\alpha^0$ represent some constant equilibrium density, temperature and pressure of the α-constituent, furthermore

$$L=c_0\Upsilon,\quad c_0=\sqrt{\tfrac{5}{3}\tfrac{\frac{k_B}{m_1}\rho_1^0T_1^0+\frac{k_B}{m_2}\rho_2^0T_2^0}{\rho_1^0+\rho_2^0}},\quad \Upsilon=\tfrac{1}{\nu_1^{12}}\tfrac{m_1+m_2}{\rho_1^0+\rho_2^0}, \tag{6}$$

L is the reference length, c_0 denotes the speed of sound in the mixture and Υ has the dimension of a time and it is defined in terms of the Maxwellian coefficient of the mixture. By insertion of the dimensionless quantities (5) into the field equations, the following dimensionless equations are obtained

$$
\begin{aligned}
&\frac{\partial\hat{\rho}_\alpha}{\partial\hat{t}}+\hat{\rho}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{r}}+\hat{v}_\alpha\frac{\partial\hat{\rho}_\alpha}{\partial\hat{r}}+\frac{2}{\hat{r}}\hat{\rho}_\alpha\hat{v}_\alpha=0,\\
&\hat{\rho}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{t}}+\hat{\rho}_\alpha\hat{v}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{r}}+\varkappa_\alpha\left[\frac{\partial\hat{\sigma}_\alpha}{\partial\hat{r}}+\frac{\partial\hat{p}_\alpha}{\partial\hat{r}}+\frac{3}{\hat{r}}\hat{\sigma}_\alpha\right]=\hat{\Sigma}_\alpha,\\
&\frac{\partial\hat{p}_\alpha}{\partial\hat{t}}+\hat{v}_\alpha\frac{\partial\hat{p}_\alpha}{\partial\hat{r}}+\frac{2}{3}\frac{\partial\hat{q}_\alpha}{\partial\hat{r}}+\frac{5}{3}\hat{p}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{r}}+\frac{2}{3}\hat{\sigma}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{r}}+\\
&\qquad+\frac{10}{3}\frac{\hat{p}_\alpha\hat{v}_\alpha}{\hat{r}}+\frac{4}{3}\frac{\hat{q}_\alpha}{\hat{r}}-\frac{2}{3}\frac{\hat{\sigma}_\alpha\hat{v}_\alpha}{\hat{r}}=\frac{\hat{U}_\alpha}{3},\\
&\frac{\partial\hat{\sigma}_\alpha}{\partial\hat{t}}+\hat{v}_\alpha\frac{\partial\hat{\sigma}_\alpha}{\partial\hat{r}}+\frac{8}{15}\frac{\partial\hat{q}_\alpha}{\partial\hat{r}}+\frac{4}{3}\hat{p}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{r}}+\frac{7}{3}\hat{\sigma}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{r}}+\\
&\qquad-\frac{8}{15}\frac{\hat{q}_\alpha}{\hat{r}}+\frac{8}{3}\hat{\sigma}_\alpha\frac{\hat{v}_\alpha}{\hat{r}}-\frac{4}{3}\hat{p}_\alpha\frac{\hat{v}_\alpha}{\hat{r}}=\hat{Z}_\alpha,\\
&\frac{\partial\hat{q}_\alpha}{\partial\hat{t}}+\hat{v}_\alpha\frac{\partial\hat{q}_\alpha}{\partial\hat{r}}+\frac{16}{5}\hat{q}_\alpha\frac{\partial\hat{v}_\alpha}{\partial\hat{r}}+\varkappa_\alpha\left[\frac{\hat{p}_\alpha-\hat{\sigma}_\alpha}{\hat{\rho}_\alpha}\frac{\partial\hat{\sigma}_\alpha}{\partial\hat{r}}+\frac{5}{2}\frac{\hat{p}_\alpha+\hat{\sigma}_\alpha}{\hat{\rho}_\alpha}\frac{\partial\hat{p}_\alpha}{\partial\hat{r}}+\right.\\
&\qquad\left.-\frac{\hat{p}_\alpha}{\hat{\rho}_\alpha^2}\left(\frac{7\hat{\sigma}_\alpha}{2}+\frac{5\hat{p}_\alpha}{2}\right)\frac{\partial\hat{\rho}_\alpha}{\partial\hat{r}}+\frac{3}{\hat{r}}\frac{\hat{p}_\alpha-\hat{\sigma}_\alpha}{\hat{\rho}_\alpha}\hat{\sigma}_\alpha\right]+\frac{14}{5}\frac{\hat{q}_\alpha}{\hat{r}}\hat{v}_\alpha=\hat{W}_\alpha,
\end{aligned}
\tag{7}
$$

with the dimensionless productions

$$
\begin{aligned}
&\hat{\Sigma}_\alpha=\hat{f}_1\left(\hat{T}_1,\hat{T}_2\right)\hat{C}\left(\hat{v}_\beta-\hat{v}_\alpha\right),\\
&\hat{U}_\alpha=\hat{f}_2\left(\hat{T}_1,\hat{T}_2\right)\hat{D}\left[3\chi_\beta\left(\hat{T}_\beta-\hat{T}_\alpha\right)+\left(\hat{v}_2-\hat{v}_1\right)^2\right],\\
&\hat{Z}_\alpha=\hat{f}_3\left(\hat{T}_1,\hat{T}_2\right)\left[-\hat{E}_\alpha\hat{\sigma}_\alpha+\hat{G}_\alpha\hat{\sigma}_\beta+\frac{2}{3}\frac{\hat{\rho}_\beta}{\varkappa_\beta}\hat{G}_\alpha\left(\hat{v}_2-\hat{v}_1\right)^2\right],\\
&\hat{W}_\alpha=\hat{f}_4\left(\hat{T}_1,\hat{T}_2\right)\left\{-\hat{H}_\alpha\hat{q}_\alpha+\hat{I}_\alpha\hat{q}_\beta+\right.\\
&\left.+\hat{\rho}_\beta\frac{\hat{v}_\beta-\hat{v}_\alpha}{2}\hat{I}_\alpha\left[5\left(\hat{T}_\beta-\hat{T}_\alpha\right)+\frac{\left(\hat{v}_2-\hat{v}_1\right)^2}{\chi_\beta}\right]+\left(\hat{v}_\beta-\hat{v}_\alpha\right)\left(\hat{L}_\alpha\hat{\sigma}_\alpha+\hat{I}_\alpha\hat{\sigma}_\beta\right)\right\},
\end{aligned}
\tag{8}
$$

and the coefficients

$$
\begin{aligned}
&\hat{C}=\frac{\rho_\beta^0\hat{\rho}_1\hat{\rho}_2}{\rho^0}, && \hat{D}=2\hat{C}\frac{m_1m_2}{m_1+m_2}\frac{(\hat{c}_0)^2}{k_BT_0},\\
&\hat{E}_\alpha=\frac{3}{4}\frac{m_1+m_2}{m_\alpha}\frac{\nu_2^{\alpha\alpha}}{\nu_1^{12}}\frac{\rho_\alpha^0\hat{\rho}_\alpha}{\rho^0}+\frac{4m_\alpha+3m_\beta\frac{\nu_2^{12}}{\nu_1^{12}}}{2(m_1+m_2)}\frac{\rho_\beta^0\hat{\rho}_\beta}{\rho^0}, && \hat{G}_\alpha=\frac{m_\alpha\left(4-3\frac{\nu_2^{12}}{\nu_1^{12}}\right)}{2(m_1+m_2)}\frac{\rho_\beta^0\hat{\rho}_\alpha}{\rho^0},\\
&\hat{H}_\alpha=\frac{m_1+m_2}{2m_\alpha}\frac{\nu_2^{\alpha\alpha}}{\nu_1^{12}}\frac{\rho_\alpha^0\hat{\rho}_\alpha}{\rho^0}+\frac{3m_\alpha^2+m_\beta^2+2m_1m_2\frac{\nu_2^{12}}{\nu_1^{12}}}{(m_1+m_2)^2}\frac{\rho_\beta^0\hat{\rho}_\beta}{\rho^0}, && \hat{I}_\alpha=\frac{2m_\alpha m_\beta\left(2-\frac{\nu_2^{12}}{\nu_1^{12}}\right)}{(m_1+m_2)^2}\frac{\rho_\beta^0\hat{\rho}_\alpha}{\rho^0},\\
&\hat{L}_\alpha=\frac{-8m_\alpha m_\beta+\left(m_\alpha-3m_\beta\right)m_\beta\frac{\nu_2^{12}}{\nu_1^{12}}}{2(m_1+m_2)^2}\frac{\rho_\beta^0\hat{\rho}_\beta}{\rho^0}.
\end{aligned}
\tag{9}
$$

System (7)-(9) is not autonomous, since the independent variable $\hat{r}$ appears explicitly in the field equations. In the following, by employing a well-known procedure based on the Lie group method [20, 19], we transform system (7) in an autonomous form and then we derive a system of ODEs depending only on a single variable, the so-called similarity variable.

3 Similarity solutions

In order to transform the set of field equations into an autonomous form, we first require the system (7)–(9) to be invariant under the following group of dilatation transformations

$$\begin{aligned} &t^* = \varepsilon^{\upsilon}\hat{t}, \qquad r^* = \varepsilon\hat{r}, \qquad \rho^*_\alpha = \varepsilon^{\gamma_\alpha}\hat{\rho}_\alpha, \quad v^*_\alpha = \varepsilon^{g_\alpha}\hat{v}_\alpha, \\ &p^*_\alpha = \varepsilon^{\delta_\alpha}\hat{p}_\alpha, \quad \sigma^*_\alpha = \varepsilon^{b_\alpha}\hat{\sigma}_\alpha, \quad q^*_\alpha = \varepsilon^{a_\alpha}\hat{q}_\alpha, \end{aligned} \tag{10}$$

where $\varepsilon \neq 0$ is an arbitrary parameter that characterizes the dilatation and the exponents υ, γ_α, g_α, δ_α, b_α and a_α must be determined. The requested invariance of equations (7)-(9) with respect to the transformation (10) conducts to the following relations

$$\begin{aligned} \gamma_\alpha = \gamma, \qquad\qquad & g_\alpha = 1-\nu, \\ \delta_\alpha = b_\alpha = \gamma + 2\left(1-\nu\right), \quad & a_\alpha = \gamma + 3\left(1-\nu\right), \end{aligned} \tag{11}$$

together with the following functional forms for the arbitrary constitutive functions $\hat{f}_i$ involved in the production terms

$$\hat{f}_i\left(\hat{T}_1, \hat{T}_2\right) = \hat{T}_1^{\frac{\nu+\gamma}{2(\nu-1)}}\mathcal{F}_i\left(\frac{\hat{T}_2}{\hat{T}_1}\right), \tag{12}$$

with $\mathcal{F}_i$ arbitrary functions. In particular, for arbitrary constants k_i and parameter s, the choice

$$\left\{ \begin{aligned} &\mathcal{F}_i = \frac{k_i}{2}\left(\frac{\hat{T}_2}{\hat{T}_1}+1\right)^{1-s} \\ &\frac{\nu+\gamma}{2(\nu-1)} = 1-s \end{aligned} \right. \quad \Rightarrow \quad \hat{f}_i\left(\hat{T}_1, \hat{T}_2\right) = k_i\left(\frac{\hat{T}_1+\hat{T}_2}{2}\right)^{1-s}, \tag{13}$$

so that response functions obeying power laws that for $\hat{T}_1 = \hat{T}_2 = \hat{T}$ have the same functional dependence as in the theory of Gilbarg Paolucci are obtained [20, 21].

According to the Lie group theory, let us introduce the new variables

$$\begin{aligned}
&\tau = \tfrac{1}{\nu}\ln\hat{t}, && \xi = \hat{r}\hat{t}^{-\frac{1}{\nu}}, && \mathcal{R}_\alpha = \hat{\rho}_\alpha\hat{t}^{-\frac{\gamma}{\nu}},\\
&\mathcal{V}_\alpha = \hat{v}_\alpha\hat{t}^{-\frac{1-\nu}{\nu}}, && \mathcal{P}_\alpha = \hat{p}_\alpha\hat{t}^{-\frac{\gamma+2(1-\nu)}{\nu}}, && \mathcal{S}_\alpha = \hat{\sigma}_\alpha\hat{t}^{-\frac{\gamma+2(1-\nu)}{\nu}},\\
&\mathcal{Q}_\alpha = \hat{q}_\alpha\hat{t}^{-\frac{\gamma+3(1-\nu)}{\nu}}, && \times_\alpha = \hat{T}_\alpha\hat{t}^{-\frac{2(1-\nu)}{\nu}}, &&
\end{aligned} \tag{14}$$

and let us recast the system of field equations in the following form

$$\begin{aligned}
&\tfrac{1}{\nu}\tfrac{\partial\mathcal{R}_\alpha}{\partial\tau} + \mathcal{R}_\alpha\tfrac{\partial\mathcal{V}_\alpha}{\partial\xi} + \left(\mathcal{V}_\alpha - \tfrac{\xi}{\nu}\right)\tfrac{\partial\mathcal{R}_\alpha}{\partial\xi} = -\tfrac{2}{\xi}\mathcal{R}_\alpha\mathcal{V}_\alpha - \tfrac{\gamma}{\nu}\mathcal{R}_\alpha,\\
&\tfrac{1}{\nu}\mathcal{R}_\alpha\tfrac{\partial\mathcal{V}_\alpha}{\partial\tau} + \mathcal{R}_\alpha\left(\mathcal{V}_\alpha - \tfrac{\xi}{\nu}\right)\tfrac{\partial\mathcal{V}_\alpha}{\partial\xi} + \varkappa_\alpha\tfrac{\partial\mathcal{S}_\alpha}{\partial\xi} + \varkappa_\alpha\tfrac{\partial\mathcal{P}_\alpha}{\partial\xi} =\\
&\quad = \tfrac{\nu-1}{\nu}\mathcal{R}_\alpha\mathcal{V}_\alpha - \tfrac{3\varkappa_\alpha}{\xi}\mathcal{S}_\alpha + \times_1^{\frac{\nu+\gamma}{2(\nu-1)}}\mathcal{F}_1\left(\tfrac{\times_2}{\times_1}\right)\hat{\Sigma}_\alpha,\\
&\tfrac{1}{\nu}\tfrac{\partial\mathcal{P}_\alpha}{\partial\tau} + \left(\mathcal{V}_\alpha - \tfrac{\xi}{\nu}\right)\tfrac{\partial\mathcal{P}_\alpha}{\partial\xi} + \tfrac{2}{3}\tfrac{\partial\mathcal{Q}_\alpha}{\partial\xi} + \left(\tfrac{5}{3}\mathcal{P}_\alpha + \tfrac{2}{3}\mathcal{S}_\alpha\right)\tfrac{\partial\mathcal{V}_\alpha}{\partial\xi} =\\
&\quad = -\tfrac{10}{3}\tfrac{\mathcal{P}_\alpha\mathcal{V}_\alpha}{\xi} - \tfrac{4}{3}\tfrac{\mathcal{Q}_\alpha}{\xi} + \tfrac{2}{3}\tfrac{\mathcal{S}_\alpha\mathcal{V}_\alpha}{\xi} - \tfrac{\gamma+2(1-\nu)}{\nu}\mathcal{P}_\alpha + \tfrac{\times_1^{\frac{\nu+\gamma}{2(\nu-1)}}}{3}\mathcal{F}_2\left(\tfrac{\times_2}{\times_1}\right)\hat{U}_\alpha,\\
&\tfrac{1}{\nu}\tfrac{\partial\mathcal{S}_\alpha}{\partial\tau} + \left(\mathcal{V}_\alpha - \tfrac{\xi}{\nu}\right)\tfrac{\partial\mathcal{S}_\alpha}{\partial\xi} + \tfrac{8}{15}\tfrac{\partial\mathcal{Q}_\alpha}{\partial\xi} + \left(\tfrac{4}{3}\mathcal{P}_\alpha + \tfrac{7}{3}\mathcal{S}_\alpha\right)\tfrac{\partial\mathcal{V}_\alpha}{\partial\xi} =\\
&\quad = \tfrac{4}{3}\tfrac{\mathcal{P}_\alpha\mathcal{V}_\alpha}{\xi} + \tfrac{8}{15}\tfrac{\mathcal{Q}_\alpha}{\xi} - \tfrac{8}{3}\tfrac{\mathcal{S}_\alpha\mathcal{V}_\alpha}{\xi} - \tfrac{\gamma+2(1-\nu)}{\nu}\mathcal{S}_\alpha + \times_1^{\frac{\nu+\gamma}{2(\nu-1)}}\mathcal{F}_3\left(\tfrac{\times_2}{\times_1}\right)\hat{Z}_\alpha,\\
&\tfrac{1}{\nu}\tfrac{\partial\mathcal{Q}_\alpha}{\partial\tau} + \left(\mathcal{V}_\alpha - \tfrac{\xi}{\nu}\right)\tfrac{\partial\mathcal{Q}_\alpha}{\partial\xi} + \tfrac{16}{5}\mathcal{Q}_\alpha\tfrac{\partial\mathcal{V}_\alpha}{\partial\xi} + \varkappa_\alpha\tfrac{\mathcal{P}_\alpha-\mathcal{S}_\alpha}{\mathcal{R}_\alpha}\tfrac{\partial\mathcal{S}_\alpha}{\partial\xi}\\
&\quad + \tfrac{5\varkappa_\alpha}{2}\tfrac{\mathcal{P}_\alpha+\mathcal{S}_\alpha}{\mathcal{R}_\alpha}\tfrac{\partial\mathcal{P}_\alpha}{\partial\xi} - \tfrac{\varkappa_\alpha\mathcal{P}_\alpha}{\mathcal{R}_\alpha^2}\left(\tfrac{7\mathcal{S}_\alpha}{2} + \tfrac{5\mathcal{P}_\alpha}{2}\right)\tfrac{\partial\mathcal{R}_\alpha}{\partial\xi} =\\
&\quad = -\tfrac{3\varkappa_\alpha}{\xi}\tfrac{\mathcal{P}_\alpha-\mathcal{S}_\alpha}{\mathcal{R}_\alpha}\mathcal{S}_\alpha - \tfrac{14}{5}\tfrac{\mathcal{Q}_\alpha}{\xi}\mathcal{V}_\alpha - \tfrac{\gamma+3(1-\nu)}{\nu}\mathcal{Q}_\alpha + \times_1^{\frac{\nu+\gamma}{2(\nu-1)}}\mathcal{F}_4\left(\tfrac{\times_2}{\times_1}\right)\hat{W}_\alpha.
\end{aligned} \tag{15}$$

The corresponding productions can be easily obtained from (8)-(9). Particular solutions of (15) depending only on the similarity variable ξ, represent the similarity solutions of the basic equations (7)-(9).

As it can be easily seen, the new system (15) is still not-autonomous but the invariance with respect to a new dilatation group of transformation, that is

$$\begin{aligned}
&\tilde{\xi} = \omega\xi, && \tilde{\mathcal{R}}_\alpha^* = \omega^{k_\alpha}\mathcal{R}_\alpha, && \tilde{\mathcal{V}}_\alpha^* = \omega^{n_\alpha}\mathcal{V}_\alpha,\\
&\tilde{\mathcal{P}}_\alpha^* = \omega^{c_\alpha}\mathcal{P}_\alpha, && \tilde{\mathcal{S}}_\alpha^* = \omega^{d_\alpha}\mathcal{S}_\alpha, && \tilde{\mathcal{Q}}_\alpha^* = \omega^{e_\alpha}\mathcal{Q}_\alpha,
\end{aligned} \tag{16}$$

allows the reduction of (15) to an autonomous form [19]. Therefore, the requested invariance with respect to this new group furnishes

$$k_\alpha = k = \tfrac{\nu+\gamma}{1-\nu}, \qquad n_\alpha = 1, \qquad c_\alpha = d_\alpha = k+2, \qquad e_\alpha = k+3. \tag{17}$$

Finally, let us introduce the transformations

$$\tau = \tau, \qquad \eta = \ln \xi, \qquad \mathcal{R}_\alpha = \xi^k \bar{\rho}_0^\alpha R_\alpha, \qquad \mathcal{V}_\alpha = \frac{\xi}{\nu}\left(V_\alpha + 1\right),$$
$$\mathcal{P}_\alpha = \bar{\rho}_0^\alpha \frac{\xi^{k+2}}{\nu^2} P_\alpha, \quad \mathcal{S}_\alpha = \bar{\rho}_0^\alpha \frac{\xi^{k+2}}{\nu^2} S_\alpha, \quad \mathcal{Q}_\alpha = \bar{\rho}_0^\alpha \frac{\xi^{k+3}}{\nu^3} Q_\alpha, \quad \times_\alpha = \frac{\xi^2}{\nu^2}\vartheta_\alpha, \tag{18}$$

with $\bar{\rho}_0^\alpha$ a dimensionless constant introduced for further purpose. System (15) is then recast in the autonomous form

$$\begin{aligned}
&\frac{\partial R_\alpha}{\partial \tau} + R_\alpha \frac{\partial V_\alpha}{\partial \eta} + V_\alpha \frac{\partial R_\alpha}{\partial \eta} = -\left(\gamma - k\right) R_\alpha - \left(k+3\right) R_\alpha \left(V_\alpha + 1\right), \\
&\frac{\partial V_\alpha}{\partial \tau} + V_\alpha \frac{\partial V_\alpha}{\partial \eta} + \frac{\varkappa_\alpha}{R_\alpha}\frac{\partial P_\alpha}{\partial \eta} + \frac{\varkappa_\alpha}{R_\alpha}\frac{\partial S_\alpha}{\partial \eta} + \\
&\quad = \nu\left(V_\alpha + 1\right) - \left(V_\alpha + 1\right)^2 - \left(k+5\right)\frac{\varkappa_\alpha S_\alpha}{R_\alpha} - \left(k+2\right)\frac{\varkappa_\alpha P_\alpha}{R_\alpha} + \frac{\bar{\Sigma}_\alpha}{R_\alpha}, \\
&\frac{\partial P_\alpha}{\partial \tau} + V_\alpha \frac{\partial P_\alpha}{\partial \eta} + \frac{5}{3} P_\alpha \frac{\partial V_\alpha}{\partial \eta} + \frac{2}{3} S_\alpha \frac{\partial V_\alpha}{\partial \eta} + \frac{2}{3}\frac{\partial Q_\alpha}{\partial \eta} = \\
&\quad = -\left(\gamma - 2\nu - k\right) P_\alpha - \left(k+7\right) P_\alpha \left(V_\alpha + 1\right) - \frac{2}{3}\left(k+5\right) Q_\alpha + \frac{\bar{U}_\alpha}{3}, \\
&\frac{\partial S_\alpha}{\partial \tau} + V_\alpha \frac{\partial S_\alpha}{\partial \eta} + \frac{8}{15}\frac{\partial Q_\alpha}{\partial \eta} + \frac{4}{3} P_\alpha \frac{\partial V_\alpha}{\partial \eta} + \frac{7}{3} S_\alpha \frac{\partial V_\alpha}{\partial \eta} = \\
&\quad = -\left(\gamma - 2\nu - k\right) S_\alpha - \left(k+7\right) S_\alpha \left(V_\alpha + 1\right) - \frac{8}{15}\left(k+2\right) Q_\alpha + \bar{Z}_\alpha, \\
&\frac{\partial Q_\alpha}{\partial \tau} + V_\alpha \frac{\partial Q_\alpha}{\partial \eta} + \frac{16}{5} Q_\alpha \frac{\partial V_\alpha}{\partial \eta} + \varkappa_\alpha \frac{P_\alpha - S_\alpha}{R_\alpha}\frac{\partial S_\alpha}{\partial \eta} + \frac{5\varkappa_\alpha}{2}\frac{P_\alpha + S_\alpha}{R_\alpha}\frac{\partial P_\alpha}{\partial \eta} \\
&\quad - \frac{\varkappa_\alpha P_\alpha}{R_\alpha^2}\left(\frac{7S_\alpha}{2} + \frac{5P_\alpha}{2}\right)\frac{\partial R_\alpha}{\partial \eta} = -\left(\gamma - 3\nu - k\right) Q_\alpha - 10\varkappa_\alpha \frac{P_\alpha S_\alpha}{R_\alpha} \\
&\quad - \left(k+9\right) Q_\alpha \left(V_\alpha + 1\right) - 5\varkappa_\alpha \frac{P_\alpha^2}{R_\alpha} + \left(k+5\right)\frac{\varkappa_\alpha S_\alpha^2}{R_\alpha} + \bar{W}_\alpha,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&\bar{\Sigma}_\alpha = \frac{\nu^{k+1}}{\sqrt{\vartheta_1}^k}\mathcal{F}_1\left(\frac{\vartheta_2}{\vartheta_1}\right)\bar{C}\left(V_\beta - V_\alpha\right), \\
&\bar{U}_\alpha = \frac{\nu^{k+1}}{\sqrt{\vartheta_1}^k}\mathcal{F}_2\left(\frac{\vartheta_2}{\vartheta_1}\right)\bar{D}\left[3\chi_\beta\left(\vartheta_\beta - \vartheta_\alpha\right) + \left(V_2 - V_1\right)^2\right], \\
&\bar{Z}_\alpha = \frac{\nu^{k+1}}{\sqrt{\vartheta_1}^k}\mathcal{F}_3\left(\frac{\vartheta_2}{\vartheta_1}\right)\left[-\bar{E}_\alpha S_\alpha + \bar{G}_\alpha S_\beta + \frac{2}{3}\frac{1}{\chi_\beta}\bar{G}_\alpha R_\beta\left(V_2 - V_1\right)^2\right], \\
&\bar{W}_\alpha = \frac{\nu^{k+1}}{\sqrt{\vartheta_1}^k}\mathcal{F}_4\left(\frac{\vartheta_2}{\vartheta_1}\right)\left\{-\bar{H}_\alpha Q_\alpha + \bar{I}_\alpha Q_\beta + \right. \\
&\left. + \bar{I}_\alpha R_\beta \frac{V_\beta - V_\alpha}{2}\left[5\left(\vartheta_\beta - \vartheta_\alpha\right) + \frac{1}{\chi_\beta}\left(V_2 - V_1\right)^2\right] + \left(\bar{L}_\alpha S_\alpha + \bar{I}_\alpha S_\beta\right)\left(V_\beta - V_\alpha\right)\right\},
\end{aligned} \tag{20}$$

with the coefficients

$$\bar{C} = \frac{\rho_\beta^0 R_1 R_2 \bar{\rho}_0^\beta}{\rho^0}, \qquad \bar{D} = 2\bar{C}\frac{m_1 m_2}{m_1+m_2}\frac{\hat{c}_0^2}{k_B T_0},$$
$$\bar{G}_\alpha = \frac{m_\alpha\left(4-3\frac{\nu_2^{12}}{\nu_1^{12}}\right)}{2(m_1+m_2)}\frac{\rho_\beta^0\bar{\rho}_0^\beta R_\alpha}{\rho^0}, \quad \bar{I}_\alpha = \frac{2m_\alpha m_\beta\left(2-\frac{\nu_2^{12}}{\nu_1^{12}}\right)}{(m_1+m_2)^2}\frac{\rho_\beta^0\bar{\rho}_0^\beta R_\alpha}{\rho^0}, \tag{21}$$

and

$$\bar{E}_\alpha = \frac{3}{4}\frac{m_1+m_2}{m_\alpha}\frac{\nu_2^{\alpha\alpha}}{\nu_1^{12}}\frac{\rho_\alpha^0\bar{\rho}_0^\alpha R_\alpha}{\rho^0} + \frac{4m_\alpha+3m_\beta\frac{\nu_2^{12}}{\nu_1^{12}}}{2(m_1+m_2)}\frac{\rho_\beta^0\bar{\rho}_0^\beta R_\beta}{\rho^0},$$
$$\bar{H}_\alpha = \frac{1}{2}\frac{m_1+m_2}{m_\alpha}\frac{\nu_2^{\alpha\alpha}}{\nu_1^{12}}\frac{\rho_\alpha^0\bar{\rho}_0^\alpha R_\alpha}{\rho^0} + \frac{3m_\alpha^2+m_\beta^2+2m_1m_2\frac{\nu_2^{12}}{\nu_1^{12}}}{(m_1+m_2)^2}\frac{\rho_\beta^0\bar{\rho}_0^\beta R_\beta}{\rho^0}, \tag{22}$$
$$\bar{L}_\alpha = \frac{-8m_\alpha m_\beta+(m_\alpha-3m_\beta)m_\beta\frac{\nu_2^{12}}{\nu_1^{12}}}{2(m_1+m_2)^2}\frac{\rho_\beta^0\bar{\rho}_0^\beta R_\beta}{\rho^0}.$$

As it can be easily seen, when written in terms of these new variables, the system of fields equations becomes autonomous, as required. The variable η is called the similarity variable and assuming that the fields don't depend on τ we get a system of ODEs that depend on t and r only through η.

In the next section, we will study the evolution of acceleration waves allowed by these equations. For further purpose, we write the dimensionless physical variables in terms of the new quantities, i.e.

$$\tau = \frac{1}{\nu}\ln\hat{t}, \quad \eta = \ln\frac{\hat{r}}{\hat{t}^{\frac{1}{\nu}}}, \tag{23}$$

for the independent variables and

$$\hat{\rho}_\alpha = \bar{\rho}_0^\alpha\hat{r}^k\hat{t}^{\frac{\gamma-k}{\nu}}R_\alpha, \qquad \hat{v}_\alpha = \frac{\hat{r}}{\nu\hat{t}}(V_\alpha+1), \qquad \hat{p}_\alpha = \bar{\rho}_0^\alpha\frac{\hat{r}^{k+2}}{\nu^2\hat{t}^2}\hat{t}^{\frac{\gamma-k}{\nu}}P_\alpha,$$
$$\hat{\sigma}_\alpha = \bar{\rho}_0^\alpha\frac{\hat{r}^{k+2}}{\nu^2\hat{t}^2}\hat{t}^{\frac{\gamma-k}{\nu}}S_\alpha, \quad \hat{q}_\alpha = \bar{\rho}_0^\alpha\frac{\hat{r}^{k+3}}{\nu^3\hat{t}^3}\hat{t}^{\frac{\gamma-k}{\nu}}Q_\alpha, \quad \hat{T}_\alpha = \frac{\hat{r}^2}{\nu^2\hat{t}^2}\vartheta_\alpha, \tag{24}$$

for the field variables.

4 Acceleration waves in a state characterized by similarity solutions

This section is devoted to the study of the evolution of weak discontinuities compatible with system (7) propagating in a non-constant state characterized by a similarity solution of set (19). To this aim, it is convenient to consider the transformed system (19) expressed in terms of the canonical variables τ and η and, following a consolidated approach [1, 33], to examine the non-constant solution obtained from (24), which corresponds to a steady state of system (19). For further convenience, let's recast system (19) in the vector form

$$\frac{\partial \mathbf{W}}{\partial \tau} + \mathrm{A}\,\frac{\partial \mathbf{W}}{\partial \eta} = \mathbf{B}(\mathbf{W}) \tag{25}$$

with

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \quad \mathrm{A} = \begin{bmatrix} \mathrm{A}_1 & 0_5 \\ 0_5 & \mathrm{A}_2 \end{bmatrix}. \tag{26}$$

In (26) 0_5 denotes the 5×5 null matrix and

$$\begin{aligned}
\mathbf{W}_\alpha &= \begin{bmatrix} R_\alpha & V_\alpha & P_\alpha & S_\alpha & Q_\alpha \end{bmatrix}^{\mathrm{T}}, \\
\mathbf{B}_\alpha &= \begin{bmatrix}
-(\gamma-k)\,R_\alpha - (k+3)\,R_\alpha\,(V_\alpha+1) \\
\nu\,(V_\alpha+1) - (V_\alpha+1)^2 - (k+5)\,\frac{\varkappa_\alpha S_\alpha}{R_\alpha} - (k+2)\,\frac{\varkappa_\alpha P_\alpha}{R_\alpha} + \frac{\bar{\Sigma}_\alpha}{R_\alpha} \\
-(\gamma-2\nu-k)\,P_\alpha - (k+7)\,P_\alpha\,(V_\alpha+1) - \frac{2}{3}\,(k+5)\,Q_\alpha + \frac{\bar{U}_\alpha}{3} \\
-(\gamma-2\nu-k)\,S_\alpha - (k+7)\,S_\alpha\,(V_\alpha+1) - \frac{8}{15}\,(k+2)\,Q_\alpha + \bar{Z}_\alpha \\
-(\gamma-3\nu-k)\,Q_\alpha - (k+9)\,Q_\alpha\,(V_\alpha+1) + \\
-10\varkappa_\alpha\frac{P_\alpha S_\alpha}{R_\alpha} - 5\varkappa_\alpha\frac{P_\alpha^2}{R_\alpha} + (k+5)\,\frac{\varkappa_\alpha S_\alpha^2}{R_\alpha} + \bar{W}_\alpha
\end{bmatrix}, \\
A_\alpha &= \begin{bmatrix}
V_\alpha & R_\alpha & 0 & 0 & 0 \\
0 & V_\alpha & \frac{\varkappa_\alpha}{R_\alpha} & \frac{\varkappa_\alpha}{R_\alpha} & 0 \\
0 & \frac{5P_\alpha+2S_\alpha}{3} & V_\alpha & 0 & \frac{2}{3} \\
0 & \frac{4P_\alpha+7S_\alpha}{3} & 0 & V_\alpha & \frac{8}{15} \\
-\frac{\varkappa_\alpha P_\alpha(7S_\alpha+5P_\alpha)}{2R_\alpha^2} & \frac{16}{5}Q_\alpha & \frac{5\varkappa_\alpha(P_\alpha+S_\alpha)}{2R_\alpha} & \frac{\varkappa_\alpha(P_\alpha-S_\alpha)}{R_\alpha} & V_\alpha
\end{bmatrix}.
\end{aligned} \tag{27}$$

In [3] the authors investigated the mathematical properties of the model (19) and the strict hyperbolicity has been proved in a neighborhood of the equilibrium state. The transformed system (25) maintains the hyperbolic character of the original model, so that it admits only real and distinct eigenvalues Λ_s to which there correspond right $\mathbf{d}^{(\Lambda_s)}$ and left eigenvectors $\mathbf{l}^{(\Lambda_s)}$, linearly independent, solutions of

$$(\mathrm{A} - \Lambda\,\mathrm{I})\mathbf{d}^{(\Lambda)} = 0, \qquad \mathbf{l}^{(\Lambda)}(\mathrm{A} - \Lambda\,\mathrm{I}) = 0. \tag{28}$$

In detail, the characteristic equation reads

$$\prod_{\alpha=1,2}(\Lambda - V_\alpha)\left[(\Lambda - V_\alpha)^4 - \frac{\varkappa_\alpha}{R_\alpha}\left(\frac{26}{5}P_\alpha + \frac{62}{15}S_\alpha\right)(\Lambda - V_\alpha)^2 - \tfrac{96}{25}\tfrac{\varkappa_\alpha}{R_\alpha}Q_\alpha(\Lambda - V_\alpha) + 3\tfrac{\varkappa_\alpha^2}{R_\alpha^2}(P_\alpha^2 + 2P_\alpha S_\alpha + \tfrac{7}{5}S_\alpha^2)\right] = 0, \tag{29}$$

whereas the right and left eigenvectors

$$\mathbf{d}^{(\Lambda)} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix}, \qquad \mathbf{l}^{(\Lambda)} = \begin{bmatrix} \mathbf{l}_1 & \mathbf{l}_2 \end{bmatrix}, \tag{30}$$

are obtained from

$$\Lambda \neq V_\alpha \quad \Rightarrow \quad \begin{cases} \mathbf{d}_\alpha = \begin{bmatrix} 1 \\ \frac{\Lambda - V_\alpha}{R_\alpha} \\ \frac{5(\Lambda - V_\alpha)^2}{9\varkappa_\alpha} - \frac{S_\alpha}{R_\alpha} \\ \frac{4(\Lambda - V_\alpha)^2}{9\varkappa_\alpha} + \frac{S_\alpha}{R_\alpha} \\ \frac{5(\Lambda - V_\alpha)}{2}\left(\frac{(\Lambda - V_\alpha)^2}{3\varkappa_\alpha} - \frac{P_\alpha + S_\alpha}{R_\alpha}\right) \end{bmatrix}, \\ \mathbf{l}_\alpha = \mathcal{M}_\alpha \begin{bmatrix} 1 \\ \frac{R_\alpha^2(\Lambda - V_\alpha)\left[\varkappa_\alpha(11P_\alpha + 17S_\alpha) - 5R_\alpha(\Lambda - V_\alpha)^2\right]}{3\varkappa_\alpha^2 P_\alpha(5P_\alpha + 7S_\alpha)} \\ -\frac{R_\alpha\left[4\varkappa_\alpha(3P_\alpha + 7S_\alpha) + 15R_\alpha(\Lambda - V_\alpha)^2\right]}{9\varkappa_\alpha P_\alpha(5P_\alpha + 7S_\alpha)} \\ \frac{5R_\alpha\left[\varkappa_\alpha(3P_\alpha + 7S_\alpha) - 3R_\alpha(\Lambda - V_\alpha)^2\right]}{9\varkappa_\alpha P_\alpha(5P_\alpha + 7S_\alpha)} \\ \frac{2R_\alpha^2(\Lambda - V_\alpha)}{\varkappa_\alpha P_\alpha(5P_\alpha + 7S_\alpha)} \end{bmatrix}^T, \end{cases}$$

$$\Lambda = V_\alpha \quad \Rightarrow \quad \begin{cases} \mathbf{d}_\alpha = \begin{bmatrix} 1 \\ 0 \\ \frac{P_\alpha(7S_\alpha+5P_\alpha)}{R_\alpha(7S_\alpha+3P_\alpha)} \\ -\frac{P_\alpha(7S_\alpha+5P_\alpha)}{R_\alpha(7S_\alpha+3P_\alpha)} \\ 0 \end{bmatrix}, \qquad \mathbf{d}_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ \mathbf{l}_\alpha = \mathcal{L}_\alpha \begin{bmatrix} \frac{9S_\alpha}{5R_\alpha} & 0 & \frac{4}{5} & -1 & 0 \end{bmatrix}, \\ \mathbf{l}_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ \mathit{if}\, V_\alpha = V_\beta, \\ \mathbf{l}_\beta = \begin{bmatrix} 1 \\ \frac{R_\beta^2(V_\alpha-V_\beta)\left[\varkappa_\beta(11P_\beta+17S_\beta)-5R_\beta(V_\alpha-V_\beta)^2\right]}{3\varkappa_\beta^2 P_\beta(5P_\beta+7S_\beta)} \\ -\frac{R_\beta\left[4\varkappa_\beta(3P_\beta+7S_\beta)+15R_\beta(V_\alpha-V_\beta)^2\right]}{9\varkappa_\beta P_\beta(5P_\beta+7S_\beta)} \\ \frac{5R_\beta\left[\varkappa_\beta(3P_\beta+7S_\beta)-3R_\beta(V_\alpha-V_\beta)^2\right]}{9\varkappa_\beta P_\beta(5P_\beta+7S_\beta)} \\ \frac{2R_\beta^2(V_\alpha-V_\beta)}{\varkappa_\beta P_\beta(5P_\beta+7S_\beta)} \end{bmatrix}^T, \ \mathit{if}\, V_\alpha \neq V_\beta, \end{cases}$$

The factors $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$ are obtained through the orthonormality condition $\mathbf{l}^{(\Lambda)} \cdot \mathbf{d}^{(\Lambda)} = 1$.

Let's assume that the field variables $\mathbf{W}(\tau, \eta)$ suffer a jump in the first order derivatives across the curve $\phi(\tau, \eta) = 0$, called the wave front, in the $(\tau, \eta)-$plane to which there corresponds the curve

$$\phi\left(\frac{1}{\nu}\ln \hat{t}, \ln\left(\frac{\hat{r}}{\hat{t}^{\frac{1}{\nu}}}\right)\right) = \phi^*(\hat{r}, \hat{t}) = 0, \tag{31}$$

in the physical $(\hat{r}, \hat{t})-$plane.

By (23), the following relation between $\lambda = -\frac{\phi^*_{\hat{t}}}{\phi^*_{\hat{r}}}$, characteristic velocity of the original system (7), and $\Lambda = -\frac{\phi_\tau}{\phi_\eta}$ is obtained

$$\lambda = \frac{1}{\nu}\frac{\hat{r}}{\hat{t}}\left(\Lambda + 1\right). \tag{32}$$

Next, let's consider as unperturbed field the steady state solutions of the autonomous system (19) satisfying $\mathbf{B}(\mathbf{W}_0) = 0$, that, bearing in mind $(17)_1$

and the relations $P_\alpha = R_\alpha \vartheta_\alpha$, is obtained from

$$V_\alpha = -1 + \frac{\nu(k+1)}{k+3},$$

$$S_\alpha = \frac{2(k+1)\nu^2}{(k+3)(k+5)} \frac{R_\alpha}{\varkappa_\alpha} - \frac{k+2}{k+5} R_\alpha \vartheta_\alpha,$$

$$Q_\alpha = \frac{3(1-k)\nu}{(k+5)(k+3)} R_\alpha \vartheta_\alpha + \frac{3\, d_\alpha R_1 R_2}{2(k+5)} (\vartheta_\beta - \vartheta_\alpha) \Psi_2(\vartheta_1, \vartheta_2),$$

$$\left[\frac{2(1-k)\nu}{k+3} + (g_\alpha R_\alpha + h_\alpha R_\beta)\Psi_3(\vartheta_1, \vartheta_2)\right] S_\alpha + l_\alpha \Psi_3(\vartheta_1, \vartheta_2) R_\alpha S_\beta \qquad (33)$$

$$- \frac{8(k+2) Q_\alpha}{15} = 0,$$

$$\left[\frac{3(1-k)\nu}{k+3} + (p_\alpha R_\alpha + q_\alpha R_\beta)\Psi_4(\vartheta_1, \vartheta_2)\right] Q_\alpha + n_\alpha \Psi_4(\vartheta_1, \vartheta_2) R_\alpha Q_\beta +$$

$$+ \frac{\varkappa_\alpha}{R_\alpha} \left[(k+5) S_\alpha^2 - 5 R_\alpha^2 \vartheta_\alpha^2 - 10 R_\alpha \vartheta_\alpha S_\alpha\right] = 0,$$

with

$$d_\alpha = \frac{2m_1 m_2}{m_1+m_2} \frac{\hat{c}_0^2}{k_B T_0} \frac{\rho_\beta^0 \bar{\rho}_0^\beta}{\rho^0} \varkappa_\beta\,, \quad g_\alpha = -\frac{3}{4} \frac{m_1+m_2}{m_\alpha} \frac{\nu_2^{\alpha\alpha}}{\nu_1^{12}} \frac{\rho_\alpha^0 \bar{\rho}_0^\alpha}{\rho^0}\,,$$

$$h_\alpha = -\frac{4m_\alpha + 3m_\beta \frac{\nu_2^{12}}{\nu_1^{12}}}{2(m_1+m_2)} \frac{\rho_\beta^0 \bar{\rho}_0^\beta}{\rho^0}, \quad l_\alpha = \frac{m_\alpha \left(4 - 3\frac{\nu_2^{12}}{\nu_1^{12}}\right)}{2(m_1+m_2)} \frac{\rho_\beta^0 \bar{\rho}_0^\beta}{\rho^0},$$

$$p_\alpha = -\frac{1}{2} \frac{m_1+m_2}{m_\alpha} \frac{\nu_2^{\alpha\alpha}}{\nu_1^{12}} \frac{\rho_\alpha^0 \bar{\rho}_0^\alpha}{\rho^0}, \quad q_\alpha = -\frac{3m_\alpha^2 + m_\beta^2 + 2m_1 m_2 \frac{\nu_2^{12}}{\nu_1^{12}}}{(m_1+m_2)^2} \frac{\rho_\beta^0 \bar{\rho}_0^\beta}{\rho^0}, \qquad (34)$$

$$n_\alpha = \frac{2m_\alpha m_\beta \left(2 - \frac{\nu_2^{12}}{\nu_1^{12}}\right)}{(m_1+m_2)^2} \frac{\rho_\beta^0 \bar{\rho}_0^\beta}{\rho^0}, \quad \Psi_i = \nu^{k+1} \vartheta_1^{-\frac{k}{2}} \mathcal{F}_i\left(\frac{\vartheta_2}{\vartheta_1}\right).$$

Furthermore, according to (13), we choose

$$\Psi_i = k_i \left(\frac{\vartheta_1 + \vartheta_2}{2}\right)^{1-s}, \qquad (35)$$

with k_i arbitrary constants and

$$k = 2(s-1), \qquad \frac{1}{2} < s < 1. \qquad (36)$$

By denoting with $\mathbf{d}_0 = \mathbf{d}(\mathbf{W}_0)$ and $\mathbf{l}_0 = \mathbf{l}(\mathbf{W}_0)$ the right and left eigenvectors evaluated at the steady state (33), it follows

$$\delta \mathbf{W} = \pi \mathbf{d}_0, \qquad \frac{\mathrm{d}\pi}{\mathrm{d}\sigma} + a_0 \pi^2 + b_0 \pi = 0, \qquad (37)$$

with

$$\delta = \frac{\partial}{\partial \phi}\bigg|_{\phi=0^+} - \frac{\partial}{\partial \phi}\bigg|_{\phi=0^-}, \quad \frac{\mathrm{d}}{\mathrm{d}\sigma} = \frac{\partial}{\partial \tau} + \Lambda_0 \frac{\partial}{\partial \eta},$$

$$a_0 = \left(\nabla_{\mathbf{W}} \Lambda \cdot \mathbf{d}\right)_0 \phi_\eta, \qquad b_0 = -\mathbf{l}_0 \left(\nabla_{\mathbf{W}} \mathbf{B} \cdot \mathbf{d}\right)_0, \tag{38}$$

$$\tau = \sigma, \qquad \eta = \eta_0 + \Lambda_0(\tau - \tau_0).$$

Consequently, from (37), it results

$$\pi(\sigma) = \frac{\pi_0 e^{-b_0(\sigma-\sigma_0)}}{1 + \frac{\pi_0 a_0}{b_0}\left(1 - e^{-b_0(\sigma-\sigma_0)}\right)}. \tag{39}$$

The original field variables $\mathbf{U}_\alpha$

$$\mathbf{U}_\alpha = \begin{bmatrix} \hat{\rho}_\alpha & \hat{v}_\alpha & \hat{p}_\alpha & \hat{\sigma}_\alpha & \hat{q}_\alpha \end{bmatrix}^T, \tag{40}$$

exhibit discontinuities in the first derivatives across the curve

$$\hat{r} = \hat{r}_0 \left(\frac{\hat{t}}{\hat{t}_0}\right)^{\frac{\Lambda_0+1}{\nu}}, \tag{41}$$

related to the jump of $\mathbf{W}_\alpha$ through

$$\delta \mathbf{U}_\alpha = \begin{bmatrix} \frac{\overline{\rho}_0^\alpha}{\hat{t}}\left(\frac{\hat{r}}{\hat{t}}\right)^k & 0 & 0 & 0 & 0 \\ 0 & \frac{\hat{r}}{\nu \hat{t}} & 0 & 0 & 0 \\ 0 & 0 & \frac{\overline{\rho}_0^\alpha}{\nu^2 \hat{t}}\left(\frac{\hat{r}}{\hat{t}}\right)^{k+2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\overline{\rho}_0^\alpha}{\nu^2 \hat{t}}\left(\frac{\hat{r}}{\hat{t}}\right)^{k+2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\overline{\rho}_0^\alpha}{\nu^3 \hat{t}}\left(\frac{\hat{r}}{\hat{t}}\right)^{k+3} \end{bmatrix} \delta \mathbf{W}_\alpha. \tag{42}$$

Finally, from (39), the amplitude π in the original variables reads

$$\pi(\hat{t}) = \frac{\pi_0 \left(\frac{\hat{t}}{\hat{t}_0}\right)^{-\frac{b_0}{\nu}}}{1 + \frac{a_0 \pi_0}{b_0}\left[1 - \left(\frac{\hat{t}}{\hat{t}_0}\right)^{-\frac{b_0}{\nu}}\right]}, \tag{43}$$

and the critical time is given by

$$\hat{t}_c = \hat{t}_0 \left(\frac{\pi_0 a_0 + b_0}{\pi_0 a_0}\right)^{\frac{b_0}{\nu}}. \tag{44}$$

The propagation of acceleration waves in a mixture of Helium and Argon is numerically evaluated. A solution of system (33) with the choice $\gamma = -1.5$, $\nu = 1.75$, that imply $s = 0.83$, and $\bar{\rho}_0^\alpha = 1$ is $\vartheta_1 = 0.1027, \vartheta_2 = 0.9046, R_1 = 2.5705 \times 10^{-4}, R_2 = 0.5311, V_1 = V_2 = -0.5625, S_1 = 2.220 \times 10^{-4}, S_2 = 0.155, Q_1 = 3.5646 \times 10^{-4}, Q_2 = 0.2703$. This is a constant state of the transformed system (19) whose corresponds by (24) a non-constant state for the initial set (7). The greatest eigenvalue of matrix A reads $\Lambda = 0.615317$, whose correspond by (32) a non-constant propagation velocity λ. Furthermore, the two coefficients a_0 and b_0 are numerically evaluated as $a_0 = -2.5139$ and $b_0 = 1.3175 \times 10^{-4}$. In Figure 1, the evolution of the

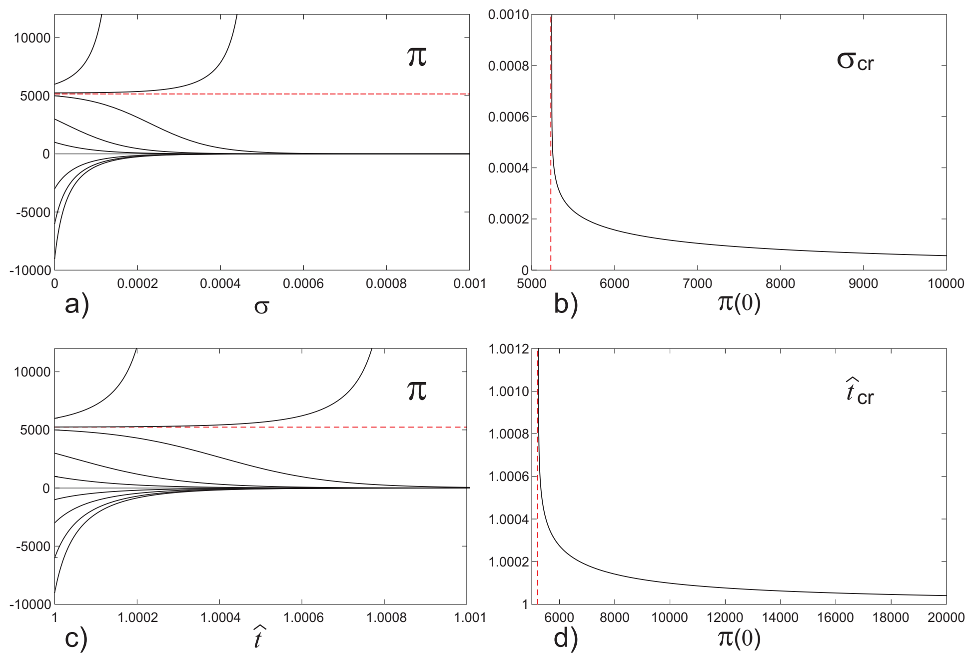

Figure 1: Evolution of the amplitude Π in terms of σ (**a**) and in terms of $\hat{t}$ (**c**) for different initial value $\Pi(0)$ The lines in red represent the critical initial amplitude. (**b**)-(**d**): dependence of the critical value of σ and $\hat{t}$, respectively, on the initial amplitude.

discontinuity is depicted together with the critical times. More precisely, Fig.1_a refers to the evolution of π in terms of the transformed variable σ, as described by (39) with $\sigma_0 = 0$. It can be easily seen that, initial perturbations smaller than the critical value $\pi_0 < 5242$ (illustrated in the figure with red dashes line) are damped, Therefore, the system is stable with respect to these initial perturbations. Instead, for $\pi_0 > 5242$, the initial discontinuity cannot be damped and it evolves into a shock. Fig.1_b illustrates the critical value of σ obtained setting the denominator of (39) equal to zero. It corresponds to the value of σ when the perturbation becomes infinite. Clearly, the critical value of σ exists only for $\pi_0 > 5242$. Fig.$1_{c,d}$ illustrate the discontinuity π in terms of the dimensionless time $\hat{t}$, obtained from (43) with $\hat{t}_0 = 1$. Clearly, the results are completely equivalent to those illustrated in Fig.$1_{a,b}$.

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