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A CHARACTERIZATION OF THE TRIANGLE $3-4-5^*$

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Dedicated to Prof. Liliana Restuccia on the occasion of her 70th anniversary

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Abstract

A Pythagorean triangle is a right triangle where the lengths of the three sides are all integers. The most famous example of a Pythagorean triangle is the $3-4-5$ triangle, where the lengths of the sides are $3, 4$, and 5 units, respectively. In this short note, we will study the class of Pythagorean triangles that may be associated with a square by means of a real parameter k. It turns out that the triangle $3-4-5$ holds exclusive property in terms of the rationality of k.

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1 Introduction

Let $\mathcal{Q}_l = (ABCD)$ be a square with side l. Then, for every real number $k \geq 1$, the segments DM_k (with $M_k \in DC$) and AN_k (with $N_k \in AD$), both of length l/k, determine two segments M_kA and BN_k whose intersection E_k is the vertex of a triangle $\mathcal{T}_k(\mathcal{Q}_l) := BE_kM_k$, see Figure 1.

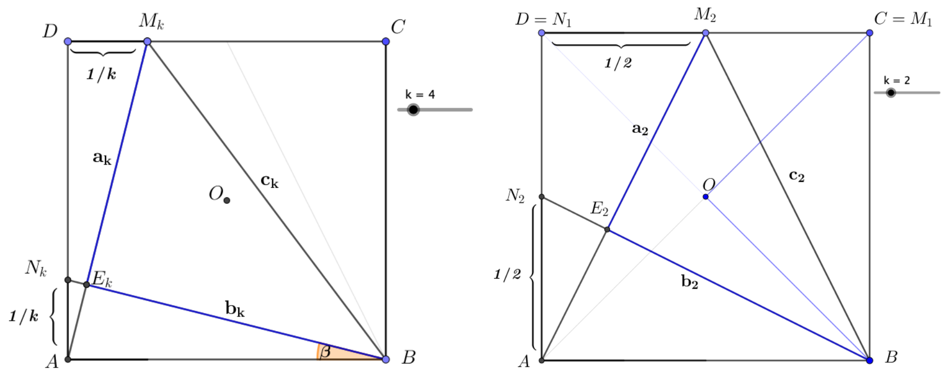

Figure 1: Left, the construction of the k-right-triangle $\mathcal{T}_k(\mathcal{Q}_1)$ associated with the unit square; right, the construction with $k = 2$, giving the triangle $\mathcal{T}_2(\mathcal{Q}_1)$, which is similar to the Pythagorean triangle $3-4-5$.

We note that BE_kM_k is a right triangle. In fact, the triangles ABN_k and AM_kD are congruent, and the segment BN_k overlaps AM_k by a rotation of $90°$ clockwise around the centre O of the square (see Figure 1, Left). In the following, we will call $\mathcal{T}_k(\mathcal{Q}_l)$ the *k-right-triangle associated with the square* $\mathcal{Q}_l$. When $l = 1$ and there are no ambiguities, we will denote it by simply $\mathcal{T}_k$.

The limit case $k = 1$ gives $M_1 = C$ and $N_1 = D$, thus the triangle $\mathcal{T}_1(\mathcal{Q}_l)$ is the isosceles triangle BOC (see Figure 1, Right). We also note that when k goes to $+\infty$, then M_k goes to D, and $\mathcal{T}_k(\mathcal{Q}_l)$ goes to the isosceles right triangle DAB.

In Lemma 1 we explicitly compute the lengths of the edges of the triangle $\mathcal{T}_k(\mathcal{Q}_l)$, from which it is immediate to see that, for $k = 2$, the triangle $\mathcal{T}_2(Q_l) = BEM_2$ is similar to the Pythagorean triangle $3-4-5$ (see Figure 1, right).

The converse statement represents the main result of this note. In Theorem 1, we prove that, for every choice of the initial square $\mathcal{Q}_l$, the integer 2 is the *unique* rational number greater than 1 such that $T_k(\mathcal{Q}_l)$ is similar to a Pythagorean triangle. This result gives a characterization of the $3-4-5$

triangle as the only Pythagorean triangle that can be realized as $\mathcal{T}_k(\mathcal{Q}_l)$ for some choices of $\ell > 0$ and $k \geq 1$.

Despite the elementary nature of the question - the idea of the above research arose from Problem n.9, 30 - January-2023, Flatlandia (see [5]) -, proving this result requires more advanced techniques. Indeed, in Lemma 2 we show that the existence of a Pythagorean triangle similar to $\mathcal{T}_k$, for some $k \in \mathbb{Q}_{\geq 1}$, implies the existence of a *rational point* - that is a point whose coordinates are rational numbers - with certain properties on the elliptic curve

$$\mathcal{E}\colon y^2 = x^3 - 7x - 6 \ .$$

In order to show our result, we compute all the rational points of this curve, exploiting the group structure of these points. In this way, we find out that the elliptic curve has only one rational point corresponding to a Pythagorean triangle similar to $\mathcal{T}_k$, proving our result.

Computing the rational points of an elliptic curve is a complex task, as it involves computing what is called *the rank* of the elliptic curve. In the last two centuries, this problem proved to be very challenging and it is the motivation of the Birch–Swinnerton-Dyer conjecture, the first of the Millennium problems, still standing to this day.

The paper is structured as follows. In Section 2 we compute the edges of the triangle $\mathcal{T}_k$, showing that if $\mathcal{T}_k$ is similar to a Pythagorean triangle, then the diophantine equation given by (3) admits a positive solution. In Section 3, after showing that the above equation defines an elliptic curve $\mathcal{E}$, we compute all the rational points of $\mathcal{E}$; in particular, we perform an explicit complete 2-descent to compute the rank of $\mathcal{E}$, showing that there is only one point coming from a triangle $\mathcal{T}_k$ that is similar to a Pythagorean triangle. In the fourth and last section, we show that, on the one hand, if we allow k to be real and not just rational, then there are more values of $k \in \mathbb{R}$ for which $\mathcal{T}_k$ is similar to a Pythagorean triangle; on the other hand, we also show that there are Pythagorean triangles that are *not* similar to $\mathcal{T}_k$, for any $k \in \mathbb{R}$.

2 The lengths of the edges

Recall that, given a point O in the Euclidean space $\mathbb{E}^3$ and a positive real number ρ, a *homothety with center O and ratio ρ* is the mapping of $\mathbb{E}^3$ onto itself that associates to each point P in $\mathbb{E}^3$ the point P' such that

$OP' = \rho\, OP$.

Remark 1. Let $\mathcal{Q}_l = (ABCD)$ be a square of side l, and let f be a homothety of ratio ρ and center O (the center of the square). Then, for every $k \geq 1$, the right triangles $\mathcal{T}_k(\mathcal{Q}_l)$ and $\mathcal{T}_k(\mathcal{Q}_{\rho l})$ are similar. In particular, the ratio between their legs is the same.

The following lemma characterizes the ratio between the legs of any triangle $\mathcal{T}_k(\mathcal{Q}_l)$.

Lemma 1. *Let $\mathcal{Q}_l = (ABCD)$ be a square of side l, and consider $k \in \mathbb{R}_{\geq 1}$. Let a_k, b_k, c_k be the lengths of the legs M_kE_k, BE_k, and the hypotenuse BM_k of the triangle $BE_kM_k = \mathcal{T}_k(\mathcal{Q}_l)$ constructed as in Section 1, respectively. Then*

$$a_k = \frac{k^2-k+1}{k\sqrt{k^2+1}}\,, \qquad b_k = \frac{k}{\sqrt{k^2+1}}\,, \qquad c_k = \frac{\sqrt{2k^2-2k+1}}{k}\,. \tag{1}$$

In particular, we have

$$a_k/b_k = \frac{k^2-k+1}{k^2} \tag{2}$$

and hence $a_k < b_k$.

Proof. By virtue of Remark 1, we may assume $l = 1$. Referring to the left-hand side of Figure 1, we have

$$b_k = \frac{1}{\sqrt{1+\tan^2(\beta)}} = \frac{1}{\sqrt{1+1/k^2}} = \frac{k}{\sqrt{k^2+1}}\,.$$

From this, it follows

$$\begin{aligned} a_k = M_kA - E_kA &= \sqrt{1+\frac{1}{k^2}} - \sqrt{1-\frac{k^2}{k^2+1}} = \\ &= \frac{\sqrt{k^2+1}}{k} - \frac{1}{\sqrt{k^2+1}} = \frac{k^2+1-k}{k\sqrt{k^2+1}}\,. \end{aligned}$$

The length c_k of the hypotenuse of the triangle $\mathcal{T}_k(\mathcal{Q}_l)$ is obtained directly as a simple application of the Pythagorean Theorem (see Figure 1)

$$c_k = \sqrt{1+\left(1-\frac{1}{k}\right)^2} = \sqrt{1+\left(\frac{k-1}{k}\right)^2} = \frac{\sqrt{2k^2-2k+1}}{k}\,,$$

completing the proof. □

Lemma 2. *Let $\mathcal{Q}_l$ be a square of side l, and let $k \geq 1$ be a rational number. If $\mathcal{T}_k(\mathcal{Q}_l)$ is similar to a Pythagorean triangle, then there exists a rational positive number y such that*

$$(k^2+1)(2k^2-2k+1) = y^2. \tag{3}$$

Proof. By Remark 1 we may assume that $l = 1$. By hypothesis, $\mathcal{T}_k := \mathcal{T}_k(\mathcal{Q}_1)$ is similar to a Pythagorean triangle, say Δ, whose lengths of the legs and the hypotenuse are the positive numbers $a < b < c$. Let $a_k < b_k$ and c_k be the lengths of the legs and the hypotenuse of $\mathcal{T}_k$, respectively. Then there exists a positive real number ρ such that

$$a = \rho a_k, \qquad b = \rho b_k, \qquad c = \rho c_k. \tag{4}$$

By Lemma 1,

$$a = \rho a_k = \rho \frac{k^2-k+1}{k\sqrt{k^2+1}}, \quad b = \rho b_k = \rho \frac{k}{\sqrt{k^2+1}}, \quad c = \rho c_k = \rho \frac{\sqrt{2k^2-2k+1}}{k}. \tag{5}$$

In particular,the ratio $y := \dfrac{k^2 c}{b}$ is a positive rational number

$$y = \sqrt{(k^2+1)(2k^2-2k+1)}\,.$$

Therefore, if $\mathcal{T}_k$ is similar to a Pythagorean triangle, then there exists at least one positive rational number y such that the equation (3) is satisfied. □

Remark 2.A computer-aided exploration suggests that the pair $(2, 5)$ is the only solution in positive integers for the equation (3). Theorem 1 will show that in fact $(2, 5)$ is the only solution in positive rational numbers. The proof of the theorem uses some classical results in the algebraic theory of elliptic curves. For the definitions and the results used below, we refer to [2, Chapters III, VIII].

3 The main result

Let $\mathcal{Q}_l$ be a square with side l and let $k \geq 1$ be a rational number. Let $\mathcal{T}_k(\mathcal{Q}_l)$ be the right triangle associated with $\mathcal{Q}_l$ constructed as in Section 1.

Theorem 1. *The triangle* $\mathcal{T}_k(\mathcal{Q}_l)$ *is similar to a Pythagorean triangle if and only if* $k=2$. *Moreover, in this case,* $\mathcal{T}_k(\mathcal{Q}_l)$ *is similar to the triangle* $3-4-5$.

For the convenience of the reader, we will split the proof into several steps. In the following, recalling Remark 1, without loss of generality we may and do assume that $l=1$ and we will denote the triangle $\mathcal{T}_k(\mathcal{Q}_l)$ by simply $\mathcal{T}_k$. First, we prove the implication from right to left.

Lemma 3. *The triangle* $\mathcal{T}_2$ *is similar to the Pythagorean triangle* $3-4-5$.

Proof. From Lemma 1 it follows that the ratio of the legs of $\mathcal{T}_2$ is $a_2/b_2 = 3/4$. As a right triangle is determined, up to homotheties, by the ratio of its legs, the statement follows. □

We are left to prove the converse implication. Recall that Lemma 2 shows that if $\mathcal{T}_k$ is similar to a Pythagorean triangle, then there exists a rational positive number u satisfying the equation (3). We are then interested in its (positive) rational solutions.

In the following we will denote by $\mathbb{A}^1(k)$ and $\mathbb{A}^2(k,u)$ the affine line with coordinate k and the affine plane with coordinates k, u, respectively.

Let $\mathcal{C} \subset \mathbb{A}^2(k,u)$ be the curve defined by the equation (3)

$$\mathcal{C}\colon (k^2+1)(2k^2-2k+1) = u^2 \ .$$

Remark 3. It is easy to see that $\mathcal{C}$ has at least four rational points:

$$(0,\pm 1), (2,\pm 5) \ .$$

Lemma 4. The curve $\mathcal{C}$ is isomorphic, over $\mathbb{Q}$, to the elliptic curve with equation

$$\mathcal{E}\colon y^2 = x^3 - 7x - 6 \ . \tag{6}$$

Proof. The projection $(k,u) \mapsto k$ endows $\mathcal{C}$ with the structure of a double cover of $\mathbb{A}^1(k)$ ramified above 4 points. Hence, $\mathcal{C}$ is a curve of genus 1. By fixing the point $(0,1)$, the curve $\mathcal{C}$ can be viewed as an elliptic curve. By means of translation and rotations, for example by following the procedure presented in [2, Section III.1], one can bring $\mathcal{C}$ into a short Weierstrass form, obtaining (6). As all the translations and rotations involved can be defined over $\mathbb{Q}$, we get the desired isomorphism between $\mathcal{C}$ and $\mathcal{E}$, proving the statement. □

The existence of an isomorphism between $\mathcal{C}$ and $\mathcal{E}$ that is defined over $\mathbb{Q}$ implies that the rational points of $\mathcal{C}$ correspond to the rational points of $\mathcal{E}$. We are then left to study the rational points of $\mathcal{E}$. Studying the rational points of an elliptic curve is a classical problem in arithmetic geometry and it is the subject of one (in fact, the first) of the Millennium Prize Problems: the Birch–Swinnerton-Dyer conjecture. One of the main features of an elliptic curve (if not *the* main feature), is that the set of its points can be endowed with a group structure. The main tool to study this group structure is given by the Mordell–Weil theorem [2, Section VIII.4], stating that the set of rational points of an elliptic curve is a finitely generated abelian group. This means that

$$\mathcal{E}(\mathbb{Q}) := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \ : \ y^2 = x^3 - 7x - 6 \ \}$$

is isomorphic, as a group, to $\mathbb{Z}^r \oplus T$, for some integer $r \in \mathbb{Z}_{\geq 0}$ and some finite group T.

Remark 4. The integer r is called *the rank* of $\mathcal{E}(\mathbb{Q})$, while the subgroup T is called *the torsion part* of $\mathcal{E}(\mathbb{Q})$.

In general, computing the torsion part of an elliptic curve is easy, while computing the rank is hard. For this reason, we will perform these two tasks separately and we will eventually show that

$$\mathcal{E}(\mathbb{Q}) = \{\mathcal{O}, (-1, 0), (-2, 0), (3, 0)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ ,$$

where $\mathcal{O}$ is *the neutral element* of $\mathcal{E}$ and corresponds to the point at infinity $(0 : 1 : 0)$ in the projective closure of $\mathcal{E}$.

Lemma 5. *The torsion part of* $\mathcal{E}(\mathbb{Q})$ *is* $T = \{\mathcal{O}, (-1, 0), (-2, 0), (3, 0)\}$ *and it is isomorphic as a group to* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. The point $\mathcal{O} = (0 : 1 : 0)$ always belongs to the projective closure of an elliptic curve given in a short Weierstrass form as it serves as the neutral element of the group given by its rational points. By the definition of group law on an elliptic curve in short Weierstrass form, we know that the points with second coordinate equal to 0 are the points of order 2. It is easy to see that the equation $x^3 - 7x - 6 = 0$ has solutions $x = -1, -2, 3$, yielding the 2-torsion points $(-1, 0), (-2, 0), (3, 0)$. Hence, the points $\{\mathcal{O}, (-1, 0), (-2, 0), (3, 0)\}$ form a subgroup of T that is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We are left to show that T is exactly $\{\mathcal{O}, (-1, 0), (-2, 0), (3, 0)\}$.

As we have shown that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a subgroup of T, Mazur theorem (see [2, Theorem VIII.7.5]) implies that T is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}$

with $N \in \{1,2,3,4\}$. Nevertheless, using the reduction modulo 3 (see [2, Section VII.3], we obtain that T is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}$ with $N \in \{1,3\}$. Using the same approach with the reduction modulo 23 we obtain that T is a subgroup of an abelian group of order 20. This means that T contains no elements of order 3 and hence

$$T = \{\mathcal{O}, (-1,0), (-2,0), (3,0)\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ ,$$

concluding the proof. □

We are now ready for the most challenging part of our result, that is, computing the rank of $\mathcal{E}$. There is no general algorithm to compute the rank of a given elliptic curve, but in our case, we can exploit the presence of four 2-torsion points and explicitly perform a complete 2-descent on $\mathcal{E}$. We refer to [2, Sections VIII.3] for all the details. See also [2, Sections X.1] for an explicit example that is very similar to the case treated here.

Lemma 6. *The rank of* $\mathcal{E}(\mathbb{Q})$ *is* $r = 0$.

Proof. Consider the quotient group $A := \mathcal{E}(\mathbb{Q})/2\mathcal{E}(\mathbb{Q})$. If $\mathcal{E}(\mathbb{Q}) = \mathbb{Z}^r \oplus T$ with $T \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $A \cong (\mathbb{Z}/2\mathbb{Z})^{r+2}$. Let S be the set of places of $\mathbb{Q}$ given by ∞ and the primes dividing the discriminant $\Delta(\mathcal{E})$ of $\mathcal{E}$. As $\Delta(\mathcal{E}) = 6400$, we obtain $S = \{\infty, 2, 5\}$. We then consider the multiplicative subgroup of $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ given by

$$K(S,2) := \{b \in \mathbb{Q}^*/(\mathbb{Q}^*)^2 \ : \ \mathrm{ord}_v \equiv 0 \bmod 2 \ \ \forall v \in S \ \} \ .$$

In our case, we have that

$$K(S,2) = \{\pm 1, \pm 2, \pm 5, \pm 10 \bmod (\mathbb{Q}^*)^2\} \subset \mathbb{Q}^*/(\mathbb{Q}^*)^2 \ .$$

One can then define an injective group homomorphism

$$\iota \colon A \hookrightarrow K(S,2) \times K(S,2)$$

such that an element $(b_1, b_2) \in K(S,2) \times K(S,2)$ is the image of an element of A different from the classes of $\mathcal{O}, (-1,0), (-2,0), (3,0)$ if and only if the equations

$$\begin{cases} b_1 z_1^2 - b_2 z_2^2 = -1 \\ b_1 z_1^2 - b_1 b_2 z_3^2 = 4 \end{cases} \tag{7}$$

have a solution $(z_1, z_2, z_3) \in \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}$.

By looking at the explicit definition of ι given in [2, Proposition X.1.4], one can easily see that the elements of $K(S,2) \times K(S,2)$ corresponding to the images of the classes of $\mathcal{O}, (-1,0), (-2,0), (3,0)$ in A are

$$(1,1), (-1,1), (-1,5), (1,5),$$

respectively. We want to show that all the other elements of $K(S,2) \times K(S,2)$ are not in the image of ι. To do so, we have to proceed with a case-by-case analysis on the elements of $K(S,2) \times K(S,2)$.

Clearly, the group $K(S,2) \times K(S,2)$ contains 64 elements, four of which we already know are contained in the image of ι. We are left to check 60 elements. This tedious chore can be mitigated by some useful observations:

- For $b_2 < 0$ and $b_1 > 0$, the first equation of (7) admits no solutions over $\mathbb{R}$ (and hence over $\mathbb{Q}$) as the sum of two positive quantities cannot be equal to -1; analogously, for $b_2 < 0$ and $b_1 < 0$ the second equation of (7) does not admit solutions over $\mathbb{R}$ (and hence over $\mathbb{Q}$); combining these two observations we have that every element $(b_1, b_2) \in K(S,2) \times K(S,2)$ with $b_2 < 0$ is not in the image of ι. This remark takes care of 32 of the remaining 60 elements to be checked.

- Using the group structure of $K(S,2) \times K(S,2)$ and the fact that ι is a group homomorphism, one immediately deduces that if (b_1, b_2) is not in the image of ι, then neither are the elements

$$(-b_1, b_2), (-b_1, 5b_2), (b_1, 5b_2) .$$

  This observation reduces our work to checking "only" 7=28/4 cases.

- If z_1, z_2, z_3 are rational solutions of (7), then we can write

$$z_1 = a/d \ , \quad z_2 = b/d \ , \quad z_3 = c/d$$

  with $a, b, c, d \in \mathbb{Z}$ and no common factors, that is, $\gcd(a, b, c, d) = 1$. This means that we have integral coprime solutions to the following homogenous quadratic equations:

$$\begin{cases} b_1 a^2 - b_2 b^2 = -d^2 \\ b_1 a^2 - b_1 b_2 c^2 = 4d^2 \end{cases} . \tag{8}$$

We then proceed to the case-by-case analysis of the remaining seven cases.

- $(b_1, b_2) = (1, 2)$. In this case, (8) becomes

$$\begin{cases} a^2 - 2b^2 = -d^2 \\ a^2 - 2c^2 = 4d^2 \end{cases} . \tag{9}$$

By looking at the second equation modulo 4 one obtains that $a, c \equiv 0 \bmod 2$. This means that a is even and hence, from the first equation, also d must be even. This means that d^2 is divisible by 4. Then, looking at the first equation modulo 4, it follows that also b is even. In this way, we have shown that a, b, c, d are all even, contradicting the hypothesis of coprimality between them, hence concluding that (8) has no solution over $\mathbb{Q}$. Therefore, $(1, 2)$ is not in the image of ι. Using the observation above, we deduce that $(-1, 2), (-1, 10), (1, 10)$ are not in the image of ι either.

- $(b_1, b_2) = (2, 1)$. In this case, (8) becomes

$$\begin{cases} 2a^2 - b^2 = -d^2 \\ 2a^2 - 2c^2 = 4d^2 \end{cases} . \tag{10}$$

Looking at the two equations modulo 2 it immediately follows that $a \equiv c$ and $b \equiv d \bmod 2$. Then, looking at the first equation modulo 4 it follows that d (and hence b) is even. Then from the second equation, also reduced modulo 4, we deduce that also a, and hence c are even, again contradicting the hypothesis of coprimality. As in the previous case, we conclude that $(2, 1), (-2, 1), (-2, 5), (2, 5)$ are not in the image of ι.

- $(b_1, b_2) = (2, 2)$. In this case, (8) becomes

$$\begin{cases} 2a^2 - 2b^2 = -d^2 \\ 2a^2 - 4c^2 = 4d^2 \end{cases} . \tag{11}$$

By looking at the equation we immediately see that a and d must be even. Then again reasoning modulo 4 we obtain that also b and c must be even, again contradicting the hypothesis of coprimality. We conclude that $(2, 2), (-2, 2), (-2, 10), (2, 10)$ are not in the image of ι.

- $(b_1, b_2) = (5, 1)$. In this case, (8) becomes

$$\begin{cases} 5a^2 - b^2 = -d^2 \\ 5a^2 - 5c^2 = 4d^2 \end{cases} . \tag{12}$$

Reducing the two equations modulo 5 we obtain that d and b are both divisible by 5. Then from the first equation it follows that so is a too. In turn, analogously, the same follows for c from the second, contradicting the hypothesis of coprimality for a, b, c, d. We conclude that $(5, 1), (-5, 1), (-5, 5), (5, 5)$ are not in the image of ι.

- $(b_1, b_2) = (5, 2), (10, 1), (10, 2)$. All these cases are analogous to the one above: we look at the equations modulo 5 to conclude that if a, b, c, d solve the equations then they are all divisible by 5, contradicting their coprimality. We conclude that all the remaining elements are not in the image of ι.

We have then shown that the image of ι consists of exactly 4 elements. As ι is injective, this means that $A \cong (\mathbb{Z}/2\mathbb{Z})^{r+2}$ consists of exactly 4 elements, yielding $r = 0$ and concluding the proof. □

Corollary 1. $\mathcal{E}(\mathbb{Q}) = \{\mathcal{O}, (-1, 0), (-2, 0), (3, 0)\}$.

Proof. Recalling the Mordell–Weil theorem, it follows immediately from lemmas 5 and 6. □

We have now everything we need to prove our main result.

Proof. [Proof of Theorem 1] Corollary 1 tells us that $\mathcal{E}$ has only four points over $\mathbb{Q}$. Lemma 4 tells us that the rational points of $\mathcal{E}$ correspond to the rational points of $\mathcal{C}$, hence (3) has only four pairs of solutions over $\mathbb{Q}$. This means that the four points of $\mathcal{C}$ observed in Remark 3 are the only solutions of (3) over $\mathbb{Q}$

$$(0, \pm 1), (2, \pm 5).$$

Among these solutions, the only one given by positive rational numbers is $(2, 5)$. Hence $k = 2$ is the only rational value of k for which $\mathcal{T}_k$ is similar to a Pythagorean triangle. Keeping in mind Lemma 3, the proof is then complete. □

Remark 5. Although there is no general algorithm to compute the rank of an elliptic curve, there are several algorithms able to compute the rank in some cases, as the one treated in this note. Our computations can be quickly performed using a computer algebra for elliptic curves, like SAGE [3] or MAGMA [1]. In fact, the elliptic curve $\mathcal{E}$ is very well known and one can find many pieces of information about it, including its rank and torsion points, in the online database [4]: it is the elliptic curve with label 40.a2.

4 Further remarks

Remark 6. The previous characterization proves that the only rational value for which $\mathcal{T}_k$ is similar to a Pythagorean triangle is $k = 2$. This does not exclude the possibility that there may exist irrational values of k for which the triangle $\mathcal{T}_k$ is similar to other Pythagorean triangles. For example, consider the solution

$$k = \frac{55}{14} + \frac{3\sqrt{165}}{14}$$

of the equation $\frac{k^2-k+1}{k^2} = 48/55$. Then, by Lemma 1, $\mathcal{T}_k$ is similar to the Pythagorean triangle $48-55-73$.

The reader can easily find other (irrational) values of k for which $\mathcal{T}_k(\mathcal{Q}_1)$ is a right triangle similar to some Pythagorean triangle.
At this point, it is natural to ask whether every Pythagorean triangle is similar to $\mathcal{T}_k$, for some real number $k > 1$. The answer is negative, since the ratio of the sides of a triangle $\mathcal{T}_k$ lies within a bounded interval of the real field.

Proposition 1. *Let $k > 1$ be a real number, and let $a_k < b_k$ be the legs of the k-triangle $\mathcal{T}_k$. Then $3/4 \leq a_k/b_k < 1$.*

Proof. Put $y(k) := a_k/b_k$. By Lemma 1 $y(k) = \frac{k^2-k+1}{k^2}$. Note that for every $k \in [1, +\infty]$, the function $y(k)$ assumes values belonging to $[3/4, 1]$ (see Figure 2). Precisely:
$min\{(k^2-k+1)/k^2\} = 3/4$ at $k = 2$ and $\lim_{k\to\infty} \frac{1-k+k^2}{k^2} = 1$. □

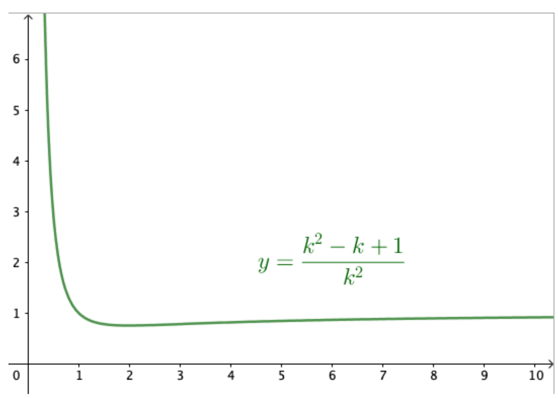

Figure 2: The function $y = \frac{k^2-k+1}{k^2}$ describes the ratio of the leg a_k and b_k of the triangle T_k.

Example 1. As pointed out in Remark 6, there are several examples of Pythagorean triangles that are similar to some $\mathcal{T}_k$. However, as suggested by Proposition 1, we can easily find examples of Pythagorean triangles that do not enjoy this property. For instance, the triangle $11-60-61$, as the (minor) ratio of its legs is $11/60 < 3/4$.

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