

A REVERSED ANALYSIS OF STAR-MENGER AND ALLIED SPACES*

Prasenjit Bal[†]

Communicated by G. Moroşanu

DOI 10.56082/annalsarscimath.2025.2.115

Abstract

This paper presents a comprehensive characterization of Menger spaces and star-Menger spaces through the lens of families of closed sets, employing nuanced modifications of the classical finite intersection property. By introducing and analyzing specific intersection patterns within these families, we develop conditions that encapsulate the essence of the Menger and star-Menger covering properties. Furthermore, we explore the associated selection principles and demonstrate how they can be systematically reversed to reconstruct the topological structure of Menger and star-Menger spaces. This dual perspective not only offers an alternative viewpoint on classical results but also contributes to the ongoing effort to bridge the gap between topological covering properties and combinatorial selection theory. Our results provide a new framework that enhances the theoretical understanding of these spaces and may inspire further investigations into related classes of topological spaces.

Keywords: Menger Space, star-Menger Space, selection principles, finite intersection property.

MSC: 54D20, 54D30, 54D40.

*Accepted for publication on July 11, 2025

[†]balprasenjit177@gmail.com, Department of Mathematics, ICAFI University Tripura, Kamalghat, 799210, India

1 Introduction and preliminaries

Menger property and Rothberger property are the most fascinating sequential covering features for topologists worldwide.

Definition 1. A space X is said to have the Menger covering property [17] if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X , there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n \in \mathbb{N}} (\bigcup \mathcal{V}_n) = X$.

In 1924, Karl Menger [17] introduced the concept of Mengeress or Menger covering property. There are many generalizations of Mengeress in the literature. We find the St-Mengeress to be the most interesting one which was introduced by Kočinac [15, 16] in 1999.

If M is a subset of a set X and $\mathcal{U}$ is a collection of subsets of X , then the star of M with respect to $\mathcal{U}$ is the set $St(M, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap M \neq \emptyset\}$ [11].

The star operator was used to generalize the concept of compactness and Lindelöfness by E.K. van Douwen [11] in 1991. Then Kočinac [15] used it for the generalization of Menger space, Rothberger space and selection principles. Some recent usage of St-operator can be found [1–10, 19].

Definition 2. A space X is said to have the star-Menger property [15] if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open cover of X , there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n \in \mathbb{N}} St(\bigcup \mathcal{V}_n, \mathcal{U}_n)$.

Although these sequential covering properties are extensively studied by many researchers [14, 18, 21–28], their representation with the help of family of closed sets has not received much attention.

Recall that a collection $\mathcal{F}$ of subsets of a set X has the finite intersection property (FIP) if the intersection of any finite subcollection of $\mathcal{F}$ is non empty. A topological space is compact if and only if every collection of closed subsets meeting the FIP has a non empty intersection itself. The use of FIP makes this alternative notion of compactness achievable [12].

In our research, we find such types of representations for Mengeress and St-Mengeress with a little variation in finite intersection property (FI property).

Throughout the paper, a space X denotes a topological space X equipped with the corresponding topology τ . For a space X we adopt the following symbols:

$\mathcal{O}$: the collection of all open covers of X .

$\mathcal{C}_X$: the collection of all families $\mathcal{F}$ of closed sets for which $\cap \mathcal{F} = \emptyset$.

$\mathcal{A}, \mathcal{B}$: represents collections of families of subsets of a space X .

Selection principles are other ways to describe sequential covering properties.

Definition 3. The symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ it is a finite subset of $\mathcal{U}_n$ and $\{\cup \mathcal{V}_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$ [20].

Definition 4. The symbol $S_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ it is a finite subset of $\mathcal{U}_n$ and $\{St(\cup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$ [13].

It is crucial to note that the type of characterization of the selection principle for Menger and St-Menger is given by $S_{fin}(\mathcal{O}, \mathcal{O})$ and $S_{fin}^*(\mathcal{O}, \mathcal{O})$, respectively.

In our investigation, we search for some new selection principles which can characterize Menger spaces and star Menger spaces through the family $\mathcal{C}_X$.

2 Main result

Definition 5. Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of families of subsets of X . This sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is said to have the sequential finite intersection property (SFI property) if for every sequence $\{\mathcal{E}_n : n \in \mathbb{N}\}$ where $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite, we have $\cap_{n \in \mathbb{N}} (\cap \mathcal{E}_n) \neq \emptyset$.

Theorem 1. The following conditions are equivalent:

- (1) X is a Menger space;
- (2) For every sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of families of closed sets with sequential finite intersection property (SFI property), there exists a $n_0 \in \mathbb{N}$, $\cap \mathcal{F}_{n_0} \neq \emptyset$.

Proof. Let X be a Menger space and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of families of closed sets having the sequential finite intersection property (SFI property). If possible let $\cap \mathcal{F}_n = \emptyset \quad \forall n \in \mathbb{N}$ and suppose $\mathcal{G}_n = \{X \setminus F : F \in \mathcal{F}_n\}$, $\forall n \in \mathbb{N}$.

Therefore, $\cup \mathcal{G}_n = \cup \{X \setminus F : F \in \mathcal{F}_n\}$, for all $n \in \mathbb{N}$

$$\begin{aligned}
&= X \setminus \cap_{F \in \mathcal{F}_n}(F), \quad \text{for all } n \in \mathbb{N} \\
&= X \setminus \cap \mathcal{F}_n, \quad \text{for all } n \in \mathbb{N} \\
&= X \setminus \emptyset, \quad \text{for all } n \in \mathbb{N} \\
&= X, \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Therefore $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open covers of X . But X is a Menger space. Therefore there exists a sequence $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $\mathcal{H}_n \subseteq \mathcal{G}_n$ is finite for all $n \in \mathbb{N}$ and $\cup_{n \in \mathbb{N}}(\cup \mathcal{H}_n) = X$.

Now, we construct the sequence $\{\mathcal{E}_n : n \in \mathbb{N}\}$ where,

$$\mathcal{E}_n = \{X \setminus H : H \in \mathcal{H}_n\} \text{ for all } n \in \mathbb{N}.$$

Clearly $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite for all $n \in \mathbb{N}$ and

$$\begin{aligned}
&\cap_{n \in \mathbb{N}}(\cap \mathcal{E}_n) \\
&= \cap_{n \in \mathbb{N}}(\cap \{X \setminus H : H \in \mathcal{H}_n\}) \\
&= \cap_{n \in \mathbb{N}}(X \setminus (\cup \{H : H \in \mathcal{H}_n\})) \\
&= X \setminus (\cup_{n \in \mathbb{N}}(\cup \mathcal{H}_n)) \\
&= X \setminus X \\
&= \emptyset.
\end{aligned}$$

Therefore, $\{\mathcal{E}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite for all $n \in \mathbb{N}$ but $\cap_{n \in \mathbb{N}}(\cap \mathcal{E}_n) = \emptyset$, contradicts the fact that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ has sequential finite intersection property (SFI property). Therefore, there must exists a $n_0 \in \mathbb{N}$ such that $\mathcal{F}_{n_0} \neq \emptyset$.

$$(2) \implies (1)$$

Let the condition (2) hold and assume that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be an arbitrary sequence of open covers for a topological space X . Therefore, $\cup \mathcal{G}_n = X \quad \forall n \in \mathbb{N}$.

If we take, $\mathcal{F}_n = \{X \setminus G : G \in \mathcal{G}_n\}$, for all $n \in \mathbb{N}$, then

$$\begin{aligned}
\cap \mathcal{F}_n &= \cap \{X \setminus G : G \in \mathcal{G}_n\}, \text{ for all } n \in \mathbb{N} \\
&= X \setminus \cup \mathcal{G}_n, \quad \text{for all } n \in \mathbb{N} \\
&= X \setminus X, \quad \text{for all } n \in \mathbb{N} \\
&= \emptyset, \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Therefore, $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of families of closed sets such that $\cap \mathcal{F}_n = \emptyset \quad \forall n \in \mathbb{N}$. So by contraposition of the statement (2), $\{\mathcal{F}_n : n \in \mathbb{N}\}$ must not have the sequential finite intersection property (SFI property). Consequently, there exists a sequence $\{\mathcal{E}_n : n \in \mathbb{N}\}$ such that $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite and $\forall n \in \mathbb{N}$ with $\cap_{n \in \mathbb{N}}(\cap \mathcal{E}_n) = \emptyset$.

Suppose $\mathcal{H}_n = \{X \setminus E : E \in \mathcal{E}_n\}$, for all $n \in \mathbb{N}$. Clearly, $\mathcal{H}_n \subseteq \mathcal{G}_n$ is finite $\forall n \in \mathbb{N}$. Therefore, $\cup_{n \in \mathbb{N}}(\cup \mathcal{H}_n)$

$$\begin{aligned}
&= \cup_{n \in \mathbb{N}}(\cup \{X \setminus E : E \in \mathcal{E}_n\}) \\
&= \cup_{n \in \mathbb{N}}(X \setminus \cap \mathcal{E}_n)
\end{aligned}$$

$$\begin{aligned}
&= X \setminus (\cap_{n \in \mathbb{N}} (\cap \mathcal{E}_n)) \\
&= X \setminus \emptyset \\
&= X.
\end{aligned}$$

Therefore, $\{\mathcal{H}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{H}_n \subseteq \mathcal{G}_n$ is finite for all $n \in \mathbb{N}$ and $\cup_{n \in \mathbb{N}} (\cup \mathcal{H}_n) = X$. Therefore, X is a Menger space. $\square$

Corollary 1. $S_{fin}(\mathcal{O}, \mathcal{O})$ and $S_{fin}(\mathcal{C}_X, \mathcal{C}_X)$ are equivalent.

Proof. Similarly to the proof of the above theorem. Hence omitted. $\square$

Definition 6. A sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of families of subsets of X is said to have modified sequential finite intersection property (MSFI property) if for all sequences $\{\mathcal{E}_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite and $\mathcal{H}_n \subseteq \mathcal{F}_n$, $\forall n \in \mathbb{N}$ either $(\cap \mathcal{E}_n) \cup F = X$ for some $F \in \mathcal{H}_n$ or

$$\bigcap_{n \in \mathbb{N}} (\cap \mathcal{H}_n) \neq \emptyset.$$

Theorem 2. The following conditions are equivalent:

- (1) (X, τ) is a St-Menger space;
- (2) If the sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of families of closed sets has modified sequential finite intersection property (MSFI property) then there exists a $n_0 \in \mathbb{N}$ such that $\cap \mathcal{F}_{n_0} \neq \emptyset$.

Proof. Let (X, τ) be a St-Menger space and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a sequence of families of closed sets having modified sequential finite intersection property (MSFI property). Suppose that $\cap \mathcal{F}_n = \emptyset$, $\forall n \in \mathbb{N}$. Now we assume $\mathcal{G}_n = \{X \setminus F : F \in \mathcal{F}_n\}$ for all $n \in \mathbb{N}$. Therefore, $\cup \mathcal{G}_n = \bigcup \{X \setminus F : F \in \mathcal{F}_n\} = X$, for all $n \in \mathbb{N}$. Therefore, $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open covers. But (X, τ) is a St-Menger space, therefore there exists a sequence $\{\mathcal{G}'_n : n \in \mathbb{N}\}$ where $\mathcal{G}'_n \subseteq \mathcal{G}_n$ is finite for all $n \in \mathbb{N}$ such that

$$\begin{aligned}
&\bigcup_{n \in \mathbb{N}} \{St(\bigcup \mathcal{G}'_n, \mathcal{G}_n)\} = X \\
&\implies \bigcup_{n \in \mathbb{N}} \{\bigcup \{G \in \mathcal{G}_n : (\bigcup \mathcal{G}'_n) \cap G \neq \emptyset\}\} = X \\
&\implies \bigcup_{n \in \mathbb{N}} \{\bigcup \{(X \setminus F) \in \mathcal{G}_n : (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset\}\} = X \\
&\implies \bigcup_{n \in \mathbb{N}} \{\bigcup \{(X \setminus F) : F \in \mathcal{F}_n \text{ and } (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset\}\} = X
\end{aligned}$$

$$\begin{aligned}
&\implies \bigcup_{n \in \mathbb{N}} \{X \setminus \bigcap \{F : F \in \mathcal{F}_n \text{ and } (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset\}\} = X \\
&\implies X \setminus \bigcap_{n \in \mathbb{N}} \{\bigcap \{F \in \mathcal{F}_n : (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset\}\} = X \\
&\implies \bigcap_{n \in \mathbb{N}} \{\bigcap \{F \in \mathcal{F}_n : (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset\}\} = \emptyset.
\end{aligned}$$

Now, let $\mathcal{H}_n = \{F \in \mathcal{F}_n : (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset\}$ for all $n \in \mathbb{N}$. Therefore, $\{\mathcal{H}_n : n \in \mathbb{N}\}$ is a sequence such that $\mathcal{H}_n \subseteq \mathcal{F}_n$, $\forall n \in \mathbb{N}$. Now consider another sequence $\{\mathcal{E}_n : n \in \mathbb{N}\}$ such that $\mathcal{E}_n = \{X \setminus G : G \in \mathcal{G}'_n\}$ $\forall n \in \mathbb{N}$. Since, $\mathcal{G}'_n \subseteq \mathcal{G}_n$ is finite $\forall n \in \mathbb{N}$, therefore $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite $\forall n \in \mathbb{N}$. But,

$$\bigcap_{n \in \mathbb{N}} (\bigcap \mathcal{H}_n) = \emptyset, \quad \forall n \in \mathbb{N}$$

$$\text{and } \forall F \in \mathcal{H}_n, (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset, \quad \forall n \in \mathbb{N}$$

$$\implies (X \setminus (\bigcap \mathcal{E}_n)) \cap (X \setminus F) \neq \emptyset, \quad \forall n \in \mathbb{N}$$

$$\implies X \setminus \{(\bigcap \mathcal{E}_n) \cup F\} \neq \emptyset, \quad \forall n \in \mathbb{N}$$

$$((\bigcap \mathcal{E}_n) \cup F) \neq X.$$

This contradicts the fact that $\{\mathcal{F}_n : n \in \mathbb{N}\}$ has Modified Sequential Finite Intersection property (MSFI property). Therefore, there exists a $n_0 \in \mathbb{N}$ such that $\mathcal{F}_{n_0} \neq \emptyset$.

$$(2) \implies (1)$$

Let condition (2) hold and assume that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ be an arbitrary sequence of open covers of a topological space (X, τ) . Therefore $\bigcup \mathcal{G}_n = X$, $\forall n \in \mathbb{N}$. Let $\mathcal{F}_n = \{X \setminus G : G \in \mathcal{G}_n\}$, $\forall n \in \mathbb{N}$.

Therefore, $\bigcap \mathcal{F}_n = \bigcap \{X \setminus G : G \in \mathcal{G}_n\} = \emptyset$ $\forall n \in \mathbb{N}$. So, $\{\mathcal{F}_n; n \in \mathbb{N}\}$ is a sequence of families of closed sets such that $\bigcap \mathcal{F}_n = \emptyset$, $\forall n \in \mathbb{N}$. By contraposition of the statement (2), $\{\mathcal{F}_n : n \in \mathbb{N}\}$ must not have modified sequential finite intersection property (MSFI property). So, there exist sequences $\{\mathcal{E}_n : n \in \mathbb{N}\}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite and $\mathcal{H}_n \subseteq \mathcal{F}_n$, $\forall n \in \mathbb{N}$ with $((\bigcap \mathcal{E}_n) \cup F) \neq X$ for all $F \in \mathcal{H}_n$ or

$$\bigcap_{n \in \mathbb{N}} (\bigcap \mathcal{H}_n) = \emptyset.$$

Now we assume $\mathcal{G}'_n = \{X \setminus F : F \in \mathcal{E}_n\}$ is a sequence for all $n \in \mathbb{N}$. Since $\mathcal{E}_n \subseteq \mathcal{F}_n$ is finite $\forall n \in \mathbb{N}$, therefore $\mathcal{G}'_n \subseteq \mathcal{G}_n$ is also finite $\forall n \in \mathbb{N}$.

Now,

$$\implies ((\cap \mathcal{E}_n) \cup F) \neq X, \quad \forall F \in \mathcal{H}_n \quad \text{and} \quad \mathcal{E}_n \subseteq \mathcal{F}_n \quad \text{is finite} \quad \forall n \in \mathbb{N}$$

$$\implies X \setminus \{(\cap \mathcal{E}_n) \cup F\} \neq \emptyset, \quad \forall F \in \mathcal{H}_n \quad \text{and} \quad \mathcal{E}_n \subseteq \mathcal{F}_n \quad \text{is finite} \quad \forall n \in \mathbb{N}$$

$$\implies (X \setminus \mathcal{E}_n) \cap (X \setminus F) \neq \emptyset, \quad \forall F \in \mathcal{H}_n \quad \text{and} \quad \mathcal{E}_n \subseteq \mathcal{F}_n \quad \text{is finite} \quad \forall n \in \mathbb{N}$$

$$\implies (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset, \quad \forall F \in \mathcal{H}_n \quad \text{and} \quad \forall n \in \mathbb{N}.$$

And

$$\bigcap_{n \in \mathbb{N}} (\bigcap \mathcal{H}_n) = \emptyset$$

$$\implies \bigcap_{n \in \mathbb{N}} (\bigcap \{F \in \mathcal{F}_n : (\bigcup \mathcal{G}'_n) \cap (X \setminus F) \neq \emptyset\}) = \emptyset$$

$$\implies \bigcap_{n \in \mathbb{N}} (\bigcap \{X \setminus G : G \in \mathcal{G}'_n \quad \text{and} \quad (\bigcup \mathcal{G}'_n) \cap G \neq \emptyset\}) = \emptyset$$

$$\implies X \setminus (\bigcup_{n \in \mathbb{N}} \bigcup \{G : G \in \mathcal{G}_n \quad \text{and} \quad (\bigcup \mathcal{G}'_n) \cap G \neq \emptyset\}) = \emptyset$$

$$\implies \bigcup_{n \in \mathbb{N}} \bigcup \{G : G \in \mathcal{G}_n \quad \text{and} \quad (\bigcup \mathcal{G}'_n) \cap G \neq \emptyset\} = X \setminus \emptyset$$

$$\implies \bigcup_{n \in \mathbb{N}} \bigcup \{G : G \in \mathcal{G}_n \quad \text{and} \quad (\bigcup \mathcal{G}'_n) \cap G \neq \emptyset\} = X$$

$$\text{Therefore, } \bigcup_{n \in \mathbb{N}} St(\bigcup \mathcal{G}'_n, \mathcal{G}_n) = X.$$

So, the pair (X, τ) is a star-Menger space. □

Example 1. $S_{fin}^*(\mathcal{O}, \mathcal{O})$ and $S_{fin}^*(\mathcal{C}_X, \mathcal{C}_X)$ are not equivalent.

Let $X = B_1(0) = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 < 1\}$, $\tau = \{X, \emptyset\} \cup \{B_r(0) : r \in [0, 1]\}$. Clearly, τ is a topology on X .

For every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X we can choose a sequence $\{U_n : n \in \mathbb{N}\}$ such that $0 \in U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$.

So, $St(U_n, \mathcal{U}_n) = X$ for all $n \in \mathbb{N}$,

$\implies \{St(U_n, \mathcal{U}_n) : n \in \mathbb{N}\} = \{X\} \in \mathcal{O}$. Hence, the selection principle $S_{fin}^*(\mathcal{O}, \mathcal{O})$ holds.

Now consider a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ where $\mathcal{F}_n = \{F_{n_m} : m \in \mathbb{N}\}$ and $F_{n_m} = \{(a, b) \in \mathbb{R}^2 : 1 - \frac{1}{m} \leq a^2 + b^2 < 1\}$ for every $n \in \mathbb{N}$. Clearly, $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a sequence of families of closed sets where $\bigcap \mathcal{F}_n = \bigcap_{m \in \mathbb{N}} F_{n_m} = \emptyset$.

Also, for every selection $F_n \in \mathcal{F}_n$, $St(F_n, \mathcal{F}_n) = X$ and hence $\{St(F_n, \mathcal{F}_n) : n \in \mathbb{N}\} = \{X\}$,

$\implies \bigcap_{n \in \mathbb{N}} St(F_n, \mathcal{F}_n) = X \neq \emptyset$.

Thus, $S_{fin}^*(\mathcal{C}_X, \mathcal{C}_X)$ does not hold.

Definition 7. The symbol $S_{fin,S}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exist sequences $(\mathcal{E}_n : n \in \mathbb{N})$ and $(\mathcal{H}_n : n \in \mathbb{N})$ such that $\mathcal{E}_n \subset \mathcal{U}_n$ is finite and $\mathcal{H}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $(\bigcap \mathcal{E}_n) \cup F \neq X$ for any $F \in \mathcal{H}_n$ and $\{\bigcap \mathcal{H}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Corollary 2. $S_{fin}^*(\mathcal{O}, \mathcal{O})$ and $S_{fin,S}^*(\mathcal{C}_X, \mathcal{C}_X)$ are equivalent.

Proof. The proof is similar to the proof of Theorem 2, hence omitted. $\square$

3 Conclusion

In this paper, we have formulated some alternative representations for star-Menger spaces by means of $S_{fin,S}^*(\mathcal{A}, \mathcal{B})$ selection principle and MSFI property. These representations can further be utilized for in-depth analysis of topological star-Menger games.

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