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COEFFICIENT ESTIMATES FOR ANALYTIC FUNCTIONS SUBORDINATED TO THE FOUR LEAF FUNCTION*

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In memory of Professor Haim Brezis

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Abstract

The purpose of this research is to unify and extend the study of the well-known concept of coefficient estimates for some subclasses of analytic functions. We define the new subclass $\mathcal{A}_{4L}(\vartheta)$ of analytic functions related to the four-leaf domain to increase the adaptability of our investigation. The initial findings are the bound estimates for the coefficients $|a_n|$, $n = 2, 3$. Furthermore, we obtain the Fekete–Szegő functional and provide an estimation of the Krushkal inequality for the function class $\mathcal{A}_{4L}(\vartheta)$. In addition, we discussed initial coefficients and Fekete-Szegő type inequalities for functions of the form f^{-1} and $\frac{z}{f(z)}$ and $\frac{1}{2}\log\frac{f(z)}{z}$ linked to the function of the four leaves.

Keywords: univalent functions, starlike and convex functions of some order, subordination, four-leaf function, coefficient inequalities, Krushkal inequality, Fekete-Szegő functional.

MSC: 30C45, 30C50, 30C80.

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1 Introduction and preliminaries

We let $\mathcal{A}$ denote the class of analytic functions defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, having the power-series expansion of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathbb{D}. \tag{1}$$

Also, we let $\mathcal{S}$ denote the class of all functions of $\mathcal{A}$ that are univalent in $\mathbb{D}$.

If F and G are analytic functions in $\mathbb{D}$, and if there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$, such that $F = G \circ w$, then we say that F is *subordinated* to G, written $F(z) \prec G(z)$ (see, for example, [16] p. 368). Using the *Schwarz lemma*, it is easy to show that $F(z) \prec G(z)$ implies $F(0) = G(0)$ and $F(\mathbb{D}) \subset G(\mathbb{D})$, and assuming that G is univalent in $\mathbb{D}$ then the next equivalence holds:

$$F(z) \prec G(z) \Leftrightarrow F(0) = G(0) \text{ and } F(\mathbb{D}) \subset G(\mathbb{D}). \tag{2}$$

The classic Fekete–Szegő problem [14] involves finding the exact limits of the functional $\left|a_3 - \mu a_2^2\right|$ for a compact-function family or $f \in \mathcal{A}$ with any $\mu \in \mathbb{C}$; for further details, one may refer to [30].

The sharp bounds of this inequality for a few subclasses $\mathcal{S}^*$ and $\mathcal{K}$ were found in [12].

Gandhi in [15] introduced a set of bounded turning functions connected to a *three-leaf function*. In 2022, in the articles [4,31] the authors introduced and studied different subclasses of analytic functions defined by subordination to the *four-leaf function* (see Figure 1, made with MAPLE™ computer software) that is given by

$$\Lambda_{4L}(z) := 1 + \frac{5}{6}z + \frac{1}{6}z^5, \ z \in \mathbb{D}.$$

With the aid of a four-leaf function, we define the following subclass of $\mathcal{A}$, using the notion of subordination, as follows:

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{A}_{4L}(\vartheta)$ if*

$$\Psi_\vartheta f(z) := \frac{f(z)}{(1-\vartheta)z + \vartheta z f'(z)} \prec \Lambda_{4L}(z), \tag{3}$$

where $0 \leq \vartheta \leq 1$.

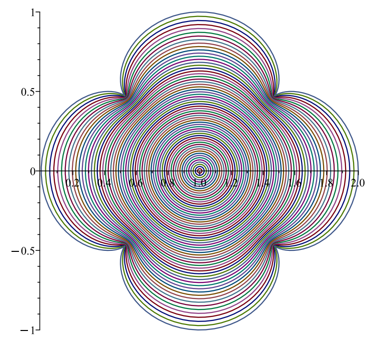


Figure 1: The image of $\Lambda_{4L}(\mathbb{D})$.

Remark 1. *Some relevant special cases of the class $\mathcal{A}_{4L}(\vartheta)$ could be obtained as follows:*

(i) For $\vartheta = 1$, the class $\mathcal{A}_{4L}(1, \Lambda) =: \mathcal{Y}_{4L}$ will be

$$\mathcal{Y}_{4L} = \left\{ f \in \mathcal{A} : \frac{f(z)}{zf'(z)} \prec \Lambda_{4L}(z) \right\}.$$

(ii) Fixing $\vartheta = 0$ in (3), we obtain the class $\mathcal{A}_{4L}(0, \Lambda) =: \mathcal{N}_{4L}$, which is

$$\mathcal{N}_{4L} = \left\{ f \in \mathcal{A} : \frac{f(z)}{z} \prec \Lambda_{4L}(z) \right\}.$$

The reason for taking the above left-hand-side expression consisted in the fact that we could obtain a subordination condition where appeared the usual expressions $\dfrac{f(z)}{z}$ and $\dfrac{f(z)}{zf'(z)}$. For special values of the parameters ϑ, some of these functions vanished or the formula became more simple and, as we can see in the further Remark 1, we could simply obtain expressions subordinated to the four-leaf function.

Many results regarding some subclasses defined by subordinations with different functions with significant geometrical properties (e.g., the limaçon function, convex functions in one direction, the cosine function, the nephroid function, etc.) were studied by the second author in many papers (see, for example, [6, 23, 24, 29]). The novelty of these subclasses and of this paper consists in the fact that such subordinations with similar expressions to the left-hand side of the subordination (3) were not studied in some other previous articles.

Remark 2. *(i) If φ is an analytic function in $\mathbb{D}$ then φ is said to be a* starlike function *with respect to $w_0 = \varphi(0)$ if φ is univalent in $\mathbb{D}$ and $\varphi(\mathbb{D})$ is a starlike domain with respect to w_0, that is the segment $[w_0, \varphi(z)]$ lies in $\varphi(\mathbb{D})$ for all $z \in \mathbb{D}$. It is well known that the function φ is starlike with respect to $w_0 = \varphi(0)$ if and only if $\varphi'(0) \neq 0$ and*

$$\operatorname{Re} \frac{z\varphi'(z)}{\varphi(z) - w_0} > 0,\ z \in \mathbb{D}.$$

Since $\Lambda_{4L}(0) = 1$, $\Lambda'_{4L}(0) = 5/6 \neq 0$ and

$$\operatorname{Re} \frac{z\Lambda'_{4L}(z)}{\Lambda_{4L}(z) - \Lambda_{4L}(0)} = 5 \operatorname{Re} \frac{1 + z^4}{5 + z^4} > 0,\ z \in \mathbb{D},$$

it follows that the four-leaf function Λ_{4L} is starlike (univalent) in $\mathbb{D}$ with respect to $w_0 = \Lambda_{4L}(0) = 1$. Moreover, from the fact that $\big(\Lambda_{4L}(1) + \Lambda_{4L}(-1)\big)/2 = 1$ it follows that the domain $\Lambda_{4L}(\mathbb{D})$ is symmetric with respect to the point $w_0 = 1$, and because $\overline{\Lambda_{4L}(z)} = \Lambda_{4L}(\overline{z})$, $z \in \mathbb{D}$ the domain $\Lambda_{4L}(\mathbb{D})$ is symmetric with respect to the real axis.

We have $\operatorname{Re} \Lambda_{4L}(z) > 0$, $z \in \mathbb{D}$ because

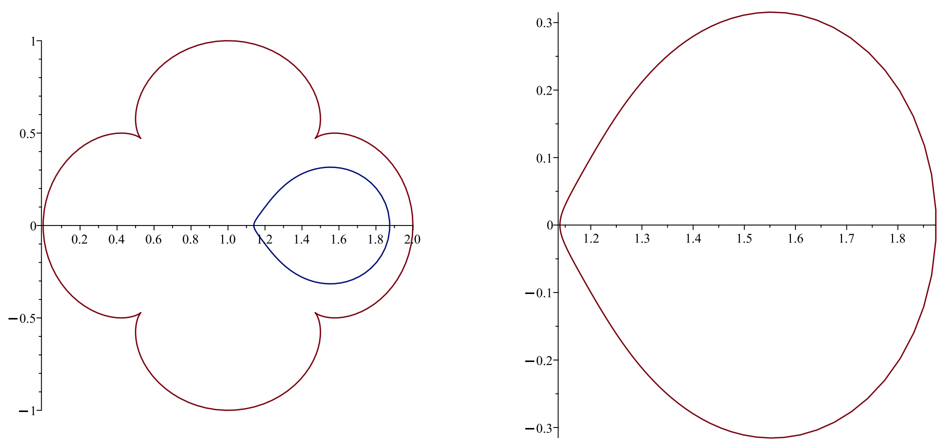
$$\begin{aligned}\operatorname{Re} \Lambda_{4L}(z) &= \operatorname{Re}\left(1 + \frac{5}{6}z + \frac{1}{6}z^5\right) = 1 + \operatorname{Re}\left(\frac{5}{6}z + \frac{1}{6}z^5\right) \geq 1 - \left|\frac{5}{6}z + \frac{1}{6}z^5\right| \\ &\geq 1 - \frac{5}{6}|z| - \frac{1}{6}|z^5| > 1 - \frac{5}{6} - \frac{1}{6} = 0,\ z \in \mathbb{D},\end{aligned}$$

hence, $\operatorname{Re} \Lambda_{4L}(z) > 0$, $z \in \mathbb{D}$.

(ii) We will emphasize that the class $\mathcal{A}_{4L}(\vartheta)$ is not empty. Considering $\widetilde{f}(z) = z + az^2 + bz^3$, for the particular case $a = 2$, $b = 0.1$, $\vartheta = 0.4$, using the 2D plot *of the MAPLE^{TM} computer software we obtain the images of the boundary $\partial\mathbb{D}$ by the functions $\Psi_\vartheta \widetilde{f}$ and Λ_{4L}, shown in Figure 2(a). Since Λ_{4L}, as we showed above, is univalent in $\mathbb{D}$, the equivalence (2) yields that the subordination $\Psi_\vartheta \widetilde{f}(z) \prec \Lambda_{4L}(z)$ holds whenever $\Psi_\vartheta \widetilde{f}(0) = \Lambda_{4L}(0) = 1$ and $\Psi_\vartheta \widetilde{f}(\mathbb{D}) \subset \Lambda_{4L}(\mathbb{D})$ (see Figure 2(b)). In conclusion, $\widetilde{f} \in \mathcal{A}_{4L}(\vartheta)$ for the above values of the parameters; hence, the class $\mathcal{A}_{4L}(\vartheta)$ is not empty for non-trivial values of the parameters.*

The following univalence theorem on the boundary is well known *(see, for example, [25] Lemma 1.1, p. 13): Let f be analytic in $\overline{\mathbb{D}}$ and injective on the boundary $\partial\mathbb{D}$. Then, f is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the inner domain of the (closed) Jordan curve $J = f(\partial\mathbb{D})$.*

For the function $\widetilde{f}$ defined by the above item (ii), we have $\widetilde{f} \in \mathcal{A}_{4L}(\vartheta)$. Using the 2D plot *of the MAPLE^{TM} computer software, the image of the* *boundary $\partial\mathbb{D}$ by the functions $\widetilde{f}$ (see Figure 2(b)), we see that $\widetilde{f}(\partial\mathbb{D})$ is a simple curve; hence, $\widetilde{f}$ is univalent on $\partial\mathbb{D}$. Therefore, according to the above result, we conclude that $\widetilde{f} \in \mathcal{S}$; hence, $\mathcal{A}_{4L}(\vartheta) \cap \mathcal{S} \neq \emptyset$ for some values of the parameter $\vartheta \in [0,1]$.*



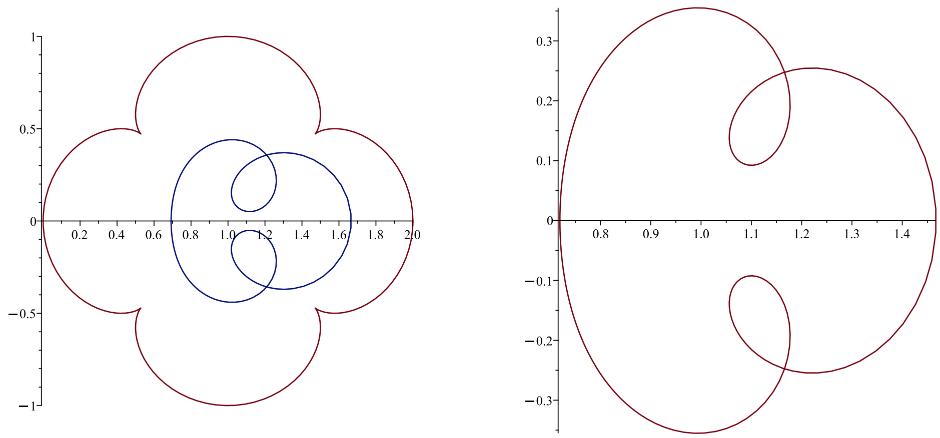
(a) The images of $\Psi_\vartheta \widetilde{f}\left(e^{i\theta}\right)$ (blue color) and $\Lambda_{4L}\left(e^{i\theta}\right)$ (red color), $\theta \in [0,2\pi)$.

(b) The image of $\widetilde{f}(\partial\mathbb{D})$.

Figure 2: Figures for Remark 2(ii)

(iii) Let us consider the function $\widehat{f}(z) = z + az^2 + bz^3 + cz^4$ for $a = 0.5$, $b = 0.2$, $c = 0.1$ and let us take $\vartheta = 0.8$. From the 2D plot *of the MAPLE^{TM} computer software we represent the images of the boundary $\partial\mathbb{D}$ by the functions $\Psi_\vartheta \widehat{f}$ and Λ_{4L} in Figure 3(a). For similar reasons, like item (ii) we have $\Psi_\vartheta \widehat{f}(z) \prec \Lambda_{4L}(z)$. In conclusion, $\widehat{f} \in \mathcal{A}_{4L}(\vartheta)$ for the above given values of the parameters. But, representing with a* 2D plot *of the MAPLE^{TM} computer software the image of the circle $|z| = 0.9$ by the functions $\widehat{f}$ (see Figure 3(b)), we see that $\widehat{f}\left(0.9\,e^{i\theta}\right)$, $\theta \in [0,2\pi)$ is not a simple curve; hence, $\widehat{f}$ is not univalent in $\mathbb{D}$. Consequently, we have $\mathcal{A}_{4L}(\vartheta) \not\subset \mathcal{S}$ for the general choices of the parameter $\vartheta \in [0,1]$.*

(iv) Not only polynomial functions belong to these classes $\mathcal{A}_{4L}(\vartheta)$, as can we see in the next examples. Taking $f_c(z) = z \cdot \dfrac{1+az}{1+bz}$ for the particular case $a = 0.5$, $b = 0.1$, $\vartheta = 0.7$, we similarly obtain the images of the boundary $\partial\mathbb{D}$ by the functions $\Psi_\vartheta f_c$ and Λ_{4L}, shown in Figure 4(a), and, for the same reasons as in the above item, we conclude that $f_c \in \mathcal{A}_{4L}(\vartheta)$ for these values of the parameters. We could mention the same property for the transcendental

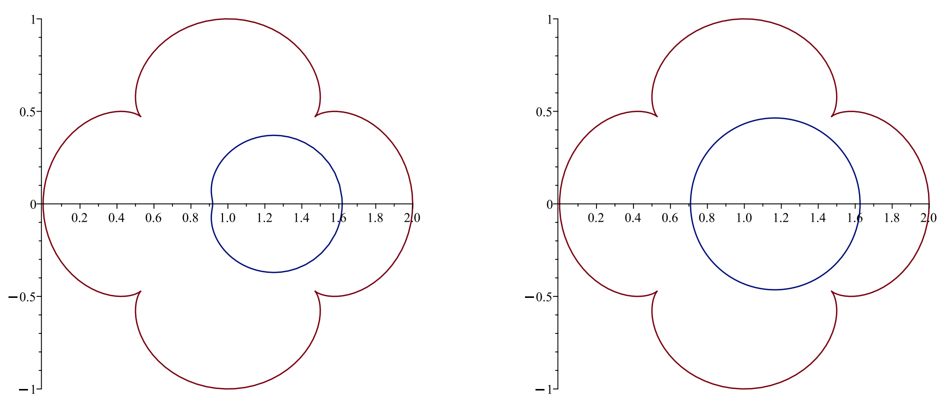


(a) The images of $\Psi_\vartheta \widehat{f}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (blue color) and $\Lambda_{4L}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (red color), $\theta \in [0, 2\pi)$.

(b) The image of $\widehat{f}\left(0.9\,\mathrm{e}^{\mathrm{i}\theta}\right)$, $\theta \in [0, 2\pi)$.

Figure 3: Figures for Remark 2(iii)

function $f_{\mathrm{e}}(z) = z\,\mathrm{e}^{az}$ *with* $a = 0.5$*, where for* $\vartheta = 0.9$*, using a proof similar to those of item (ii) (see Figure 4(b)), we obtain* $f_{\mathrm{e}} \in \mathcal{A}_{4L}(\vartheta)$*.*



(a) The images of $\Psi_\vartheta f_{\mathrm{c}}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (red color) and $\Lambda_{4L}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (blue color), $\theta \in [0, 2\pi)$.

(b) The image of $\Psi_\vartheta f_{\mathrm{e}}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (red color) and $\Lambda_{4L}\left(\mathrm{e}^{\mathrm{i}\theta}\right)$ (blue color), $\theta \in [0, 2\pi)$.

Figure 4: Figures for Remark 2(iv)

(vi) Definition 1 of the class $\mathcal{A}_{4L}(\vartheta)$ generates the next natural question: whether for every function $f \in \mathcal{A}$ there exists $\vartheta \in [0,1]$, such that the function f belongs to the class $\mathcal{A}_{4L}(\vartheta)$.

We will provide below a negative answer to this question, i.e., there exists a function $g \in \mathcal{A}$, such that for any $\vartheta \in [0,1]$ we have $\Psi_\vartheta g(z) \not\prec \Lambda_{4L}(z)$. The proof of this fact will be presented below, where we provide an example of such a function.

Letting $g(z) := \dfrac{\sin(5z)}{5} \in \mathcal{A}$, from Formula (3) we easily obtain

$$H_\vartheta(z) := \Psi_\vartheta g(z) = \frac{\sin(5z)}{\big(1-\vartheta+\vartheta\cos(5z)\big)\,5z},\ z \in \mathbb{D}.$$

The point $z_* = 0$ is a regular point for the function H_ϑ. If for the value $\vartheta_0 \in [0,1]$ such that H_{ϑ_0} is not analytic in some point $z_0 \in \mathbb{D} \setminus \{0\}$, then z_0 will be a pole for H_{ϑ_0} hence the function H_{ϑ_0} will not be analytic in $\mathbb{D}$. This implies $\Psi_{\vartheta_0} g(z) \not\prec \Lambda_{4L}(z)$ that is $g \notin \mathcal{A}_{4L}(\vartheta)$, and moreover $H_{\vartheta_0}(\mathbb{D}) \not\subset \Lambda_{4L}(\mathbb{D})$ because $H_{\vartheta_0}(\mathbb{D})$ is an unbounded domain while $\Lambda_{4L}(\mathbb{D})$ is a bounded one.

We will prove that there not exists any values of $\vartheta \in [0,1]$ such that H_ϑ is analytic in $\mathbb{D}$ and $H_\vartheta(z) \not\prec \Lambda_{4L}(z)$. Contrary, if there exists a $\vartheta^* \in [0,1]$ such that $H_{\vartheta^*}(z) \prec \Lambda_{4L}(z)$, since Λ_{4L} is a univalent function in $\mathbb{D}$, this subordination is equivalent to $H_{\vartheta^*}(0) = \Lambda_{4L}(0) = 1$ and

$$H_{\vartheta^*}(\mathbb{D}) \subset \Lambda_{4L}(\mathbb{D}). \tag{4}$$

It's easy to check that

$$H_\vartheta(-1) = \frac{-\sin 5}{5(\vartheta - 1 - \vartheta\cos 5)} \in \left[\frac{\sin 5}{5\cos 5}, \frac{\sin 5}{5}\right] = [-0.676\ldots, -0.191\ldots]$$

for all $\vartheta \in [0,1]$, while $\operatorname{Re}\Lambda_{4L}(z) > 0$, $z \in \mathbb{D}$. This implies that there exists a neighborhood $\mathcal{V}$ of $z = -1$ such that $H_{\vartheta^*}(\mathcal{V} \cap \mathbb{D}) \not\subset \Lambda_{4L}(\mathbb{D})$. Hence, $H_{\vartheta^*}(\mathbb{D}) \not\subset \Lambda_{4L}(\mathbb{D})$ that contradicts the inclusion (4).

Thus, for the function $g(z) = \dfrac{\sin(5z)}{5} \in \mathcal{A}$ there does not exist $\vartheta \in [0,1]$ such that $g \in \mathcal{A}_{4L}(\vartheta)$, therefore

$$\mathcal{A} \not\subset \Big\{\mathcal{A}_{4L}(\vartheta) : \vartheta \in [0,1]\Big\}.$$

To prove our main results, we will use the next preliminary results.

We say a function p belongs to the class $\mathcal{P}$ of *Carathéodory functions* (see [9, 10]) if and only if it has the series expansion

$$p(z) = 1 + \sum_{k=1}^{\infty} c_n z^k, \ z \in \mathbb{D}, \tag{5}$$

and $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$.

Lemma 1. *Let $p \in \mathcal{P}$ be of the form (5). Then:*

(i) For $n \geq 1$

$$|c_n| \leq 2. \tag{6}$$

The inequality holds for all $n \geq 1$ if and only if $p(z) = (1+\lambda z)/(1-\lambda z)$, $|\lambda| = 1$.

(ii) Also, if $\mu \in \mathbb{C}$ then

$$|c_{n+k} - \mu c_n c_k| \leq 2 \max\{1; |2\mu - 1|\} \tag{7}$$

for all $n, k \geq 1$.

If $0 < \mu < 1$ the inequality is sharp for the function $p(z) = \left(1 + z^{n+k}\right) \Big/ \left(1 - z^{n+k}\right)$. In the other cases, the inequality is sharp for the function $p(z) = (1+z)\big/(1-z)$.

(iii) Moreover, if $\mathsf{B} \in [0, 1]$ with $\mathsf{B}(2\mathsf{B} - 1) \leq \mathsf{D} \leq \mathsf{B}$, we have

$$\left|c_3 - 2\mathsf{B} c_1 c_2 + \mathsf{D} c_1^3\right| \leq 2. \tag{8}$$

We note that inequality (6) is the well-known result of the *Carathéodory lemma* [9] (see also [25, Corollary 2.3, p. 41], [11, Carathéodory's Lemma, p. 41]).

Inequality (7) represents Theorem 1 of [13], while for $\mu \geq 0$ and sharpness for the case $0 < \mu < 1$ is [26, Lemma 2.3]. This last result for particular case $\mu = 1$ was proved in a more general form for $p(0) = c_0$ in [22, Lemma 1, p. 546]. Inequality (8) refers to [1, Lemma 3, p. 66.].

2 Initial coefficient estimates for class $\mathcal{A}_{4L}(\vartheta)$

The first theorem gives us the upper bounds for the first five coefficients $|a_n|$ for the functions belonging to $\mathcal{A}_{4L}(\vartheta)$ as follows:

Theorem 1. *If the function $f \in \mathcal{A}_{4L}(\vartheta)$ is given by (1), then*

$$|a_2| \leq \frac{5}{6|1-2\vartheta|},\ \vartheta \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{9}$$

$$|a_3| \leq \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\vartheta}{3|1-2\vartheta|}\right\},\ \vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}, \tag{10}$$

$$|a_4| \leq \frac{5}{6(1-4\vartheta)},\ \vartheta \in [0, x_0 \simeq 0.2028090285]\,, \tag{11}$$

where x_0 is the solution in $[0,1]$ of the equation

$$792x^4 - 2820x^3 + 2339x^2 - 720x + 72 = 0. \tag{12}$$

Proof. Supposing that $f \in \mathcal{A}_{4L}(\vartheta)$ has the form (1), then there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, satisfying

$$\frac{f(z)}{(1-\vartheta)z + \vartheta z f'(z)} = \Lambda_{4L}\left(w(z)\right),\ z \in \mathbb{D}. \tag{13}$$

It is easy to check that

$$\begin{aligned}\frac{f(z)}{(1-\vartheta)z + \vartheta z f'(z)} &= 1 + (1-2\vartheta)a_2 z + \left[(1-3\vartheta)a_3 - 2\vartheta(1-2\vartheta)a_2^2\right] z^2 \\ &+ \left[(1-4\vartheta)a_4 - \vartheta(5-12\vartheta)a_2 a_3 + 4\vartheta^2(1-2\vartheta)a_2^3\right] z^3 + \ldots,\ z \in \mathbb{D}.\end{aligned} \tag{14}$$

Letting the function l defined by

$$l(z) := \frac{1+w(z)}{1-w(z)} = 1 + \sum_{n=1}^{\infty} l_n\, z^n,\ z \in \mathbb{D},$$

since $|w(z)| < 1$ in $\mathbb{D}$, it follows that $l \in \mathcal{P}$.

A simple computation gives

$$w(z) = \frac{l(z)-1}{l(z)+1} = \frac{1}{2} l_1 z + \frac{1}{2}\left(l_2 - \frac{1}{2} l_1^2\right) z^2 + \frac{1}{2}\left(l_3 - l_1 l_2 + \frac{1}{4} l_1^3\right) z^3 + \ldots, \tag{15}$$

and by replacing the power series expansion of (15) in the right-hand side of the relation (13) we obtain

$$\begin{aligned}\frac{f(z)}{(1-\vartheta)z + \vartheta z f'(z)} =& 1 + \frac{5}{12} l_1 z + \left(\frac{5l_2}{12} - \frac{5l_1^2}{24}\right) z^2 \\ &+ \left(\frac{5}{12} l_3 - \frac{5}{12} l_1 l_2 + \frac{5}{48} l_1^3\right) z^3 + \ldots,\ z \in \mathbb{D}.\end{aligned} \tag{16}$$

Equating the first three coefficients of (14) and (16) we have

$$a_2 = \frac{5}{12(1-2\vartheta)} l_1,\ \vartheta \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{17}$$

$$a_3 = \frac{5}{12(1-3\vartheta)} \left[l_2 - \frac{1}{2}\left(1 - \frac{5\vartheta}{3(1-2\vartheta)}\right) l_1^2\right],\ \vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}, \tag{18}$$

$$\begin{aligned} a_4 =& \frac{5}{12(1-4\vartheta)} \left\{\left[\frac{5\vartheta(5-12\vartheta)}{12(1-2\vartheta)(1-3\vartheta)} - 1\right] l_1 l_2 \right. \\ & \left. - \frac{5}{24}\left[\frac{\vartheta(5-12\vartheta)(3-11\vartheta)}{3(1-2\vartheta)^2(1-3\vartheta)} + \frac{10\vartheta^2}{3(1-2\vartheta)^2} - \frac{6}{5}\right] l_1^3 + l_3\right\}, \\ & \qquad \vartheta \in [0,1] \setminus \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}. \end{aligned} \tag{19}$$

First, using the inequality (6) in (17) we obtain the inequality (9).

The equality (18) implies

$$|a_3| = \frac{5}{12|1-3\vartheta|} \left| l_2 - \frac{1}{2}\left(1 - \frac{5\vartheta}{3(1-2\vartheta)}\right) l_1^2 \right|, \tag{20}$$

and using inequality (7) for $n = k = 1$ and $\mu = \dfrac{1}{2}$ we obtain (10).

From (19) it follows that

$$\begin{aligned} |a_4| =& \frac{5}{12|1-4\vartheta|} \left| l_3 - 2\left[\frac{1}{2} - \frac{5\vartheta(5-12\vartheta)}{24(1-2\vartheta)(1-3\vartheta)}\right] l_1 l_2 \right. \\ & \left. + \frac{5}{24}\left(\frac{6}{5} - \frac{\vartheta(5-12\vartheta)(3-11\vartheta)}{3(1-2\vartheta)^2(1-3\vartheta)} - \frac{10\vartheta^2}{3(1-2\vartheta)^2}\right) l_1^3 \right| \\ =& \frac{5}{12|1-4\vartheta|} \left| l_3 - 2\left[\frac{1}{2} + \frac{5\vartheta(12\vartheta-5)}{24(2\vartheta-1)(3\vartheta-1)}\right] l_1 l_2 \right. \\ & \left. + \frac{5}{24}\left(\frac{6}{5} + \frac{\vartheta(12\vartheta-5)(11\vartheta-3)}{3(2\vartheta-1)^2(3\vartheta-1)} - \frac{10\vartheta^2}{3(2\vartheta-1)^2}\right) l_1^3 \right|, \end{aligned}$$

and we will compare the right-hand side of the above relation with (8). Assuming that $\vartheta \in [0,1]$, then

$$0 \le \mathsf{B} = \frac{1}{2} + \frac{5\vartheta(12\vartheta-5)}{24(2\vartheta-1)(3\vartheta-1)} \le 1 \Leftrightarrow$$

$$\vartheta \in S_1 := \left[0, \frac{85}{264} - \frac{\sqrt{889}}{264} \simeq 0.209\right]$$

$$\cup \left[\frac{35}{24} - \frac{\sqrt{649}}{24} \simeq 0.396, \frac{85}{264} + \frac{\sqrt{889}}{264} \simeq 0.434\right],$$

$$\begin{aligned}
\mathsf{B} &= \frac{1}{2} + \frac{5\vartheta(12\vartheta-5)}{24(2\vartheta-1)(3\vartheta-1)} \\
\geq \mathsf{D} &= \frac{5}{24}\left(\frac{6}{5} + \frac{\vartheta(12\vartheta-5)(11\vartheta-3)}{3(2\vartheta-1)^2(3\vartheta-1)} - \frac{10\vartheta^2}{3(2\vartheta-1)^2}\right) \\
\Leftrightarrow \vartheta \in\ S_2 &:= \left[0, \frac{15}{11} - \frac{\sqrt{159}}{11} \simeq 0.217\right] \cup \left[\frac{1}{3}, \frac{1}{2}\right],
\end{aligned}$$

and

$$\mathsf{C} := \mathsf{B}(2\mathsf{B}-1) \leq \mathsf{D} \Leftrightarrow \vartheta \in\ S_3 := [0, x_0 \simeq 0.2028090285],$$

where x_0 is the solution in $[0,1]$ of the equation (12). Thus, all the above three inequalities hold if and only if

$$\vartheta \in\ S := S_1 \cap S_2 \cap S_3 = [0, x_0 \simeq 0.2028090285],$$

and all the requirements of Lemma 1(iii) are satisfied, hence (8) leads us to (11). □

By fixing the parameter $\vartheta = 0$ and $\vartheta = 1$ in Theorem 1 we state the following:

Corollary 1. *1. If the function $f \in \mathcal{N}_{4L}$ is given by (1) then*

$$|a_2| \leq \frac{5}{6}, \quad |a_3| \leq \frac{5}{6}, \quad |a_4| \leq \frac{5}{6}.$$

2. If the function $f \in \mathcal{Y}_{4L}$ is given by (1) then

$$|a_2| \leq \frac{5}{6}, \quad |a_3| \leq \frac{5}{12} \max\left\{1; \frac{5}{3}\right\} = \frac{25}{36}.$$

In [14] Fekete and Szegő proved the well-known result

$$\max\left\{\left|a_3 - \mu a_2^2\right| : f \in \mathcal{S}\right\} = 1 + 2\mathrm{e}^{-\frac{2\mu}{1-\mu}},\ \mu \in [0,1],$$

and in our next theorem we consider the corresponding problem for the family $\mathcal{A}_{4L}(\vartheta)$, as follows.

Theorem 2. *If the function $f \in \mathcal{A}_{4L}(\vartheta)$ has the form (1) and $\mu \in \mathbb{C}$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\left|4\vartheta^2 - (3\mu+2)\vartheta + \mu\right|}{6(2\vartheta-1)^2}\right\}, \tag{21}$$

whenever $\vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}$.

Proof. If $f \in \mathcal{A}_{4L}(\vartheta)$ has the form (1), as in the proof of the previous theorem, using (17) and (20), we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &= \left|\frac{5}{12(1-3\vartheta)}\left(l_2 - \frac{1}{2}\left(1 - \frac{5\vartheta}{3(1-2\vartheta)}\right) l_1^2\right) - \mu\,\frac{25}{144(1-2\vartheta)^2}\,l_1^2\right| \\ &= \frac{5}{12\left|1-3\vartheta\right|}\left|l_2 - \frac{1}{12}\,\frac{44\vartheta^2 - (15\mu+34)\,\vartheta + 5\mu + 6}{(2\vartheta-1)^2} l_1^2\right|,\end{aligned}$$

and Lemma 1(ii) lead us to (21). □

Taking the parameter $\vartheta = 0$ and $\vartheta = 1$ in Theorem 1 we get the following corollary, while for $\mu = 1$ we obtain the second one, respectively:

Corollary 2. *1. If the function $f \in \mathcal{N}_{4L}$ has the form (1) and $\mu \in \mathbb{C}$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{5}{6} \max\left\{1; \frac{5|\mu|}{6}\right\}.$$

2. If the function $f \in \mathcal{Y}_{4L}$ has the form (1) and $\mu \in \mathbb{C}$, then

$$\left|a_3 - \mu a_2^2\right| \le \frac{5}{12} \max\left\{1; \frac{5|3-\mu|}{3}\right\}.$$

Corollary 3. *If the function $f \in \mathcal{A}_{4L}(\vartheta)$ has the form (1) and $\vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}$, then*

$$\left|a_3 - a_2^2\right| \le \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\left|4\vartheta^2 - 5\vartheta + 1\right|}{6(2\vartheta-1)^2}\right\}.$$

3 Coefficient inequalities for the inverses of functions belonging to $\mathcal{A}_{4L}(\vartheta) \cap \mathcal{S}$

If $\mathcal{F}$ is a set of analytic functions in the unit disk $\mathbb{D}$, then the *Koebe domain* of the set $\mathcal{F}$ is defined as the largest domain contained in $f(\mathbb{D})$ for all $f \in \mathcal{F}$, and it is denoted by $\mathcal{K}(\mathcal{F})$.

The Koebe–Bieberbach *one quarter theorem* [16, Theorem 2, p. 49] ensures that the image of $\mathbb{D}$ for every univalent function $f \in \mathcal{A}$ contains the disk with center in the origin and of radius $\frac{1}{4}$, that is $\mathcal{K}(\mathcal{S}) = 1/4$. Thus, every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}\big(f(z)\big) = z,\ z \in \mathbb{D} \text{ and } \quad f\big(f^{-1}(w)\big) = w,\ |w| < r_0(f),\ r_0(f) \ge \frac{1}{4}.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, assuming that f^{-1} has an analytic extension tho $\mathbb{D}$. We notice that the class of bi-univalent functions defined in the unit disk $\mathbb{D}$ is not empty. For example, the functions z, $\dfrac{z}{1-z}$, $-\log(1-z)$ and $\dfrac{1}{2}\log\dfrac{1+z}{1-z}$ are members of the bi-univalent function class, however the Koebe function is not a member.

Like we mentioned in the Remark 2(ii) for some values of the parameter $\vartheta \in [0,1]$ we have $\mathcal{A}_{4L}(\vartheta) \cap \mathcal{S} \neq \emptyset$, while the Remark 2(vi) shows that $\mathcal{A} \not\subset \left\{\mathcal{A}_{4L}(\vartheta) : \vartheta \in [0,1]\right\}$, hence $\mathcal{S} \not\subset \left\{\mathcal{A}_{4L}(\vartheta) : \vartheta \in [0,1]\right\}$. These facts motivate the below studies regarding the coefficients of the inverses of the functions from the classes $\mathcal{A}_{4L}(\vartheta) \cap \mathcal{S}$.

Theorem 3. *If $f \in \mathcal{A}_{4L}(\vartheta)$ and $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} d_n w^n$ is the inverse function of f in the Koebe domain $\mathcal{K}\left(\mathcal{A}_{4L}(\vartheta)\right)$ of the class $\mathcal{A}_{4L}(\vartheta)$, then*

$$\begin{aligned} &|d_2| \leq \frac{5}{6|1-2\vartheta|},\ \vartheta \in [0,1] \setminus \left\{\frac{1}{2}\right\},\\ &|d_3| \leq \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\left|2\vartheta^2-4\vartheta+1\right|}{3(2\vartheta-1)^2}\right\},\ \textit{if}\quad \vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}, \end{aligned}$$

and for any complex number $\hbar$ and $\vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}$ we have

$$\left|d_3 - \hbar d_2^2\right| \leq \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\left|4\vartheta^2-(8-3\hbar)\vartheta+2-\hbar\right|}{6(2\vartheta-1)^2}\right\}. \tag{22}$$

Proof. If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n,\ w \in \mathcal{K}\left(\mathcal{A}_{4L}(\vartheta)\right)$$

is the inverse function of f, it can be seen that

$$f^{-1}\big(f(z)\big) = z,\ z \in \mathbb{D} \quad \text{and} \quad f\big(f^{-1}(w)\big) = w,\ w \in \mathcal{K}\left(\mathcal{A}_{4L}(\vartheta)\right). \tag{23}$$

Since

$$f^{-1}\left(z + \sum_{n=2}^{\infty} a_n z^n\right) = z,\ z \in \mathbb{D}, \tag{24}$$

from (23) and (24) one can obtain

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2 d_2 + d_3)z^3 + \cdots = z,\ z \in \mathbb{D}. \tag{25}$$

By equating the corresponding coefficients of (25) we get

$$d_2 = -a_2, \tag{26}$$
$$d_3 = 2a_2^2 - a_3, \tag{27}$$

and from relations (17) together with (26) we get

$$d_2 = -\frac{5}{12(1-2\vartheta)} l_1.$$

Using Lemma 1(i) it follows

$$|d_2| \le \frac{5}{6|1-2\vartheta|},$$

and according to (27), if we set $\mu = 2$ in (21) we have

$$|d_3| = \left|a_3 - 2a_2^2\right| \le \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\left|2\vartheta^2 - 4\vartheta + 1\right|}{3(2\vartheta-1)^2}\right\}.$$

To find an upper bound of the Fekete-Szegő functional for the inverse function, from (26) and (27), for any $\hbar \in \mathbb{R}$ we have

$$\left|d_3 - \hbar d_2^2\right| = \left|2a_2^2 - a_3 - \hbar a_2^2\right| = \left|a_3 - (2-\hbar)a_2^2\right|.$$

Finally, if we let $\mu := 2 - \hbar$ in the (21) we obtain our conclusion (22). $\square$

4 Initial logarithmic coefficient bounds for functions of the classes $\mathcal{A}_{4L}(\vartheta)$

Inspired by recent works like [2,3], in this section we determine the coefficient bounds and Fekete–Szegő problem associated with the logarithmic function.

If the function $f \in \mathcal{A}$ given by (1) is analytic in $\mathbb{D}$ such that $\dfrac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, then the well-known *logarithmic coefficients* $d_n := d_n(f)$, $n \in \mathbb{N}$, of f are given by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} d_n z^n, \ z \in \mathbb{D}. \tag{28}$$

Theorem 4. *Let $f \in \mathcal{A}_{4L}(\vartheta)$ such that $\dfrac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, with the logarithmic coefficients given by (28). Then,*

$$|d_1| \leq \frac{5}{12|1-2\vartheta|}, \quad \text{if} \quad \vartheta \in [0,1] \setminus \left\{\frac{1}{2}\right\},$$
$$|d_2| \leq \frac{5}{12|1-3\vartheta|} \max\left\{1; \frac{5}{12}\frac{\left|8\vartheta^2-7\vartheta+1\right|}{(2\vartheta-1)^2}\right\}, \quad \text{if} \quad \vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\},$$

and for $\wp \in \mathbb{C}$ and $\vartheta \in [0,1] \setminus \left\{\dfrac{1}{3}, \dfrac{1}{2}\right\}$ we have

$$\left|d_2 - \wp d_1^2\right| \leq \frac{5}{12|1-3\vartheta|} \max\left\{1; \frac{5}{12}\frac{\left|8\vartheta^2-(3\wp+7)\vartheta+1+\wp\right|}{(2\vartheta-1)^2}\right\}.$$

Proof. If $f \in \mathcal{A}_{4L}(\vartheta)$ has the form (3), equating the first two coefficients of the relation (28) we obtain

$$d_1 = \frac{a_2}{2}, \quad d_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right),$$

and using the inequality (17) it follows that

$$|d_1| = \left|\frac{a_2}{2}\right| \leq \frac{5}{12|1-2\vartheta|}, \ \vartheta \in [0,1] \setminus \left\{\frac{1}{2}\right\}.$$

Since

$$|d_2| = \frac{1}{2}\left|a_3 - \frac{1}{2}a_2^2\right|,$$

taking $\mu = \dfrac{1}{2}$ in (21) the above equality leads to

$$|d_2| \leq \frac{5}{12|1-3\vartheta|} \max\left\{1; \frac{5}{12}\frac{\left|8\vartheta^2-7\vartheta+1\right|}{(2\vartheta-1)^2}\right\}$$

whenever $\vartheta \in [0,1] \setminus \left\{\dfrac{1}{3}, \dfrac{1}{2}\right\}$.

To find the Fekete-Szegő inequality for the inverse function, for any complex number $\wp$ we have

$$\left|d_2 - \wp d_1^2\right| = \left|\frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right) - \wp\frac{a_2^2}{4}\right| = \frac{1}{2}\left|a_3 - \frac{\wp+1}{2}a_2^2\right|.$$

Finally, putting $\mu = \dfrac{\wp + 1}{2}$ in the (21) we get

$$\left|d_2 - \wp d_1^2\right| = \frac{5}{12|1-3\vartheta|} \max\left\{1; \frac{5}{12}\left|\frac{(8\vartheta^2 - (3\wp + 7)\vartheta + 1 + \wp}{(2\vartheta - 1)^2}\right|\right\}.$$

□

5 Coefficient estimates for the quotient function $\dfrac{z}{f(z)}$

In this section we determine the coefficient bounds and Fekete-Szegő problem associated with the function H defined by

$$H(z) := \frac{z}{f(z)} = 1 + \sum_{t=1}^{\infty} u_t z^t,\ z \in \mathbb{D}, \tag{29}$$

where $f \in \mathcal{A}$ such that $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$.

Theorem 5. *Let $f \in \mathcal{A}_{4L}(\vartheta)$ of the form (1) such that $\dfrac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, and let H be given by (29). Then,*

$$|u_1| \le \frac{5}{6|1-2\vartheta|},\ \vartheta \in [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{30}$$

$$|u_2| \le \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\left|4\vartheta^2 - 5\vartheta + 1)\right|}{6(2\vartheta-1)^2}\right\},\ \textit{if}\quad \vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}, \tag{31}$$

$$|u_2 - \rho u_1^2| \le \frac{5}{6|1-3\vartheta|} \max\left\{1; \frac{5\left|4\vartheta^2 - (5-3\rho)\vartheta + 1 - \rho)\right|}{6(2\vartheta-1)^2}\right\}, \tag{32}$$

for $\rho \in \mathbb{C}$ and $\vartheta \in [0,1] \setminus \left\{\frac{1}{3}, \frac{1}{2}\right\}$

Proof. If $f \in \mathcal{A}_{4L}(\vartheta)$ with $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$, the function H given by (29) is well defined. A simple computation gives

$$H(z) = \frac{z}{f(z)} = 1 - a_2 z + \left(a_2^2 - a_3\right) z^2 + \left(a_2^3 + 2a_2a_3 - a_4\right) z^3 + \ldots,\ z \in \mathbb{D}, \tag{33}$$

and equating the coefficients of z and z^2 on the both sides of (29) and (33) we get

$$u_1 = -\,a_2, \tag{34}$$

$$u_2 = a_2^2 - a_3. \tag{35}$$

Using (34), the estimate (30) follows immediately from (9). The bound given in (31) for $|u_2|$ follows from Corollary 3.

For $\rho \in \mathbb{C}$, using (34) and (35) we get

$$\left|u_2 - \rho u_1^2\right| = \left|a_3 - (1-\rho)a_2^2\right|,$$

and by taking $\mu := 1-\rho$ in the inequality (21) we get the desired result (32). □

6 Krushkal inequalities for the class $\mathcal{A}_{4L}(\vartheta)$

In this section we will show that for the well-known inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \le 2^{p(n-1)} - n^p \tag{36}$$

we can find smaller upper bound for the subclass $\mathcal{A}_{4L}(\vartheta)$ and for the specific values $n = 4$ and $p = 1$. This inequality was originally presumed and proved by Krushkal for the class of normalized univalent functions $\mathcal{S}$ and the integers $n > 3$, $p \ge 1$, while it is sharp since the equality occurs for the *Koebe function* (see [19] Theorem 6.1, p. 17).

For $n = 4$ and $p = 1$ we obtain the following upper bound for the left-hand side of (36).

Theorem 6. *If the function $f \in \mathcal{A}_{4L}(\vartheta)$ has the form* (1)*, then*

$$\left|a_4 - a_2^3\right| \le \frac{5}{6|1-4\vartheta|},\ \vartheta \in [0, x_3 \simeq 0.1282573496],$$

where x_3 is the smallest solution in $[0,1]$ of the equation

$$1584x^5 - 6432x^4 + 5698x^3 - 2129x^2 + 364x - 22 = 0. \tag{37}$$

Proof. If $f \in \mathcal{A}_{4L}(\vartheta)$, from (17) and (19) we obtain

$$\begin{aligned} a_4 - a_2^3 = \frac{5}{12(1-4\vartheta)} \Bigg\{ & \left[\frac{5\vartheta(5-12\vartheta)}{12(1-2\vartheta)(1-3\vartheta)} - 1\right] l_1 l_2 \\ & - \frac{5}{24}\left(\frac{\vartheta(5-12\vartheta)(3-11\vartheta)}{3(1-2\vartheta)^2(1-3\vartheta)} + \frac{10\vartheta^2}{3(1-2\vartheta)^2} + \frac{5(1-4\vartheta)}{6(1-2\vartheta)^3} - \frac{6}{5}\right) l_1^3 + l_3 \Bigg\}, \\ & \vartheta \in [0,1] \setminus \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}, \end{aligned}$$

hence,

$$\left|a_4-a_2^3\right| = \frac{5}{12|1-4\vartheta|}\left|l_3 - 2\cdot\frac{1}{2}\left[1-\frac{5\vartheta(5-12\vartheta)}{12(1-2\vartheta)(1-3\vartheta)}\right]l_1l_2 \right.$$
$$\left. +\frac{5}{24}\left(\frac{6}{5}-\frac{\vartheta(5-12\vartheta)(3-11\vartheta)}{3(1-2\vartheta)^2(1-3\vartheta)}-\frac{10\vartheta^2}{3(1-2\vartheta)^2}-\frac{5(1-4\vartheta)}{6(1-2\vartheta)^3}\right)l_1^3\right|. \tag{38}$$

Comparing the right-hand side of the above relation with (8), for $\vartheta \in [0,1]\setminus\left\{\frac{1}{4},\frac{1}{3},\frac{1}{2}\right\}$ we get

$$\begin{aligned}
\mathsf{B} =& \frac{1}{2}\left[1-\frac{5\vartheta(5-12\vartheta)}{12(1-2\vartheta)(1-3\vartheta)}\right],\\
\mathsf{D} =& \frac{5}{24}\left(\frac{6}{5}-\frac{\vartheta(5-12\vartheta)(3-11\vartheta)}{3(1-2\vartheta)^2(1-3\vartheta)}-\frac{10\vartheta^2}{3(1-2\vartheta)^2}-\frac{5(1-4\vartheta)}{6(1-2\vartheta)^3}\right).
\end{aligned}$$

Since $\vartheta\in[0,1]$, we obtain

$$0\le \mathsf{B} = \frac{1}{2}\left[1-\frac{5\vartheta(5-12\vartheta)}{12(1-2\vartheta)(1-3\vartheta)}\right]\le 1 \Leftrightarrow$$
$$\vartheta\in M_1 := \left[0,\frac{85}{264}-\frac{\sqrt{889}}{264}\simeq 0.209\right]$$
$$\cup\left[\frac{35}{24}-\frac{\sqrt{649}}{24}\simeq 0.396,\frac{85}{264}+\frac{\sqrt{889}}{264}\simeq 0.434\right],$$
$$\mathsf{B}\ge\mathsf{D}\Leftrightarrow\vartheta\in\ M_2 := [0,x_1\simeq 0.230]\cup\left(\frac{1}{3},x_2\simeq 0.372\right]\cup\left(\frac{1}{2},1\right],$$

where x_1 and x_2 are the solution in $[0,1]$ of the equation

$$264x^4-984x^3+1230x^2-499x+61=0,$$

and

$$\mathsf{C} := \mathsf{B}(2\mathsf{B}-1)\le\mathsf{D}\Leftrightarrow$$
$$\vartheta\in\ M_3 := [0,x_3\simeq 0.1282573496]\cup\left[x_4\simeq 0.3796254135,\frac{1}{2}\right),$$

where x_3 and x_4 are the solution in $[0,1]$ of the equation (37).

Consequently, all the above three inequalities hold if and only if

$$\vartheta\in\ M := M_1\cap M_2\cap M_3 = [0,x_3\simeq 0.1282573496],$$

and all the assumptions of Lemma 1(iii) are fulfilled. Finally, combining (8) with (38) we obtain the desired result. □

7 Conclusions

In this study, we focused on a subclass of bounded turning functions associated with a four-leaf-type domain. We made some useful findings for this class, including the bounds of the first four initial coefficients, the Fekete–Szegő-type inequality, and the Krushkal inequality. The actual results do not overlap any of these nor the structure of the subclasses, because the subordinations by expressions such as the left-hand side of subordination (3) had not already appeared. Taking into account the upper bounds given in Theorem 1, an interesting open problem that could start a real challenge is to find the estimate $|a_n|$ holds for all $n \in \mathbb{N} \setminus \{1\}$ for the function class $\mathcal{A}_{4L}(\vartheta)$. In the future, this work can be applied to derive the boundaries of Hankel determinants of second, third, fourth, and fifth order for various subclasses of univalent functions as discussed in [5,7,8,17,18,20,21,27,28,32,33].

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