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# CONVERGENCE OF INFINITE PRODUCTS OF CONTRACTIVE MAPPINGS ON METRIC SPACES WITH GRAPHS*

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*In memory of Professor Haim Brezis*

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**Abstract**

We study the convergence of infinite products of contractive mappings on metric spaces with graphs. In particular, we extend analogous results of ours which have recently been obtained for powers of contractive mappings.

**Keywords:** contractive mapping, graph, infinite product, metric space.
**MSC:** 47H09, 47H10, 54E50.

## 1 Introduction

For more than sixty years now, there has been a lot of research activity regarding the theory of nonexpansive (that is, 1-Lipschitz) mappings and semigroups. See, for example, [1–3, 5, 6, 9, 11–15, 18, 19] and references cited therein. In particular, the study of nonexpansive and contractive mappings on complete metric spaces with graphs has recently become a rapidly growing area of research. See, for instance, [7, 8, 10–12, 20]. In the present paper, we study the convergence of infinite products of contractive mappings on

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metric spaces with graphs. We extend, in particular, the analogous results which have recently been obtained for powers of contractive mappings [21]. For information on infinite products of operators on metric spaces, see, for example, [4, 16, 17] and references therein.

Let $(X, \rho)$ be a metric space, and let $G$ be a (directed) graph. Let $V(G)$ be the set of its vertices and let $E(G)$ be the set of its edges. We identify the graph $G$ with the pair $(V(G), E(G))$. For each point $x \in X$ and each number $r > 0$, set

$$B_\rho(x, r) := \{y \in X : \ \rho(x, y) \leq r\}.$$

Denote by $\mathcal{M}$ the set of all mappings $T : X \to X$ such that for each $x, y \in X$ satisfying $(x, y) \in E(G)$, we have

$$(T(x), T(y)) \in E(G) \tag{1}$$

and

$$\rho(T(x), T(y)) \leq \rho(x, y). \tag{2}$$

A mapping $T \in \mathcal{M}$ is called $G$-nonexpansive. If $T \in \mathcal{M}$, $\alpha \in (0, 1)$ and for each $x, y \in X$ satisfying $(x, y) \in E(G)$, we have

$$\rho(T(x), T(y)) \leq \alpha \rho(x, y),$$

then $T$ is called a $G$-strict contraction.

A mapping $T \in \mathcal{M}$ is called $G$-contractive (or a $G$-Rakotch contraction [13]) if there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that

$$\phi(t) < 1, \ t \in [0, \infty)$$

and for each $x, y \in X$ satisfying $(x, y) \in E(G)$, we have

$$\rho(T(x), T(y)) \leq \phi(\rho(x, y))\rho(x, y).$$

In the sequel we assume that the infimum over the empty set is $\infty$, $\infty + \infty = \infty$, and that $a + \infty = \infty$ for each $a \in R^1$.

In this paper we assume that $A_t \in \mathcal{M}$, $t = 1, 2, \dots$, is a given sequence of operators, $\phi : [0, \infty) \to [0, 1]$ is a decreasing function which satisfies

$$\phi(t) < 1, \ t \in [0, \infty) \tag{3}$$

and that for each $x, y \in X$ satisfying $(x, y) \in E(G)$ and each integer $t \geq 1$, we have

$$\rho(A_t(x), A_t(y)) \leq \phi(\rho(x, y))\rho(x, y). \tag{4}$$

Such a sequence of operators $\{A_t\}_{t=1}^{\infty}$ is said to be contractive [13].

## 2 The first result

**Theorem 1.** *Let $M, \epsilon > 0$ be given. Then there exists a natural number $n_0$ such that for each $x, y \in X$ satisfying*

$$(x, y) \in E(G),\ \rho(x, y) \le M, \tag{5}$$

*each $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ and each integer $n \ge n_0$, we have*

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \le \epsilon.$$

*Proof.* Choose an integer

$$n_0 > 1 + M\epsilon^{-1}(1 - \phi(\epsilon))^{-1}. \tag{5}$$

Assume that $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, $x, y \in X$ and (5) holds. By (4) and (5), for each integer $t \ge 1$, we have

$$\begin{aligned}
&\rho(A_{r(t+1)} \cdots A_{r(1)}(x), A_{r(t+1)} \cdots A_{r(1)}(y)) \\
&\le \phi(\rho(A_{r(t)} \cdots A_{r(1)}(x), A_{r(t)} \cdots A_{r(1)}(y))) \\
&\times \rho(A_{r(t)} \cdots A_{r(1)}(x), A_{r(t)} \cdots A_{r(1)}(y)) \le \rho(x, y) \le M.
\end{aligned} \tag{6}$$

In view of (6), it is sufficient to show that there exists an integer $n \in \{1, \dots, n_0\}$ such that

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \le \epsilon.$$

Suppose to the contrary that this does not hold. Then in view of (6),

$$\rho(x, y) > \epsilon \tag{7}$$

and for each integer $n \in \{1, \dots, n_0\}$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) > \epsilon. \tag{8}$$

Relations (4), (7) and (8) imply that

$$\rho(A_{r(1)}(x), A_{r(1)}(y)) \le \phi(\rho(x, y))\rho(x, y),$$

$$\rho(x, y) - \rho(A_{r(1)}(x), A_{r(1)}(y)) \ge (1 - \phi(\rho(x, y)))\rho(x, y) \ge (1 - \phi(\epsilon))\epsilon \tag{9}$$

and that for each $n \in \{1, \dots, n_0\}$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y))$$

$$
\begin{aligned}
&-\rho(A_{r(n+1)}\cdots A_{r(1)}(x), A_{r(n+1)}\cdots A_{r(1)}(y)) \\
&\geq (1-\phi(\rho(A_{r(n)}\cdots A_{r(1)}(x), A_{r(n)}\cdots A_{r(1)}(y)))) \\
&\times\rho(A_{r(n)}\cdots A_{r(1)}(x), A_{r(n)}\cdots A_{r(1)}(y)) \geq (1-\phi(\epsilon))\epsilon. \qquad (10)
\end{aligned}
$$

It follows from (6) and (10) that

$$
\begin{aligned}
&M \geq \rho(x,y) \geq \rho(A_{r(1)}(x), A_{r(1)}(y)) \\
&\geq \rho(A_{r(1)}(x), A_{r(1)}(y)) - \rho(A_{r(n_0)}\cdots A_{r(1)}(x), A_{r(n_0)}\cdots A_{r(1)}(y)) \\
&= \sum_{n=1}^{n_0-1} (\rho(A_{r(n)}\cdots A_{r(1)}(x), A_{r(n)}\cdots A_{r(1)}(y)) \\
&-\rho(A_{r(n+1)}\cdots A_{r(1)}(x), A_{r(n+1)}\cdots A_{r(1)}(y))) \\
&\geq (n_0-1)(1-\phi(\epsilon))\epsilon
\end{aligned}
$$

and

$$n_0 \leq 1 + M\epsilon^{-1}(1-\phi(\epsilon))^{-1}.$$

This, however, contradicts (5). The contradiction we have reached proves Theorem 1. □

**Corollary 1.** *Assume that $x, y \in X$ and that there exist a natural number $q$ and points $x_i \in X$, $i = 0, \dots, q$, such that*

$$x_0 = x,\ y = x_q,$$

$$(x_i, x_{i+1}) \in E(G),\ i = 0, \dots, q-1.$$

*Let $\gamma > 0$. Then there exists a natural number $n_0$ such that for each $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ and each integer $n \geq n_0$, we have*

$$\rho(A_{r(n)}\cdots A_{r(1)}(x), A_{r(n)}\cdots A_{r(1)}(y)) \leq \gamma.$$

*Proof.* Set

$$M = \max\{\rho(x_i, x_{i+1}) :\ i = 0, \dots, q-1\},$$

$$\epsilon = \gamma/q$$

and let a natural number $n_0$ be as guaranteed by Theorem 1.

Assume that $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ and that $n \geq n_0$ is an integer. In view of the choice of $n_0$, for each $i \in \{0, \dots, q-1\}$, we have

$$\rho(A_{r(n)}\cdots A_{r(1)}(x_i), A_{r(n)}\cdots A_{r(1)}(x_{i+1})) \leq \epsilon = \gamma/q$$

and this implies that

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \gamma.$$

This competes the proof of Corollary 1. $\square$

**Corollary 2.** *Assume that*

$$(x, x) \in E, \ x \in X,$$

$$\sup\{\rho(x, y) : \ x, y \in X\} < \infty$$

*and that there exists an integer $q \geq 1$ such that for each $\xi, \eta \in X$, there exist points $x_i \in X$, $i = 0, \dots, q$, satisfying*

$$x_0 = \xi, \ \eta = x_q,$$

$$(x_i, x_{i+1}) \in E(G), \ i = 0, \dots, q-1.$$

*Let $\gamma > 0$ be given. Then there exists a natural number $n_0$ such that for each $x, y \in X$, each mapping $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ and each integer $n \geq n_0$, we have*

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \gamma.$$

*Proof.* Set

$$M = \sup\{\rho(x, y) : \ x, y \in X\},$$

$$\epsilon = \gamma/q$$

and let a natural number $n_0$ be as guaranteed by Theorem 1.

Assume that $x, y \in X$, $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ and that $n \geq n_0$ is an integer. Then there exist points $x_i \in X$, $i = 0, \dots, q$, such that

$$x_0 = x, \ y = x_q,$$

$$(x_i, x_{i+1}) \in E(G), \ i = 0, \dots, q-1.$$

In view of the choice of $n_0$ and Theorem 1, for each $i \in \{0, \dots, q-1\}$, we have

$$\rho(A_{r(n)} \cdots A_{r(1)}(x_i), A_{r(n)} \cdots A_{r(1)}(x_{i+1})) \leq \epsilon = \gamma/q$$

and this implies that

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \gamma.$$

This completes the proof of Corollary 2. $\square$

## 3 The second result

The following distance was introduced in [20].

For each $x, y \in X$, define

$$\rho_1(x,y) := \inf\{\sum_{i=0}^{q-1} \rho(x_i, x_{i+1}) : \; q \geq 1 \text{ is an integer},$$

$$x_i \in X, \; i = 0, \dots, q, \; x_0 = x, \; x_q = y, \; (x_i, x_{i+1}) \in E(G), \; i = 0, \dots, q-1\}. \tag{11}$$

It is not difficult to see that for each $x, y, z \in X$, $\rho_1(x,y) \in [0, \infty]$,

$$\rho_1(x,y) \geq \rho(x,y),$$

$$\rho_1(x,z) \leq \rho_1(x,y) + \rho_1(y,z),$$

and if $\rho_1(x,y) = 0$, then $x = y$. The pseudometric $\rho_1$ plays an important role in [20] and in the present paper because under certain assumptions it turns out that if a mapping $T$ is $G$-nonexpansive (respectively, a $G$-strict contraction), then it is nonexpansive (respectively, a strict contraction) with respect to the pseudometric $\rho_1$. Clearly, in general the pseudometric $\rho_1$ is not symmetric and its values are not necessarily finite.

In this section we assume that there exists a number $\bar{\Delta} > 0$ such that the following assumption holds.

(A) If $(x_0, x_1), (x_1, x_2) \in E(G)$ satisfy $\rho(x_0, x_1) \leq \bar{\Delta}$, $\rho(x_1, x_2) \leq \bar{\Delta}$, then $(x_0, x_2) \in E(G)$.

It turns out that this assumption holds for many important graphs; see, for instance, the examples below.

**Example 1.** *Assume that $\Delta > 0$ and that for each $x, y \in X$, $(x,y) \in E(G)$ if and only if $\rho(x,y) \leq \Delta$. Clearly, (A) holds with $\bar{\Delta} = \Delta/2$.*

**Example 2.** *Let $X$ be a closed set in a Banach space $(E, \|\cdot\|)$ ordered by a closed and convex cone $E_+$ such that $\|x\| \leq \|y\|$ for each $x, y \in E_+$ satisfying $x \leq y$, $\rho(x,y) = \|x - y\|$, $x, y \in X$, and $(x,y) \in E(G)$ if and only if $x \leq y$. Clearly, (A) holds.*

**Lemma 1.** *Let $0 < \epsilon < \bar{\Delta}/2$ and let $x, y \in X$ satisfy*

$$\epsilon \leq \rho_1(x,y).$$

*Then*

$$\rho_1(x,y) = \inf\{\sum_{i=0}^{q-1} \rho(x_i, x_{i+1}) : \; q \geq 1 \textit{ is an integer},$$

$$x_i \in X,\ i = 0, \dots, q,\ x_0 = x,\ x_q = y,\ (x_i, x_{i+1}) \in E(G),\ i = 0, \dots, q-1,$$
$$\max\{\rho(x_i, x_{i+1}) :\ i = 0, \dots, q-1\} \geq \epsilon/4\}.$$

For the proof of this lemma, see [21].

Clearly, for each $x, y \in X$ and each integer $t \geq 1$,

$$\rho_1(A_t(x), A_t(y)) \leq \rho_1(x, y).$$

**Lemma 2.** *Assume that* $0 < \epsilon < \bar{\Delta}/2$ *and let* $x, y \in X$ *satisfy*

$$\epsilon \leq \rho_1(x, y) < \infty.$$

*Then for each integer* $t \geq 1$*,*

$$\rho_1(A_t(x), A_t(y)) \leq \rho_1(x, y) - 4^{-1}\epsilon(1 - \phi(\epsilon/4)).$$

*Proof.* Assume that $t, q \geq 1$ are integers and that $x_i \in X,\ i = 0, \dots, q$, satisfy

$$x_0 = x,\ x_q = y, \tag{12}$$

$$(x_i, x_{i+1}) \in E(G),\ i = 0, \dots, q-1, \tag{13}$$

$$\max\{\rho(x_i, x_{i+1}) :\ i = 0, \dots, q-1\} \geq \epsilon/4. \tag{14}$$

By (4) and (11)-(13), we have

$$\rho_1(A_t(x), A_t(y)) \leq \sum_{i=0}^{q-1} \rho(A_t(x_i), A_t(x_{i+1})) \tag{15}$$

and for $i = 0, \dots, q-1$,

$$\rho(A_t(x_i), A_t(x_{i+1})) \leq \rho(x_i, x_{i+1}). \tag{16}$$

In view of (14), there exists an integer

$$j \in \{0, \dots, q-1\}$$

such that

$$\rho(x_j, x_{j+1}) \geq \epsilon/4. \tag{17}$$

It follows from (4) and (17) that

$$\rho(A_t(x_j), A_t(x_{j+1})) \leq \phi(\rho(x_j, x_{j+1}))\rho(x_j, x_{j+1}) \leq \phi(\epsilon/4)\rho(x_j, x_{j+1}),$$

$$\rho(x_j, x_{j+1}) - \rho(A_t(x_j), A_t(x_{j+1})) \geq (1 - \phi(\epsilon/4))\rho(x_j, x_{j+1})$$

$$\geq (1 - \phi(\epsilon/4))\epsilon/4.$$

By (15), (16) and the above relation, we have

$$\rho_1(A_t(x), A_t(y)) \leq \sum_{i=0}^{q-1} \rho(x_i, x_{i+1}) - (1 - \phi(\epsilon/4))\epsilon/4.$$

When combined with (4), this implies that

$$\rho_1(A_t(x), A_t)) \leq \rho_1(x, y) - (1 - \phi(\epsilon/4))\epsilon/4.$$

This completes the proof of Lemma 2. □

**Theorem 2.** *Let $M > 0$ and $\epsilon \in (0, \bar{\Delta}/2)$ be given. Then there exists a natural number $n_\epsilon$ such that for each $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, each $x, y \in X$ satisfying*

$$\rho_1(x, y) \leq M, \tag{18}$$

*and each integer $n \geq n_\epsilon$, we have*

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \epsilon. \tag{19}$$

*Proof.* Choose an integer

$$n_\epsilon > 1 + 4M\epsilon^{-1}(1 - \phi(\epsilon)/4)^{-1}. \tag{20}$$

Assume that $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, $x, y \in X$ and that (18) holds. It is sufficient to show that there exists an integer $n \in \{1, \dots, n_\epsilon\}$ such that (19) holds.

Suppose to the contrary that (19) does not hold. Then

$$\rho_1(x, y) > \epsilon \tag{21}$$

and for each integer $n \in \{1, \dots, n_\epsilon)\}$,

$$\rho_1(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) > \epsilon. \tag{22}$$

Relations (21), (22) and Lemma 2 imply that

$$\rho_1(A_{r(1)}(x), A_{r(1)}(y)) \leq \rho_1(x, y) - 4^{-1}(1 - \phi(\epsilon/4))\epsilon$$

and that for each $n \in \{1, \dots, n_\epsilon\}$,

$$\rho_1(A_{r(n+1)} \cdots A_{r(1)}(x), A_{r(n+1)} \cdots A_{r(1)}(y))$$

$$\leq \rho_1(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) - 4^{-1}(1 - \phi(\epsilon/4))\epsilon.$$

When combined with (18), the above relation implies that

$$M \geq \rho_1(x, y) \geq \rho_1(A_{r(1)}(x), A_{r(1)}(y))$$

$$\geq \rho_1(A_{r(1)}(x), A_{r(1)}(y)) - \rho_1(A_{r(n_\epsilon)} \cdots A_{r(1)}(x), A_{r(n_\epsilon)} \cdots A_{r(1)}(y))$$

$$= \sum_{n=1}^{n_\epsilon - 1} (\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y))$$

$$-\rho(A_{r(n+1)} \cdots A_{r(1)}(x), A_{r(n+1)} \cdots A_{r(1)}(y)))$$

$$\geq 4^{-1}(n_\epsilon - 1)(1 - \phi(\epsilon/4))\epsilon$$

and

$$n_\epsilon \leq 1 + 4M\epsilon^{-1}(1 - \phi(\epsilon/4))^{-1}.$$

This, however, contradicts (20). The contradiction we have reached completes the proof of Theorem 2. □

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