

IMPLICIT NEUTRAL TEMPERED ψ -CAPUTO FRACTIONAL DIFFERENTIAL COUPLED SYSTEMS WITH DELAY IN GENERALIZED BANACH SPACES*

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Abstract

This article is a subject of some results of the existence, uniqueness, and Ulam-Hyers stability of solutions for a class of implicit neutral fractional differential coupled systems involving the tempered ψ -Caputo fractional derivative with delay. The results are based on the Perov fixed-point theorem for contractions and the Krasnoselskii fixed-point theorem in generalized Banach spaces. Furthermore, the Ulam-Hyers stability of the proposed system is studied. To illustrate our results, we give an example.

Keywords: implicit neutral tempered ψ -Caputo fractional derivative, coupled system, existence, uniqueness, Ulam–Hyers stability, generalized Banach space, fixed point.

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1 Introduction

Fractional calculus, which extends differentiation and integration to non-integer orders, has garnered significant attention in both theoretical studies and practical applications across various research domains. Its versatility has made it an essential tool in the field. Recently, research on fractional calculus has notably increased, with investigations exploring various outcomes under different conditions and forms of fractional differential equations and inclusions. For more details on the applications of fractional calculus, readers are directed to the works of Baleanu et al. [6], Kilbas et al. [19], Samko et al. [36], and Zhou [42]. Additionally, Abbas et al. [1, 2] have studied several problems involving advanced fractional differential and integral equations, presenting various applications. Benchohra et al. [7–9] have demonstrated the existence, stability, and uniqueness of solutions for diverse problems using various fractional derivatives and different types of conditions.

Tempered fractional calculus has recently emerged as an important class of fractional calculus operators. This class generalizes various forms of fractional calculus and features analytic kernels, allowing it to describe the transition between normal and anomalous diffusion. Buschman initially defined fractional integration with weak singular and exponential kernels in [12], and further elaboration on this topic can be found in [16, 22, 33, 34]. A notable development occurred when Almeida [5] used the concept of the fractional derivative in the Caputo sense to introduce the ψ -Caputo derivative with respect to another function ψ , which generalizes a class of fractional derivatives. Medved et al. [23] further modified this concept by defining the tempered ψ -Caputo derivative. Additionally, Salim et al. [35] introduced a new definition for the tempered (k, ψ) -fractional operator and established various properties associated with it.

Implicit neutral problems represent a class of differential equations involving both the dependent variable and its derivatives. These problems occur in various fields such as biology, physics, and engineering, and they present significant challenges in mathematical analysis and numerical solutions. For more information, refer to the monographs by Hale [15], Hale and Verduyn Lunel [14], Hino et al. [17], Kolmanovskii and Myshkis [20], and the references therein, as well as [21]. The incorporation of tempered fractional derivatives into implicit neutral problems offers a novel perspective, providing a deeper understanding of their behavior and characteristics.

In [11], we have investigated the existence result of solutions for the following implicit neutral fractional problem involving tempered ψ -Caputo fractional differential equations with finite delay:

$${}_0^C \mathfrak{D}_\eta^{\zeta, \varpi; \psi} (\gamma(\eta) - \wp(\eta, \gamma_\eta)) = \aleph \left(\eta, \gamma_\eta, {}_0^C \mathfrak{D}_\eta^{\zeta, \varpi; \psi} (\gamma(\eta) - \wp(\eta, \gamma_\eta)) \right);$$

$$\eta \in \Lambda := [0, \kappa], \gamma(\eta) = \chi(\eta); \eta \in [-\theta, 0],$$

where $0 < \zeta \leq 1$, $\varpi \geq 0$, $0 < \kappa < \infty$, $\theta > 0$, ${}_0^C \mathfrak{D}_\delta^{\zeta, \varpi; \psi}$ is the tempered ψ -Caputo fractional derivative of order ζ , Ξ is a Banach space with the norm $\|\cdot\|$, $\wp : \Lambda \times \Pi_\theta \rightarrow \Xi$, $\aleph : \Lambda \times \Pi_\theta \times \Xi \rightarrow \Xi$ are given functions, $\chi \in \Pi_\theta$ and $\Pi_\theta := C([-\theta, 0], \Xi)$. For any $\eta \in \Lambda$, they defined $\gamma_\eta \in \Pi_\theta$ by $\gamma_\eta(s) = \gamma(\eta + s)$; for $s \in [-\theta, 0]$.

Coupled systems involving fractional differential equations are of interest in various scientific and engineering fields. These systems generally comprise multiple equations, which can be interconnected through their derivatives or the variables they describe. For more information, see publications [3, 9, 10] and the references therein. Such systems are used to model fractional-order dynamics in contexts like viscoelastic materials, biological processes, and complex networks.

In [13], the authors studied the existence and uniqueness of solutions to the following delayed coupled system of the form:

$$\begin{cases} \left({}^C D_{\varkappa_1+}^{\bar{\mu}, \psi} \xi_1 \right) (\delta) = F_1(\delta, \xi_{1\delta}, \xi_{2\delta}), \\ \left({}^C D_{\varkappa_1+}^{\mu, \psi} \xi_2 \right) (\delta) = F_2(\delta, \xi_{1\delta}, \xi_{2\delta}), \end{cases} \quad \delta \in J := [\varkappa_1, \varkappa_2],$$

with the following initial conditions

$$\begin{cases} \xi_1(\delta) = \varpi_1(\delta), \\ \xi_2(\delta) = \varpi_2(\delta), \end{cases} \quad \delta \in [\varkappa_1 - \zeta, \varkappa_1],$$

where ${}^C D_{\varkappa_1+}^{\bar{\mu}, \psi}$, ${}^C D_{\varkappa_1+}^{\mu, \psi}$ are the ψ -Caputo fractional derivative of order $\bar{\mu}$, $\mu \in (0, 1]$, respectively, $F_1, F_2 : J \times C([\varkappa_1 - \zeta, \varkappa_1], \mathbb{R}^n) \times C([\varkappa_1 - \zeta, \varkappa_1], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ are given continuous functions, $\varkappa_1$ and $\varkappa_2$ are positive constants such that $\varkappa_1 < \varkappa_2$, $\zeta > 0$ is a constant delay and $\varpi_1, \varpi_2 : [\varkappa_1 - \zeta, \varkappa_1] \rightarrow \mathbb{R}^n$ are two continuous functions. For any function z defined on $[\varkappa_1 - \zeta, \varkappa_2]$ and z_δ the element of $C([\varkappa_1 - \zeta, \varkappa_1], \mathbb{R}^n)$.

Ulam-Hyers stability is a concept in functional equation theory, originating from Ulam's 1940 question about the conditions for an approximately additive function to be close to an exact additive function, see [38]. Hyers

provided a partial answer in 1941, proving that if a function approximates an additive condition, there is an exact additive function close to the approximate one, see [18]. This concept has been extended to various functional equations and mathematical settings, significantly influencing their study and applications in various fields. Further elaboration on this topic can be found in [25, 29–31, 41].

Motivated by the papers mentioned earlier, we discuss the existence uniqueness and Ulam-Hyers stability of solutions to the following delayed coupled system involving implicit neutral tempered ψ -Caputo fractional equations, for $\delta \in \mathfrak{J}$,

$$\begin{cases} {}^C_0\mathfrak{D}_\delta^{\zeta_1, \omega_1; \psi} (\mu(\delta) - \wp_1(\delta, \mu_\delta)) = \aleph_1 \left(\delta, \mu_\delta, \bar{\mu}_\delta, {}^C_0\mathfrak{D}_\delta^{\zeta_1, \omega_1; \psi} (\mu(\delta) - \wp_1(\delta, \mu_\delta)) \right), \\ {}^C_0\mathfrak{D}_\delta^{\zeta_2, \omega_2; \psi} (\bar{\mu}(\delta) - \wp_2(\delta, \bar{\mu}_\delta)) = \aleph_2 \left(\delta, \mu_\delta, \bar{\mu}_\delta, {}^C_0\mathfrak{D}_\delta^{\zeta_2, \omega_2; \psi} (\bar{\mu}(\delta) - \wp_2(\delta, \bar{\mu}_\delta)) \right), \end{cases} \quad (1)$$

with the initial conditions

$$\begin{cases} \mu(\delta) = \phi_1(\delta), \\ \bar{\mu}(\delta) = \phi_2(\delta), \end{cases} \quad , \quad \delta \in [-r, 0], \quad (2)$$

where $0 < \zeta_j < 1$, $\omega_j \geq 0$, $\mathfrak{J} := [0, T]$, $T > 0$, ${}^C_0\mathfrak{D}_\theta^{\zeta_j, \omega_j; \psi}$ are the tempered ψ -Caputo fractional derivatives of order ζ_j ; $j = 1, 2$, $\mathbb{R}^m$; $m \in \mathbb{N}^*$ is the Euclidean Banach space with a suitable norm $\|\cdot\|$, $\wp_j : \mathfrak{J} \times \Omega_r \rightarrow \mathbb{R}^m$, $\aleph_j : \mathfrak{J} \times \Omega_r \times \Omega_r \times \mathbb{R}^m \rightarrow \mathbb{R}^m$; $j = 1, 2$, are given continuous functions, $\phi_j \in \Omega_r$; $j = 1, 2$, and $\Omega_r = C([-r, 0], \mathbb{R}^m)$. For any $\delta \in \mathfrak{J}$, we defined $\mu_\delta \in \Omega_r$ by

$$\mu_\delta(\theta) = \mu(\delta + \theta); \quad \text{for } \theta \in [-r, 0].$$

This paper is arranged as follows: Section 2 introduces some preliminaries, definitions, lemmas and auxiliary results that are used throughout this work. In Section 3, we prove the uniqueness and existence of solutions for the coupled system (1)-(2) by using Perov's and Krasnoselskii's fixed point theorems in generalized Banach spaces. Moreover we establish the Ulam-Hyers stability of this system. Finally, we present an example to show the validity of our results.

2 Preliminaries

First, we give the definitions and the notations that we will use throughout this paper.

Let $C(\mathfrak{J}) = C(\mathfrak{J}, \mathbb{R}^m)$ be the Banach space of all continuous functions μ from $\mathfrak{J}$ into $\mathbb{R}^m$ with the supremum (uniform) norm

$$\|\mu\|_\infty := \sup_{\delta \in \mathfrak{J}} \|\mu(\delta)\|.$$

As usual, $C^n(\mathfrak{J}, \mathbb{R}^m)$ be the space of continuous functions, n -times continuously differentiable functions from $\mathfrak{J}$ into $\mathbb{R}^m$.

Consider $\Omega_r := C([-r, 0], \mathbb{R}^m)$ the Banach space with the norm

$$\|\mu\|_{\Omega_r} = \sup_{\delta \in [-r, 0]} \|\mu(\delta)\|.$$

And the Banach space $\Omega := C([-r, T], \mathbb{R}^m)$ with the norm

$$\|\mu\|_\Omega = \sup_{\delta \in [-r, T]} \|\mu(\delta)\|.$$

Obviously, the product space $\Upsilon := \Omega \times \Omega$ is a generalized Banach space, endowed with the vector-valued norm

$$\|(\mu, \bar{\mu})\|_\Upsilon = \begin{pmatrix} \|\mu\|_\Omega \\ \|\bar{\mu}\|_\Omega \end{pmatrix}.$$

2.1 Tempered ψ -fractional calculus

Definition 1 (The tempered ψ -fractional integral [23]). *Let $\zeta > 0$, $\mu \in C(\mathfrak{J})$, $\omega \geq 0$ and $\psi \in C^1(\mathfrak{J}, \mathbb{R}^m)$ is an increasing differentiable function such that $\psi'(\delta) \neq 0$ for all $\delta \in [0, T]$. Then, the tempered ψ -fractional integral of order ζ is defined by:*

$$\begin{aligned} {}_0I_\delta^{\zeta, \omega; \psi} \mu(\delta) &= e^{-\omega\psi(\delta)} {}_0I_\delta^{\zeta; \psi} \left(e^{\omega\psi(\delta)} \mu(\delta) \right) \\ &= \int_0^\delta \Psi_\zeta^{\omega; \psi}(\delta, \varpi) \psi'(\varpi) \mu(\varpi) d\varpi, \end{aligned} \tag{3}$$

where $\Psi_\zeta^{\omega; \psi}(\delta, \varpi) = \frac{e^{-\omega(\psi(\delta) - \psi(\varpi))} [\psi(\delta) - \psi(\varpi)]^{\zeta-1}}{\Gamma(\zeta)}$ and ${}_0I_\delta^{\zeta; \psi}$ is the ψ -Riemann-Liouville fractional integral [23], defined by:

$${}_0I_\delta^{\zeta; \psi} \mu(\delta) = \frac{1}{\Gamma(\zeta)} \int_0^\delta \psi'(\varpi) [\psi(\delta) - \psi(\varpi)]^{\zeta-1} \mu(\varpi) d\varpi. \tag{4}$$

Definition 2 (The tempered ψ -Caputo fractional derivative [23]). Let $\psi \in C^n(\mathfrak{J}, \mathbb{R}^m)$ is an increasing differentiable function such that $\psi'(\delta) \neq 0$ for all $\delta \in [0, T]$, $n-1 < \zeta < n$; $n \in \mathbb{N}^+$, $\omega \geq 0$. The tempered ψ -Caputo fractional derivative of order ζ is defined as:

$$\begin{aligned} {}^C_0\mathfrak{D}_\delta^{\zeta, \omega; \psi} \mu(\delta) &= e^{-\omega\psi(\delta)} {}^C_0\mathfrak{D}_\delta^{\zeta; \psi} \left(e^{\omega\psi(\delta)} \mu(\delta) \right) \\ &= \frac{e^{-\omega\psi(\delta)}}{\Gamma(n-\zeta)} \int_0^\delta \psi'(\varpi) [\psi(\delta) - \psi(\varpi)]^{n-\zeta-1} \mathcal{D}_\psi^n \left(e^{\omega\psi(\varpi)} \mu(\varpi) \right) d\varpi, \end{aligned}$$

where $\mathcal{D}_\psi^n = \left[\frac{1}{\psi'(\delta)} \frac{d}{d\delta} \right]^n$ and ${}^C_0\mathfrak{D}_\delta^{\zeta; \psi}$ denotes the ψ -Caputo fractional derivative [23], given by:

$${}^C_0\mathfrak{D}_\delta^{\zeta; \psi} \mu(\delta) = \frac{1}{\Gamma(n-\zeta)} \int_0^\delta \psi'(\varpi) [\psi(\delta) - \psi(\varpi)]^{n-\zeta-1} \mathcal{D}_\psi^n \mu(\varpi) d\varpi.$$

Remark 1. If we modify the parameter ω , and the function ψ , the tempered ψ -Caputo fractional derivative interpolate the following fractional derivatives:

- The Caputo tempered fractional derivative ($\psi(\delta) = \delta$) [37];
- The ψ -Caputo fractional derivative ($\omega = 0$) [8, 9];
- The Caputo fractional derivative ($\omega = 0$, $\psi(\delta) = \delta$) [8, 9];
- the Caputo-Hadamard fractional derivative ($\omega = 0$, $\psi(\delta) = \ln \delta$) [8, 9].

Lemma 1. [23] Let $\mu \in C^n(\mathfrak{J}, \mathbb{R}^m)$, $\omega \geq 0$ and $n-1 < \zeta < n$. Then we have:

$$\begin{aligned} & {}_0I_\delta^{\zeta, \omega; \psi} \left[{}^C_0\mathfrak{D}_\delta^{\zeta, \omega; \psi} \mu(\delta) \right] \\ &= \mu(\delta) - e^{-\omega\psi(\delta)} \sum_{k=0}^{n-1} \frac{[\psi(\delta) - \psi(0)]^k}{k!} \left[\mathcal{D}_\psi^k \left(e^{\omega\psi(\delta)} \mu(\delta) \right) \Big|_{\delta=0} \right]. \end{aligned}$$

In particular, if $\mu \in C^1(\mathfrak{J}, \mathbb{R}^m)$, then, ${}^C_0\mathfrak{D}_\delta^{\zeta, \omega; \psi} \left[{}_0I_\delta^{\zeta, \omega; \psi} \mu(\delta) \right] = \mu(\delta)$.

Lemma 2. [11] Let $0 < \zeta < 1$, and $\mathfrak{h} : \mathfrak{J} \rightarrow \mathbb{R}^m$, $\mathfrak{Q} : \mathfrak{J} \rightarrow \mathbb{R}^m$ be two continuous functions. Then the problem

$${}^C_0\mathfrak{D}_\delta^{\zeta, \omega; \psi} (\gamma(\delta) - \mathfrak{h}(\delta)) = \mathfrak{Q}(\delta); \quad \delta \in \mathfrak{J}, \quad (5)$$

$$\gamma(\delta) = \phi(\delta); \quad \delta \in [-r, 0], \quad (6)$$

has a unique solution defined by

$$\gamma(\delta) = \begin{cases} e^{-\omega(\psi(\delta)-\psi(0))}[\phi(0) - \mathfrak{h}(0)] + \\ \mathfrak{h}(\delta) + \int_0^\delta \Psi_{\zeta}^{\omega;\psi}(\delta, \varpi) \psi'(\varpi) \mathfrak{Q}(\varpi) d\varpi; & \delta \in \mathfrak{J} \\ \gamma(\delta) = \phi(\delta); & \eta \in [-r, 0]. \end{cases} \quad (7)$$

Lemma 3. Let $\zeta_j \in (0, 1]$; $j = 1, 2$ be fixed and $\wp_j : \mathfrak{J} \times \Omega_r \rightarrow \mathbb{R}^m$, $\aleph_j : \mathfrak{J} \times \Omega_r \times \Omega_r \times \mathbb{R}^m \rightarrow \mathbb{R}^m$; $j = 1, 2$, are a given continuous functions. Then (1)–(2) is equivalent to

$$\mu(\delta) = \begin{cases} e^{-\omega_1(\psi(\delta)-\psi(0))}[\phi_1(0) - \wp_1(0, \phi_1)] + \wp_1(\delta, \mu_\delta) \\ + \int_0^\delta \Psi_{\zeta_1}^{\omega_1;\psi}(\delta, \varpi) \psi'(\varpi) \aleph_1(\varpi, \mu_\varpi, \bar{\mu}_\varpi, \vartheta_1(\varpi)) d\varpi; & \delta \in \mathfrak{J}, \\ \phi_1(\delta); & \delta \in [-r, 0], \end{cases} \quad (8)$$

and

$$\bar{\mu}(\delta) = \begin{cases} e^{-\omega_2(\psi(\delta)-\psi(0))}[\phi_2(0) - \wp_2(0, \phi_2)] + \wp_2(\delta, \bar{\mu}_\delta) \\ + \int_0^\delta \Psi_{\zeta_2}^{\omega_2;\psi}(\delta, \varpi) \psi'(\varpi) \aleph_2(\varpi, \mu_\varpi, \bar{\mu}_\varpi, \vartheta_2(\varpi)) d\varpi; & \delta \in \mathfrak{J}, \\ \phi_2(\delta); & \delta \in [-r, 0], \end{cases} \quad (9)$$

where $\vartheta_j(\cdot)$; $j = 1, 2$ satisfy the following functional equation

$$\vartheta_j(\delta) = \aleph_j(\delta, \mu_\delta, \bar{\mu}_\delta, \vartheta_j(\delta)).$$

2.2 Generalized Banach spaces

Let $\mu, \bar{\mu} \in \mathbb{R}^m$ with $\mu = (\mu_1, \mu_2, \dots, \mu_m)$, $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_m)$. By $\mu < \bar{\mu}$ we mean $\mu_j < \bar{\mu}_j$ for $j = 1, \dots, m$.

$$|\mu| = (|\mu_1|, |\mu_2|, \dots, |\mu_m|).$$

$$\max(\mu, \bar{\mu}) = (\max(\mu_1, \bar{\mu}_1), \max(\mu_2, \bar{\mu}_2), \dots, \max(\mu_m, \bar{\mu}_m)).$$

If $\gamma \in \mathbb{R}$ then $\mu \leq \gamma$ means $\mu_j \leq \gamma$; $j = 1, \dots, m$.

And

$$\mathbb{R}_+^m = \{\mu \in \mathbb{R}^m, \mu_j \in \mathbb{R}_+, j = 1, \dots, m\}.$$

Definition 3. [27] Let Υ be a nonempty set. By a vector-valued metric on Υ we mean a map $d : \Upsilon \times \Upsilon \rightarrow \mathbb{R}^m$ with the following properties:

- (i) $d(\mu, \bar{\mu}) \geq 0$ for all $\mu, \bar{\mu} \in \Upsilon$, and if $d(\mu, \bar{\mu}) = 0$ then $\mu = \bar{\mu}$;
- (ii) $d(\mu, \bar{\mu}) = d(\bar{\mu}, \mu)$ for all $\mu, \bar{\mu} \in \Upsilon$;
- (iii) $d(\mu, \xi) = d(\mu, \bar{\mu}) + d(\bar{\mu}, \xi)$ for all $\mu, \bar{\mu}, \xi \in \Upsilon$.

We call the pair (Υ, d) a generalized metric space with

$$d(\mu, \bar{\mu}) := \begin{pmatrix} d_1(\mu, \bar{\mu}) \\ d_2(\mu, \bar{\mu}) \\ \vdots \\ d_m(\mu, \bar{\mu}) \end{pmatrix}.$$

Notice that d is a generalized metric space on Υ if and only if d_j ; $j = 1, \dots, m$ are metrics on Υ .

Definition 4. [4] A square matrix M of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open-unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denotes the unit matrix of $M_{m \times m}(\mathbb{R})$.

Theorem 1. [40] For any nonnegative square matrix M , the following properties are equivalent

- (i) M is convergent to zero;
- (ii) $\rho(M) < 1$;
- (iii) the matrix $I - M$ is nonsingular and

$$(I - M)^{-1} = I + M + \dots + M^n + \dots;$$

- (iv) $I - M$ is nonsingular and $(I - M)^{-1}$ is a nonnegative matrix.

Example 1. [27] The matrix $A \in M_{2 \times 2}(\mathbb{R})$ defined by

$$A = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix},$$

converges to zero in the following cases:

- (1) $\xi_2 = \xi_3 = 0$, $\xi_1, \xi_4 > 0$ and $\max\{\xi_1, \xi_4\} < 1$.

(2) $\xi_3 = 0$, $\xi_1, \xi_4 > 0$, $\xi_1 + \xi_4 < 1$ and $-1 < \xi_2 < 0$.

(2) $\xi_1 + \xi_2 = \xi_3 + \xi_4 = 0$, $\xi_1 > 1$, $\xi_3 > 0$ and $|\xi_1 - \xi_3| < 1$.

Definition 5. [28] Let (Υ, d) be a generalized metric space. An operator $\mathbb{Q} : \Upsilon \rightarrow \Upsilon$ is said to be contractive if there exists a matrix M convergent to zero such that

$$d(\mathbb{Q}(\mu), \mathbb{Q}(\bar{\mu})) \leq Md(\mu, \bar{\mu}), \text{ for all } \mu, \bar{\mu} \in \Upsilon.$$

Theorem 2. [26–28] (Perov) Let (Υ, d) be a complete generalized metric space and $\mathbb{Q} : \Upsilon \rightarrow \Upsilon$ a contractive operator with Lipschitz matrix M . Then $\mathbb{Q}$ has a unique fixed point μ_0 and, for each $\mu \in \Upsilon$, we have

$$d(\mathbb{Q}^k(\mu), \mu_0) \leq M^k(M)^{-1}d(\mu, \mathbb{Q}(\mu)), \text{ for all } k \in \mathbb{N}.$$

Theorem 3. [24] (Krasnoselskii) Let B be a closed, convex, non-empty subset of a generalized Banach spaces Υ . Suppose that $\mathcal{K}$ and $\mathcal{P}$ map B into Υ and that

- (i) $\mathcal{K}\gamma + \mathcal{P}\eta \in B$ for all $\gamma, \eta \in B$;
- (ii) $\mathcal{K}$ is compact and continuous;
- (iii) $\mathcal{P}$ is an M -contraction mapping.

Then the operator equation $\mathcal{K}\gamma + \mathcal{P}\gamma = \gamma$ has at least one solution on B .

3 Main Results

3.1 Uniqueness results

In this section, we give our main uniqueness result for the coupled system (1)-(2).

Definition 6. By a solution of the coupled system (1)-(2), we mean a coupled continuous functions $(\mu, \bar{\mu}) \in \Upsilon$ satisfying the initial condition (2) and the system (1) on $\mathfrak{J}$.

The following hypotheses will be used in the sequel.

(H₁) There exist functions $a_j, b_j, c_j \in C(\mathfrak{J}, \mathbb{R}^+)$; $j = 1, 2$ where $\|c_j\|_\infty < 1$; $j = 1, 2$ such that:

$$\begin{aligned} & \|\aleph_j(\delta, \mu, \bar{\mu}, \xi) - \aleph_j(\delta, \bar{\mu}, \bar{\bar{\mu}}, \bar{\xi})\| \\ & \leq a_j(\delta)\|\mu - \bar{\mu}\|_{\Omega_r} + b_j(\delta)\|\bar{\mu} - \bar{\bar{\mu}}\|_{\Omega_r} + c_j(\delta)\|\xi - \bar{\xi}\|; \quad j = 1, 2, \\ & \text{for } \delta \in \mathfrak{J}, \text{ and each } \mu, \bar{\mu}, \bar{\bar{\mu}} \in \Omega_r, \quad \xi, \bar{\xi} \in \mathbb{R}^m. \end{aligned}$$

(H₂) There exist functions $d_j \in C(\mathfrak{J}, \mathbb{R}^+)$; $j = 1, 2$ where $\|d_j\|_\infty < 1$; $j = 1, 2$ such that:

$$\|\wp_j(\delta, \mu) - \wp_j(\delta, \bar{\mu})\| \leq d_j(\delta) \|\mu - \bar{\mu}\|_{\Omega_r}; \quad j = 1, 2,$$

for $\delta \in \mathfrak{J}$, and each $\mu, \bar{\mu} \in \Omega_r$.

Set

$$\aleph_j^* = \sup_{\delta \in \mathfrak{J}} \|\aleph_j(\delta, 0, 0, 0)\|, \quad \wp_j^* = \sup_{\delta \in \mathfrak{J}} \|\wp_j(\delta, 0)\|,$$

$$L_{\zeta_j}^\psi = \frac{[\psi(T) - \psi(0)]^{\zeta_j}}{\Gamma(\zeta_j + 1)}, \quad \kappa_{c_j}^{a_j} = \frac{\|a_j\|_\infty}{1 - \|c_j\|_\infty}, \quad \eta_{c_j}^{b_j} = \frac{\|b_j\|_\infty}{1 - \|c_j\|_\infty}; \quad j = 1, 2.$$

Theorem 4. Assume that the hypotheses (H₁) – (H₂) hold. If the matrix

$$M = \begin{pmatrix} \|d_1\|_\infty + \kappa_{c_1}^{a_1} L_{\zeta_1}^\psi & \eta_{c_1}^{b_1} L_{\zeta_1}^\psi \\ \kappa_{c_2}^{a_2} L_{\zeta_2}^\psi & \|d_2\|_\infty + \eta_{c_2}^{b_2} L_{\zeta_2}^\psi \end{pmatrix} \quad (10)$$

converges to 0, then the coupled system (1)–(2) has a unique solution.

Proof. Define the operators $\mathcal{N}_j : \Upsilon \rightarrow \Omega$; $j = 1, 2$ by

$$(\mathcal{N}_1(\mu, \bar{\mu}))(\delta) = \begin{cases} e^{-\omega_1(\psi(\delta) - \psi(0))} [\phi_1(0) - \wp_1(0, \phi_1)] + \wp_1(\delta, \mu_\delta) \\ + \int_0^\delta \Psi_{\zeta_1}^{\omega_1; \psi}(\delta, \varpi) \psi'(\varpi) \aleph_1(\varpi, \mu_\varpi, \bar{\mu}_\varpi, \vartheta_1(\varpi)) d\varpi; \quad \delta \in \mathfrak{J}, \\ \phi_1(\delta); \quad \delta \in [-r, 0], \end{cases} \quad (11)$$

and

$$(\mathcal{N}_2(\mu, \bar{\mu}))(\delta) = \begin{cases} e^{-\omega_2(\psi(\delta) - \psi(0))} [\phi_2(0) - \wp_2(0, \phi_2)] + \wp_2(\delta, \bar{\mu}_\delta) \\ + \int_0^\delta \Psi_{\zeta_2}^{\omega_2; \psi}(\delta, \varpi) \psi'(\varpi) \aleph_2(\varpi, \mu_\varpi, \bar{\mu}_\varpi, \vartheta_2(\varpi)) d\varpi; \quad \delta \in \mathfrak{J}, \\ \phi_2(\delta); \quad \delta \in [-r, 0], \end{cases} \quad (12)$$

where $\vartheta_j(\cdot)$; $j = 1, 2$ satisfy the following functional equation

$$\vartheta_j(\delta) = \aleph_j(\delta, \mu_\delta, \bar{\mu}_\delta, \vartheta_j(\delta)).$$

Consider the continuous operator $\mathcal{N} : \Upsilon \rightarrow \Upsilon$ defined by

$$(\mathcal{N}(\mu, \bar{\mu}))(\delta) = ((\mathcal{N}_1(\mu, \bar{\mu}))(\delta), (\mathcal{N}_2(\mu, \bar{\mu}))(\delta)). \quad (13)$$

Clearly, the fixed points of the operator $\mathcal{N}$ are solutions of the coupled system (1)–(2). We shall show that $\mathcal{N}$ satisfies all conditions of Theorem 2. Let $(\mu, \bar{\mu}), (\bar{\mu}, \bar{\bar{\mu}}) \in \Omega$. Then for each $\delta \in [-r, 0]$, we have

$$\|\mathcal{N}_1(\mu, \bar{\mu})(\delta) - \mathcal{N}_1(\bar{\mu}, \bar{\bar{\mu}})(\delta)\| = 0.$$

And for each $\delta \in \mathfrak{J}$, we have

$$\begin{aligned} \|\mathcal{N}_1(\mu, \bar{\mu})(\delta) - \mathcal{N}_1(\bar{\mu}, \bar{\bar{\mu}})(\delta)\| &\leq \|\wp_1(\delta, \mu_\delta) - \wp_1(\delta, \bar{\mu}_\delta)\| \\ &\quad + \int_0^\theta \Psi_{\zeta_j}^{\omega_j, \psi}(\delta, \varpi) \psi'(\varpi) \|\vartheta_1(\varpi) - \bar{\vartheta}_1(\varpi)\| d\varpi, \end{aligned} \quad (14)$$

where $\vartheta_1(\cdot), \bar{\vartheta}_1(\cdot) \in \mathbb{R}^m$ satisfy the following functional equations

$$\vartheta_1(\delta) = \aleph_1(\delta, \mu_\delta, \bar{\mu}_\delta, \vartheta_1(\delta)) \quad \text{and} \quad \bar{\vartheta}_1(\delta) = \aleph_1(\delta, \mu_\delta, \bar{\mu}_\delta, \bar{\vartheta}_1(\delta)).$$

From (H_1) , we have

$$\begin{aligned} \|\vartheta_1(\delta) - \bar{\vartheta}_1(\delta)\| &\leq a_1(\delta) \|\mu_\delta - \bar{\mu}_\delta\|_{\Omega_r} + b_1(\delta) \|\bar{\mu}_\delta - \bar{\bar{\mu}}_\delta\|_{\Omega_r} + c_1(\delta) \|\vartheta_1(\delta) - \bar{\vartheta}_1(\delta)\|. \end{aligned}$$

Then,

$$\|\vartheta_1 - \bar{\vartheta}_1\|_\Omega \leq \|a_1\|_\infty \|\mu - \bar{\mu}\|_\Omega + \|b_1\|_\infty \|\bar{\mu} - \bar{\bar{\mu}}\|_\Omega + \|c_1\|_\infty \|\vartheta_1 - \bar{\vartheta}_1\|_\Omega.$$

This gives

$$\begin{aligned} \|\vartheta_1 - \bar{\vartheta}_1\|_\Omega &\leq \frac{\|a_1\|_\infty}{1 - \|c_1\|_\infty} \|\mu - \bar{\mu}\|_\Omega + \frac{\|b_1\|_\infty}{1 - \|c_1\|_\infty} \|\bar{\mu} - \bar{\bar{\mu}}\|_\Omega \\ &= \kappa_{c_1}^{a_1} \|\mu - \bar{\mu}\|_\Omega + \eta_{c_1}^{b_1} \|\bar{\mu} - \bar{\bar{\mu}}\|_\Omega. \end{aligned} \quad (15)$$

From (H_2) , (14) and (15), we get

$$\begin{aligned} \|\mathcal{N}_1(\mu, \bar{\mu})(\delta) - \mathcal{N}_1(\bar{\mu}, \bar{\bar{\mu}})(\delta)\| &\leq d_1(\delta) \|\mu_\delta - \bar{\mu}_\delta\|_{\Omega_r} + \frac{[\psi(T) - \psi(0)]^{\zeta_1}}{\Gamma(\zeta_1 + 1)} \\ &\quad \times \left(\kappa_{c_1}^{a_1} \|\mu - \bar{\mu}\|_\Omega + \eta_{c_1}^{b_1} \|\bar{\mu} - \bar{\bar{\mu}}\|_\Omega \right) \\ &\leq \left(\|d_1\|_\infty + \kappa_{c_1}^{a_1} L_{\zeta_1}^\psi \right) \|\mu - \bar{\mu}\|_\Omega + \eta_{c_1}^{b_1} L_{\zeta_1}^\psi \|\bar{\mu} - \bar{\bar{\mu}}\|_\Omega. \end{aligned} \quad (16)$$

Hence,

$$\|\mathcal{N}_1(\mu, \bar{\mu}) - \mathcal{N}_1(\bar{\mu}, \bar{\bar{\mu}})\|_\Omega \leq \left(\|d_1\|_\infty + \kappa_{c_1}^{a_1} L_{\zeta_1}^\psi \right) \|\mu - \bar{\mu}\|_\Omega + \eta_{c_1}^{b_1} L_{\zeta_1}^\psi \|\bar{\mu} - \bar{\bar{\mu}}\|_\Omega.$$

Similarly, we get

$$\|\mathcal{N}_2(\mu, \bar{\mu}) - \mathcal{N}_2(\bar{\mu}, \bar{\bar{\mu}})\|_{\Omega} \leq \kappa_{c_2}^{a_2} L_{\zeta_2}^{\psi} \|\mu - \bar{\mu}\|_{\Omega} + \left(\|d_2\|_{\infty} + \eta_{c_2}^{b_2} L_{\zeta_2}^{\psi} \right) \|\bar{\mu} - \bar{\bar{\mu}}\|_{\Omega}.$$

Consequently,

$$\|\mathcal{N}(\mu, \bar{\mu}) - \mathcal{N}(\bar{\mu}, \bar{\bar{\mu}})\|_{\Upsilon} \leq M \|(\mu, \bar{\mu}) - (\bar{\mu}, \bar{\bar{\mu}})\|_{\Upsilon}.$$

Since M converges to zero, then Theorem 2 implies that (1)-(2) has a unique solution in Υ . $\square$

3.2 Existence results

In this section, we give our main existence result for the coupled system (1)-(2).

Theorem 5. *Assume that the hypotheses $(H_1) - (H_2)$ hold. If the matrix*

$$M = \begin{pmatrix} \|d_1\|_{\infty} + \kappa_{c_1}^{a_1} L_{\zeta_1}^{\psi} & \eta_{c_1}^{b_1} L_{\zeta_1}^{\psi} \\ \kappa_{c_2}^{a_2} L_{\zeta_2}^{\psi} & \|d_2\|_{\infty} + \eta_{c_2}^{b_2} L_{\zeta_2}^{\psi} \end{pmatrix}$$

converges to 0, then the coupled system (1)-(2) has at least one solution.

Proof. Consider the operators $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2)$, $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2) : \Upsilon \rightarrow \Upsilon$ defined by

$$\mathcal{K}(\mu, \bar{\mu}) = (\mathcal{K}_1(\mu, \bar{\mu}), \mathcal{K}_2(\mu, \bar{\mu})),$$

and

$$\mathcal{P}(\mu, \bar{\mu}) = (\mathcal{P}_1(\mu, \bar{\mu}), \mathcal{P}_2(\mu, \bar{\mu})),$$

such that

$$\begin{cases} \mathcal{K}_1(\mu, \bar{\mu})(\delta) = \wp_1(\delta, \mu_{\delta}); \\ \mathcal{K}_2(\mu, \bar{\mu})(\delta) = \wp_2(\delta, \bar{\mu}_{\delta}); \end{cases} \quad \delta \in \mathfrak{J}, \quad (17)$$

with

$$\begin{cases} \mathcal{K}_1(\mu, \bar{\mu})(\delta) = \phi_1(\delta); \\ \mathcal{K}_2(\mu, \bar{\mu})(\delta) = \phi_2(\delta); \end{cases} \quad \delta \in [-r, 0], \quad (18)$$

and

$$\begin{cases} \mathcal{P}_1(\mu, \bar{\mu})(\delta) = e^{-\omega_1(\psi(\delta) - \psi(0))} [\phi_1(0) - \wp_1(0, \phi_1)] + {}_0I_{\theta}^{\zeta_1, \omega_1; \psi} \wp_1(\delta); \\ \mathcal{P}_2(\mu, \bar{\mu})(\delta) = e^{-\omega_2(\psi(\delta) - \psi(0))} [\phi_2(0) - \wp_2(0, \phi_2)] + {}_0I_{\theta}^{\zeta_2, \omega_2; \psi} \wp_2(\delta); \end{cases} \quad \delta \in \mathfrak{J}, \quad (19)$$

with

$$\begin{cases} \mathcal{P}_1(\mu, \bar{\mu})(\delta) = 0; \\ \mathcal{P}_2(\mu, \bar{\mu})(\delta) = 0; \end{cases} \quad \delta \in [-r, 0], \quad (20)$$

where $\vartheta_j(\cdot) \in \mathbb{R}^m$; $j = 1, 2$ satisfy the following functional equation

$$\vartheta_j(\delta) = \aleph_j(\delta, \mu_\delta, \bar{\mu}_\delta, \vartheta_j(\delta)).$$

Obviously, both $\mathcal{K}$ and $\mathcal{P}$ are well defined due to (H_1) and (H_2) . Furthermore, the operator $\mathcal{N}$ given by (13) may be written as

$$\mathcal{N}(\mu, \bar{\mu}) = (\mathcal{K}_1(\mu, \bar{\mu}), \mathcal{K}_2(\mu, \bar{\mu})) + (\mathcal{P}_1(\mu, \bar{\mu}), \mathcal{P}_2(\mu, \bar{\mu})). \quad (21)$$

Let

$$\Delta_\chi = \{(\mu, \bar{\mu}) \in \Upsilon : \|(\mu, \bar{\mu})\|_\Upsilon \leq \chi\},$$

with $\chi = (\chi_1, \chi_2) \in \mathbb{R}_+^2$ such that

$$\begin{cases} \chi_1 \geq \sigma_1 \lambda_1 + \sigma_2 \lambda_2, \\ \chi_1 \geq \sigma_3 \lambda_1 + \sigma_4 \lambda_2, \end{cases}$$

where λ_1, λ_2 and σ_j ; $j = \overline{1, 4}$ are positive real numbers that will be specified later. Moreover, notice that Δ_χ is closed, convex and bounded subset of the generalized Banach space Υ . We shall prove that $\mathcal{K}$ and $\mathcal{P}$, satisfy all conditions of Theorem 3. The proof will be given in three steps.

Step 1. $\mathcal{K}(\mu, \bar{\mu}) + \mathcal{P}(\bar{\mu}, \bar{\bar{\mu}}) \in \Delta_\chi$ whenever $(\mu, \bar{\mu}), (\bar{\mu}, \bar{\bar{\mu}}) \in \Delta_\chi$.
Let $(\mu, \bar{\mu}), (\bar{\mu}, \bar{\bar{\mu}}) \in \Delta_\chi$. Then for each $\delta \in [-r, 0]$, we have

$$\|\mathcal{K}_1(\mu, \bar{\mu})(\delta)\| \leq \|\phi_1\|_{\Omega_r}.$$

Thus,

$$\|\mathcal{K}_1(\mu, \bar{\mu})\|_{\Omega_r} \leq \|\phi_1\|_{\Omega_r}. \quad (22)$$

For each $\delta \in \mathfrak{J}$, we have

$$\begin{aligned} \|\mathcal{K}_1(\mu, \bar{\mu})(\delta)\| &= \|\wp_1(\delta, \mu_\delta)\| \\ &\leq \|\wp_1(\delta, \mu_\delta) - \wp_1(\delta, 0)\| + \|\wp_1(\delta, 0)\| \\ &\leq d_1(\delta) \|\mu_\delta\|_{\Omega_r} + \wp_1^* \\ &\leq \|d_1\|_\infty \|\mu\|_\Omega + \wp_1^*. \end{aligned}$$

Hence, we get

$$\|\mathcal{K}_1(\mu, \bar{\mu})\|_\infty \leq \|d_1\|_\infty \|\mu\|_\Omega + \wp_1^*. \quad (23)$$

So from (22) and (23), we get

$$\begin{aligned}\|\mathcal{K}_1(\mu, \bar{\mu})\|_{\Omega} &= \|\mathcal{K}_1(\mu, \bar{\mu})\|_{\Omega_r} + \|\mathcal{K}_1(\mu, \bar{\mu})\|_{\infty} \\ &\leq \|\phi_1\|_{\Omega_r} + \|d_1\|_{\infty} \|\mu\|_{\Omega} + \wp_1^*.\end{aligned}$$

Similarly, we get

$$\|\mathcal{K}_2(\mu, \bar{\mu})\|_{\Omega} \leq \|\phi_2\|_{\Omega_r} + \|d_2\|_{\infty} \|\bar{\mu}\|_{\Omega} + \wp_2^*.$$

Thus the above inequalities can be written in the vectorial form as follows

$$\|\mathcal{K}(\mu, \bar{\mu})\|_{\Upsilon} \leq \begin{pmatrix} \|\mathcal{K}_1(\mu, \bar{\mu})\|_{\Omega} \\ \|\mathcal{K}_2(\mu, \bar{\mu})\|_{\Omega} \end{pmatrix} \leq A \begin{pmatrix} \|\mu\|_{\Omega} \\ \|\bar{\mu}\|_{\Omega} \end{pmatrix} + \begin{pmatrix} \|\phi_1\|_{\Omega_r} + \wp_1^* \\ \|\phi_2\|_{\Omega_r} + \wp_2^* \end{pmatrix}, \quad (24)$$

where

$$A = \begin{pmatrix} \|d_1\|_{\infty} & 0 \\ 0 & \|d_2\|_{\infty} \end{pmatrix}. \quad (25)$$

Also, from (H_2) and (15) we have

$$\begin{aligned}\|\mathcal{P}_1(\mu, \bar{\mu})(\delta)\| &\leq \|\phi_1(0)\| + \|\wp_1(0, \phi_1)\| + \int_0^{\delta} \frac{[\psi(\delta) - \psi(\varpi)]^{\zeta_1-1}}{\Gamma(\zeta_1)} \psi'(\varpi) \\ &\quad \times \|\aleph_1(\varpi, \mu_{\varpi}, \bar{\mu}_{\varpi}, \vartheta_1(\varpi))\| d\varpi \\ &\leq \|\phi_1(0)\| + \|\wp_1(0, \phi_1) - \wp_1(0, 0)\| + \|\wp_1(0, 0)\| \\ &\quad + \int_0^{\delta} \frac{[\psi(\delta) - \psi(\varpi)]^{\zeta_1-1}}{\Gamma(\zeta_1)} \psi'(\varpi) \\ &\quad \times (\|\aleph_1(\varpi, \mu_{\varpi}, \bar{\mu}_{\varpi}, \vartheta_1(\varpi)) - \aleph_1(\varpi, 0, 0, 0)\| + \|\aleph_1(\varpi, 0, 0, 0)\|) d\varpi \\ &\leq \|\phi_1\|_{\Omega_r} + \|d_1\|_{\infty} \|\phi_1\|_{\Omega_r} + \wp_1^* + \frac{[\psi(T) - \psi(0)]^{\zeta_1}}{\Gamma(\zeta_1 + 1)} \\ &\quad \times \left(\kappa_{c_1}^{a_1} \|\mu\|_{\Omega} + \eta_{c_1}^{b_1} \|\bar{\mu}\|_{\Omega} + \aleph_1^* \right) \\ &\leq (1 + \|d_1\|_{\infty}) \|\phi_1\|_{\Omega_r} + \wp_1^* + L_{\zeta_1}^{\psi} \aleph_1^* + \kappa_{c_1}^{a_1} L_{\zeta_1}^{\psi} \|\mu\|_{\Omega} + \eta_{c_1}^{b_1} L_{\zeta_1}^{\psi} \|\bar{\mu}\|_{\Omega}.\end{aligned}$$

Hence, we get

$$\|\mathcal{P}_1(\mu, \bar{\mu})\|_{\Omega} \leq (1 + \|d_1\|_{\infty}) \|\phi_1\|_{\Omega_r} + \wp_1^* + L_{\zeta_1}^{\psi} \aleph_1^* + \kappa_{c_1}^{a_1} L_{\zeta_1}^{\psi} \|\mu\|_{\Omega} + \eta_{c_1}^{b_1} L_{\zeta_1}^{\psi} \|\bar{\mu}\|_{\Omega}.$$

Similarly, we get

$$\|\mathcal{P}_2(\mu, \bar{\mu})\|_{\Omega} \leq (1 + \|d_2\|_{\infty}) \|\phi_2\|_{\Omega_r} + \wp_2^* + L_{\zeta_2}^{\psi} \aleph_2^* + \kappa_{c_2}^{a_2} L_{\zeta_2}^{\psi} \|\mu\|_{\Omega} + \eta_{c_2}^{b_2} L_{\zeta_2}^{\psi} \|\bar{\mu}\|_{\Omega}.$$

Thus, we have

$$\begin{aligned} \|\mathcal{P}(\mu, \bar{\mu})\|_{\Upsilon} &\leq \begin{pmatrix} \|\mathcal{P}_1(\mu, \bar{\mu})\|_{\Omega} \\ \|\mathcal{P}_2(\mu, \bar{\mu})\|_{\Omega} \end{pmatrix} \\ &\leq B \begin{pmatrix} \|\mu\|_{\Omega} \\ \|\bar{\mu}\|_{\Omega} \end{pmatrix} + \begin{pmatrix} (1 + \|d_1\|_{\infty})\|\phi_1\|_{\Omega_r} + \wp_1^* + L_{\zeta_1}^{\psi} \aleph_1^* \\ (1 + \|d_2\|_{\infty})\|\phi_2\|_{\Omega_r} + \wp_2^* + L_{\zeta_1}^{\psi} \aleph_2^* \end{pmatrix}. \end{aligned} \quad (26)$$

where

$$B = \begin{pmatrix} \kappa_{c_1}^{a_1} L_{\zeta_1}^{\psi} & \eta_{c_1}^{b_1} L_{\zeta_1}^{\psi} \\ \kappa_{c_2}^{a_2} L_{\zeta_2}^{\psi} & \eta_{c_2}^{b_2} L_{\zeta_2}^{\psi} \end{pmatrix}.$$

Combining (24) and (26), it follows that

$$\begin{aligned} \|\mathcal{K}(\mu, \bar{\mu})\|_{\Upsilon} + \|\mathcal{P}(\bar{\mu}, \bar{\bar{\mu}})\|_{\Upsilon} &\leq A \begin{pmatrix} \|\mu\|_{\Omega} \\ \|\bar{\mu}\|_{\Omega} \end{pmatrix} + B \begin{pmatrix} \|\bar{\mu}\|_{\Omega} \\ \|\bar{\bar{\mu}}\|_{\Omega} \end{pmatrix} \\ &\quad + \begin{pmatrix} (2 + \|d_1\|_{\infty})\|\phi_1\|_{\Omega_r} + 2\wp_1^* + L_{\zeta_1}^{\psi} \aleph_1^* \\ (2 + \|d_2\|_{\infty})\|\phi_2\|_{\Omega_r} + 2\wp_2^* + L_{\zeta_1}^{\psi} \aleph_2^* \end{pmatrix}. \end{aligned} \quad (27)$$

Now we look for $\chi = (\chi_1, \chi_2) \in \mathbb{R}_+^2$ such that $\mathcal{K}(\mu, \bar{\mu}) + \mathcal{P}(\bar{\mu}, \bar{\bar{\mu}}) \in \Delta_{\chi}$ for any $(\mu, \bar{\mu}), (\bar{\mu}, \bar{\bar{\mu}}) \in \Delta_{\chi}$. To this end, according to (27), it is sufficient to show

$$M \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \leq \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

where M is the matrix given by (10) and

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} (2 + \|d_1\|_{\infty})\|\phi_1\|_{\Omega_r} + 2\wp_1^* + L_{\zeta_1}^{\psi} \aleph_1^* \\ (2 + \|d_2\|_{\infty})\|\phi_2\|_{\Omega_r} + 2\wp_2^* + L_{\zeta_1}^{\psi} \aleph_2^* \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \leq (I - M) \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \quad (28)$$

Since the matrix M converges to zero. It yields, from Theorem 1 that the matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements. Therefore, (28) is equivalent to

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Then we obtain

$$\begin{cases} \chi_1 \geq \sigma_1 \lambda_1 + \sigma_2 \lambda_2, \\ \chi_1 \geq \sigma_3 \lambda_1 + \sigma_4 \lambda_2, \end{cases}$$

where

$$(I - M)^{-1} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix},$$

which means that $\mathcal{K}(\mu, \bar{\mu}) + \mathcal{P}(\bar{\mu}, \bar{\bar{\mu}}) \in \Delta_\chi$.

Step 2. $\mathcal{P}$ is compact and continuous.

Claim 1. $\mathcal{P}$ is compact.

Firstly, we prove that $\mathcal{P}$ is uniformly bounded on Δ_χ . From (24), and for each $(\mu, \bar{\mu}) \in \Delta_\chi$ we can get $\|\mathcal{P}(\mu, \bar{\mu})\|_\Upsilon < \infty$. This proves that $\mathcal{P}$ is uniformly bounded.

Next, Let $\delta_1, \delta_2 \in \mathfrak{J}$, be such that $\delta_1 < \delta_2$ and let $(\mu, \bar{\mu}) \in \Delta_\chi$. From (H_2) and (15), we get

$$\begin{aligned} & \|\mathcal{P}_1(\mu, \bar{\mu})(\delta_2) - \mathcal{P}_1(\mu, \bar{\mu})(\delta_1)\| \\ & \leq \int_0^{\delta_1} \left\| \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_2, \varpi) - \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_1, \varpi) \right\| \psi'(\varpi) \|\aleph_1(\varpi, \mu_\varpi, \bar{\mu}_\varpi, \vartheta_1(\varpi))\| d\varpi \\ & + \int_{\delta_1}^{\delta_2} \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_2, \varpi) \psi'(\varpi) \|\aleph_1(\varpi, \mu_\varpi, \bar{\mu}_\varpi, \vartheta_1(\varpi))\| d\varpi \\ & \leq \left(\kappa_{c_1}^{a_1} \|\mu\|_\Omega + \eta_{c_1}^{b_1} \|\bar{\mu}\|_\Omega + \aleph_1^* \right) \int_0^{\delta_1} \left\| \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_2, \varpi) - \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_1, \varpi) \right\| \psi'(\varpi) d\varpi \\ & + \frac{\kappa_{c_1}^{a_1} \|\mu\|_\Omega + \eta_{c_1}^{b_1} \|\bar{\mu}\|_\Omega + \aleph_1^*}{\Gamma(\zeta_1 + 1)} [\psi(\delta_2) - \psi(\delta_1)]^{\zeta_1} \\ & \leq \left(\kappa_{c_1}^{a_1} \chi_1 + \eta_{c_1}^{b_1} \chi_2 + \aleph_1^* \right) \left[\int_0^{\delta_1} \left\| \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_2, \varpi) - \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_1, \varpi) \right\| \psi'(\varpi) d\varpi \right. \\ & \left. + \frac{[\psi(\delta_2) - \psi(\delta_1)]^{\zeta_1}}{\Gamma(\zeta_1 + 1)} \right]. \end{aligned}$$

Similarly

$$\begin{aligned} & \|\mathcal{P}_2(\mu, \bar{\mu})(\delta_2) - \mathcal{P}_2(\mu, \bar{\mu})(\delta_1)\| \\ & \leq \left(\kappa_{c_2}^{a_2} \chi_1 + \eta_{c_2}^{b_2} \chi_2 + \aleph_2^* \right) \left[\int_0^{\delta_1} \left\| \Psi_{\zeta_2}^{\omega_2; \psi}(\delta_2, \varpi) - \Psi_{\zeta_2}^{\omega_2; \psi}(\delta_1, \varpi) \right\| \psi'(\varpi) d\varpi \right. \\ & \left. + \frac{[\psi(\delta_2) - \psi(\delta_1)]^{\zeta_2}}{\Gamma(\zeta_2 + 1)} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{P}(\mu, \bar{\mu})(\delta_2) - \mathcal{P}(\mu, \bar{\mu})(\delta_1)\| &= \left(\begin{aligned} &\|\mathcal{P}_1(\mu, \bar{\mu})(\delta_2) - \mathcal{P}_1(\mu, \bar{\mu})(\delta_1)\| \\ &\|\mathcal{P}_2(\mu, \bar{\mu})(\delta_2) - \mathcal{P}_2(\mu, \bar{\mu})(\delta_1)\| \end{aligned} \right) \\ &\leq \left(\begin{aligned} &\left(\kappa_{c_1}^{a_1} \chi_1 + \eta_{c_1}^{b_1} \chi_2 + \aleph_1^* \right) \left[\int_0^{\delta_1} \left\| \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_2, \varpi) - \Psi_{\zeta_1}^{\omega_1; \psi}(\delta_1, \varpi) \right\| \psi'(\varpi) d\varpi \right. \\ &\quad \left. + \frac{[\psi(\delta_2) - \psi(\delta_1)]^{\zeta_1}}{\Gamma(\zeta_1 + 1)} \right] \\ &\left(\kappa_{c_2}^{a_2} \chi_1 + \eta_{c_2}^{b_2} \chi_2 + \aleph_2^* \right) \left[\int_0^{\delta_1} \left\| \Psi_{\zeta_2}^{\omega_2; \psi}(\delta_2, \varpi) - \Psi_{\zeta_2}^{\omega_2; \psi}(\delta_1, \varpi) \right\| \psi'(\varpi) d\varpi \right. \\ &\quad \left. + \frac{[\psi(\delta_2) - \psi(\delta_1)]^{\zeta_2}}{\Gamma(\zeta_2 + 1)} \right] \end{aligned} \right) \\ &\longrightarrow 0, \quad \text{as } \delta_1 \rightarrow \delta_2. \end{aligned}$$

Consequently, $\mathcal{P}(\Delta_\chi)$ is equicontinuous. By Arzèla-Ascoli theorem, we conclude that $\mathcal{P}$ is compact.

Claim 1. $\mathcal{P}$ is continuous.

Let $(\mu^n, \bar{\mu}^n)$ be a sequence such that $(\mu^n, \bar{\mu}^n) \rightarrow (\mu, \bar{\mu})$ in Δ_χ . Then, we have

$$\|\mathcal{P}_1(\mu^n, \bar{\mu}^n)(\delta) - \mathcal{P}_1(\mu, \bar{\mu})(\delta)\| \leq \int_0^\delta \Psi_{\zeta_1}^{\omega_1; \psi}(\delta, \varpi) \psi'(\varpi) \|\vartheta_1^n(\varpi) - \vartheta_1(\delta)\| d\varpi,$$

where $\vartheta_1(\cdot)$, $\vartheta_1^n(\cdot)$ satisfy the following functional equations

$$\vartheta_1(\delta) = \aleph_1(\delta, \mu_\delta, \bar{\mu}_\delta, \vartheta_1(\delta)) \quad \text{and} \quad \vartheta_1^n(\delta) = \aleph_1(\delta, \mu_\delta^n, \bar{\mu}_\delta^n, \vartheta_1^n(\delta)).$$

From (15), we have

$$\begin{aligned} \|\mathcal{P}_1(\mu^n, \bar{\mu}^n)(\delta) - \mathcal{P}_1(\mu, \bar{\mu})(\delta)\| &\leq \int_0^\delta \frac{e^{-\omega[\psi(\delta) - \psi(\varpi)]} [\psi(\delta) - \psi(\varpi)]^{\zeta_1 - 1}}{\Gamma(\zeta_1)} \psi'(\varpi) \\ &\quad \times \left(\kappa_{c_1}^{a_1} \|\mu^n - \mu\|_\Omega + \eta_{c_1}^{b_1} \|\bar{\mu}^n - \bar{\mu}\|_\Omega \right) d\varpi. \end{aligned}$$

By the Lebesgue dominated convergence theorem, we get

$$\|\mathcal{P}_1(\mu^n, \bar{\mu}^n) - \mathcal{P}_1(\mu, \bar{\mu})\|_\Omega \longrightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Similarly, we can get

$$\|\mathcal{P}_2(\mu^n, \bar{\mu}^n) - \mathcal{P}_2(\mu, \bar{\mu})\|_\Omega \longrightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Hence, $\mathcal{P}(\Delta_\chi)$ is continuous.

Step 3. $\mathcal{K}$ is A -contraction mapping on Δ_χ .

Let $(\mu, \bar{\mu}), (\bar{\mu}, \bar{\bar{\mu}}) \in \Delta_\chi$, for each $\delta \in \mathfrak{J}$. By Theorem 4, we have

$$\|\mathcal{N}(\mu, \bar{\mu}) - \mathcal{K}(\bar{\mu}, \bar{\bar{\mu}})\|_{\Upsilon} \leq A\|(\mu, \bar{\mu}) - (\bar{\mu}, \bar{\bar{\mu}})\|_{\Upsilon}.$$

Since the matrix A converges to zero and thus, $\mathcal{K}$ is A -contraction mapping on Δ_χ .

Hence, all the assumptions of Theorem 3 are fulfilled. Consequently, by applying Krasnoselskii's fixed point theorem, we deduce that the operator $\mathcal{N} = \mathcal{K} + \mathcal{P}$ defined in (21) has at least one fixed point $(\mu, \bar{\mu}) \in \Delta_\chi$, which corresponds to the solution of the coupled system (1)-(2). $\square$

3.3 Stability results

In this section, we study the Ulam-Hyers stability of the coupled system (1)-(2), by means of integral representation of its solution given by $\mu(\delta) = \mathcal{N}_1(\mu, \bar{\mu})(\delta)$, $\bar{\mu}(\delta) = \mathcal{N}_2(\mu, \bar{\mu})(\delta)$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are defined by (11) and (12).

For each $\delta \in \mathfrak{J}$, we define the following nonlinear operators $\mathcal{H}_j : \Upsilon \rightarrow \Omega$; $j = 1, 2$ by:

$$\begin{cases} {}^C_0\mathfrak{D}_\delta^{\zeta_1, \omega_1; \psi} (\tilde{\mu}(\delta) - \wp_1(\delta, \tilde{\mu}_\delta)) - \aleph_1 \left(\delta, \tilde{\mu}_\delta, \tilde{\bar{\mu}}_\delta, {}^C_0\mathfrak{D}_\delta^{\zeta_1, \omega_1; \psi} (\tilde{\mu}(\delta) - \wp_1(\delta, \tilde{\mu}_\delta)) \right) \\ = \mathcal{H}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta), \\ {}^C_0\mathfrak{D}_\delta^{\zeta_2, \omega_2; \psi} (\tilde{\bar{\mu}}(\delta) - \wp_2(\delta, \tilde{\bar{\mu}}_\delta)) - \aleph_2 \left(\delta, \tilde{\mu}_\delta, \tilde{\bar{\mu}}_\delta, {}^C_0\mathfrak{D}_\delta^{\zeta_2, \omega_2; \psi} (\tilde{\bar{\mu}}(\delta) - \wp_2(\delta, \tilde{\bar{\mu}}_\delta)) \right) \\ = \mathcal{H}_2(\tilde{\mu}, \tilde{\bar{\mu}})(\delta), \end{cases} \quad (29)$$

For some $\epsilon_1, \epsilon_2 > 0$, we consider the following inequality:

$$\begin{cases} \|\mathcal{H}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta)\| \leq \epsilon_1, \\ \|\mathcal{H}_2(\tilde{\mu}, \tilde{\bar{\mu}})(\delta)\| \leq \epsilon_2, \end{cases} \quad \delta \in \mathfrak{J}. \quad (30)$$

Definition 7. [39] the coupled system (1)-(2) is Ulam-Hyers stable if there exist positive constants ϱ_j ; $j = \overline{1, 4}$ such that for each $\epsilon_1, \epsilon_2 > 0$ and for each solution $(\tilde{\mu}, \tilde{\bar{\mu}}) \in \Upsilon$ of inequality (30), there exists a solution $(\mu, \bar{\mu}) \in \Upsilon$ of (1)-(2) with

$$\begin{cases} \|\tilde{\mu}(\delta) - \mu(\delta)\| \leq \varrho_1\epsilon_1 + \varrho_2\epsilon_2, \\ \|\tilde{\bar{\mu}}(\delta) - \bar{\mu}(\delta)\| \leq \varrho_3\epsilon_1 + \varrho_4\epsilon_2, \end{cases} \quad \delta \in \mathfrak{J}. \quad (31)$$

Theorem 6. *Let the assumptions of Theorem 4 hold. Then the coupled system (1)–(2) is Ulam-Hyers stable.*

Proof. Let $(\mu, \bar{\mu}) \in \Upsilon$ be the solution of the coupled system (1)–(2) satisfying (11) and (12). Let $(\tilde{\mu}, \tilde{\bar{\mu}}) \in \Upsilon$ be any solution satisfying (30):

From (29), we get

$$\tilde{\mu}(\delta) - \mathcal{N}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta) = \int_0^\delta \Psi_{\zeta_1}^{\omega_1; \psi}(\delta, \varpi) \psi'(\varpi) \mathcal{H}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\varpi) d\varpi, \quad (32)$$

and

$$\tilde{\bar{\mu}}(\delta) - \mathcal{N}_2(\tilde{\mu}, \tilde{\bar{\mu}})(\delta) = \int_0^\delta \Psi_{\zeta_2}^{\omega_2; \psi}(\delta, \varpi) \psi'(\varpi) \mathcal{H}_2(\tilde{\mu}, \tilde{\bar{\mu}})(\varpi) d\varpi. \quad (33)$$

From (32) and (33), we have

$$\begin{aligned} & \|\tilde{\mu}(\delta) - \mathcal{N}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta)\| \\ & \leq \int_0^\delta \frac{e^{-\omega_1[\psi(\delta) - \psi(\varpi)]} [\psi(\delta) - \psi(\varpi)]^{\zeta_1 - 1}}{\Gamma(\zeta_1)} \psi'(\varpi) \|\mathcal{H}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\varpi)\| d\varpi \\ & \leq L_{\zeta_1}^\psi \epsilon_1, \end{aligned} \quad (34)$$

and

$$\begin{aligned} & \|\tilde{\bar{\mu}}(\delta) - \mathcal{N}_2(\tilde{\mu}, \tilde{\bar{\mu}})(\delta)\| \\ & \leq \int_0^\delta \frac{e^{-\omega_2[\psi(\delta) - \psi(\varpi)]} [\psi(\delta) - \psi(\varpi)]^{\zeta_2 - 1}}{\Gamma(\zeta_2)} \psi'(\varpi) \|\mathcal{H}_2(\tilde{\mu}, \tilde{\bar{\mu}})(\varpi)\| d\varpi \\ & \leq L_{\zeta_2}^\psi \epsilon_2. \end{aligned} \quad (35)$$

Thus, by (H_1) , (H_2) and inequalities (34), (35), we get

$$\begin{aligned} \|\tilde{\mu}(\delta) - \mu(\delta)\| &= \|\tilde{\mu}(\delta) - \mathcal{N}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta) + \mathcal{N}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta) - \mu(\delta)\| \\ &\leq \|\tilde{\mu}(\delta) - \mathcal{N}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta)\| + \|\mathcal{N}_1(\tilde{\mu}, \tilde{\bar{\mu}})(\delta) - \mathcal{N}_1(\mu, \bar{\mu})(\delta)\| \\ &\leq L_{\zeta_1}^\psi \epsilon_1 + \|d_1\|_\infty + \kappa_{c_1}^{a_1} L_{\zeta_1}^\psi \|\mu - \tilde{\mu}\|_\Omega + \eta_{c_1}^{b_1} L_{\zeta_1}^\psi \|\bar{\mu} - \tilde{\bar{\mu}}\|_\Omega. \end{aligned}$$

Hence we get

$$\|\tilde{\mu} - \mu\|_\Omega \leq L_{\zeta_1}^\psi \epsilon_1 + (\|d_1\|_\infty + \kappa_{c_1}^{a_1} L_{\zeta_1}^\psi) \|\mu - \tilde{\mu}\|_\Omega + \eta_{c_1}^{b_1} L_{\zeta_1}^\psi \|\bar{\mu} - \tilde{\bar{\mu}}\|_\Omega. \quad (36)$$

Similarly, we have

$$\|\tilde{\bar{\mu}} - \bar{\mu}\|_\Omega \leq L_{\zeta_2}^\psi \epsilon_2 + \kappa_{c_2}^{a_2} L_{\zeta_2}^\psi \|\mu - \tilde{\mu}\|_\Omega + (\|d_2\|_\infty + \eta_{c_2}^{b_2} L_{\zeta_2}^\psi) \|\bar{\mu} - \tilde{\bar{\mu}}\|_\Omega. \quad (37)$$

Inequalities (36) and (37) can be rewritten in matrix form as

$$(I - M) \begin{pmatrix} \|\tilde{\mu} - \mu\|_{\Omega} \\ \|\tilde{\bar{\mu}} - \bar{\mu}\|_{\Omega} \end{pmatrix} \leq \begin{pmatrix} L_{\zeta_1}^{\psi} \epsilon_1 \\ L_{\zeta_2}^{\psi} \epsilon_2 \end{pmatrix}. \quad (38)$$

where M is given by (28). From Theorem 1, we deduce that $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements. Therefore, (38) is equivalent to

$$\begin{pmatrix} \|\tilde{\mu} - \mu\|_{\Omega} \\ \|\tilde{\bar{\mu}} - \bar{\mu}\|_{\Omega} \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} L_{\zeta_1}^{\psi} \epsilon_1 \\ L_{\zeta_2}^{\psi} \epsilon_2 \end{pmatrix}. \quad (39)$$

Thus

$$\begin{cases} \|\tilde{\mu} - \mu\| \leq \sigma_1 L_{\zeta_1}^{\psi} \epsilon_1 + \sigma_2 L_{\zeta_1}^{\psi} \epsilon_2, \\ \|\tilde{\bar{\mu}} - \bar{\mu}\| \leq \sigma_3 L_{\zeta_2}^{\psi} \epsilon_1 + \sigma_4 L_{\zeta_2}^{\psi} \epsilon_2, \end{cases} \quad \delta \in \mathfrak{J}, \quad (40)$$

where σ_j ; $j = \overline{1, 4}$ are the elements of $(I - M)^{-1}$. Thus, (1)-(2) is Ulam-Hyers stable. $\square$

4 Examples

Example 2. Consider the following coupled system of implicit tempered ψ -Caputo fractional equations

$$\begin{cases} {}^C_0 \mathfrak{D}_{\delta}^{\frac{1}{2}, 2; \psi} (\mu(\delta) - \wp_1(\delta, \mu_{\delta})) = \aleph_1 \left(\delta, \mu_{\delta}, \bar{\mu}_{\delta}, {}^C_0 \mathfrak{D}_{\delta}^{\frac{1}{2}, 2; \psi} (\mu(\delta) - \wp_1(\delta, \mu_{\delta})) \right), \\ {}^C_0 \mathfrak{D}_{\delta}^{\frac{1}{2}, 3; \psi} (\bar{\mu}(\delta) - \wp_2(\delta, \bar{\mu}_{\delta})) = \aleph_2 \left(\delta, \mu_{\delta}, \bar{\mu}_{\delta}, {}^C_0 \mathfrak{D}_{\delta}^{\frac{1}{2}, 3; \psi} (\bar{\mu}(\delta) - \wp_2(\delta, \bar{\mu}_{\delta})) \right), \end{cases} \quad (41)$$

for $\delta \in \mathfrak{J} := [0, 1]$, with the initial conditions

$$\begin{cases} \mu(\delta) = \phi(\delta) = (\phi_1(\delta), \phi_2(\delta)), \\ \bar{\mu}(\delta) = \xi(\delta) = (\xi_1(\delta), \xi_2(\delta)), \end{cases} \quad \delta \in [-5, 0], \quad (42)$$

where $\psi(\delta) = \delta^2$, $r = 5$, $\aleph_1, \aleph_2 : \mathfrak{J} \times \Omega_r \times \Omega_r \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\mu = (\mu_1, \mu_2)$, $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2)$, $\vartheta = (\vartheta_1, \vartheta_2)$, $\vartheta^* = (\vartheta_1^*, \vartheta_2^*)$ with

$$\begin{aligned} \aleph_1(\delta, \mu_{\delta}, \bar{\mu}_{\delta}, \vartheta(\delta)) &= \frac{e^{-\delta-3}}{\delta + e^{\sqrt{\delta}}} \left(\frac{|\mu_{1,\delta}| + |\mu_{2,\delta}|}{\frac{\sin(|\vartheta_1(\delta)| + |\vartheta_2(\delta)|)}{\delta+5}} \right), \\ \aleph_2(\delta, \mu_{\delta}, \bar{\mu}_{\delta}, \vartheta^*(\delta)) &= \frac{1}{(\delta + 2)^2} \left(\frac{\ln(1 + |\vartheta_1^*(\delta)| + |\vartheta_2^*(\delta)|)}{|\bar{\mu}_{1,\delta}| + |\bar{\mu}_{2,\delta}|} \right), \end{aligned}$$

where $\vartheta(\cdot)$, $\vartheta^*(\cdot) \in \mathbb{R}^m$ satisfy the following functional equations

$$\vartheta(\delta) = \aleph_1(\delta, \mu_\delta, \bar{\mu}_\delta, \vartheta(\delta)) \quad \text{and} \quad \vartheta^*(\delta) = \aleph_2(\delta, \mu_\delta^n, \bar{\mu}_\delta^n, \vartheta^*(\delta)),$$

and $\wp_1, \wp_2 : \mathfrak{J} \times \Omega_r \times \rightarrow \mathbb{R}^2$ with

$$\wp_1(\delta, \mu_\delta) = \frac{e^{-\delta-3}}{\delta + e^{\sqrt{\delta}}} \left(\frac{1 + |\mu_{1,\delta}|}{|\mu_{2,\delta}| - e^{-2}} \right),$$

$$\wp_2(\delta, \mu_\delta) = e^{\delta-5} \left(\frac{\ln(1 + |\mu_{1,\delta}|)}{\sin(|\mu_{2,\delta}|)} \right).$$

Clearly, the functions $\aleph_j, \wp_j; j = 1, 2$ are continuous. Moreover, for any $\mu, \bar{\mu}, \bar{\mu}, \bar{\mu} \in \Omega_r, \vartheta, \bar{\vartheta}, \vartheta^*, \bar{\vartheta}^* \in \mathbb{R}^2$ and $\delta \in \mathfrak{J}$ we have

$$\begin{aligned} & \|\aleph_1(\delta, \mu, \bar{\mu}, \vartheta) - \aleph_1(\delta, \bar{\mu}, \bar{\mu}, \bar{\vartheta})\|_1 \\ & \leq a_1(\delta) \|\mu - \bar{\mu}\|_{\Omega_r} + b_1(\delta) \|\bar{\mu} - \bar{\mu}\|_{\Omega_r} + c_1(\delta) \|\vartheta(\delta) - \bar{\vartheta}(\delta)\|_1, \\ & \|\aleph_2(\delta, \mu, \bar{\mu}, \vartheta^*) - \aleph_2(\delta, \bar{\mu}, \bar{\mu}, \bar{\vartheta}^*)\|_1 \\ & \leq a_2(\delta) \|\mu - \bar{\mu}\|_{\Omega_r} + b_2(\delta) \|\bar{\mu} - \bar{\mu}\|_{\Omega_r} + c_2(\delta) \|\vartheta^*(\delta) - \bar{\vartheta}^*(\delta)\|_1, \end{aligned}$$

and

$$\|\wp_j(\delta, \mu) - \wp_j(\delta, \bar{\mu})\|_1 \leq d_j(\delta) \|\mu - \bar{\mu}\|_{\Omega_r}, \quad j = 1, 2.$$

where $\|\cdot\|_1$ is a norm in $\mathbb{R}^2$ defined as follows

$$\|\mu\|_1 = |\mu_1| + |\mu_2|, \quad \mu = (\mu_1, \mu_2).$$

The hypothesis (H_1) is satisfied with

$$a_1(\delta) = \frac{e^{-2\delta-4}}{\delta + e^{\sqrt{\delta}}}, \quad b_1(\delta) = 0, \quad c_1(\delta) = \frac{e^{-\delta-3}}{(\delta + 5)(\delta + e^{\sqrt{\delta}})},$$

and

$$a_2(\delta) = 0, \quad b_2(\delta) = c_2(\delta) = \frac{1}{(\delta + 2)^2}.$$

The hypothesis (H_2) is satisfied with

$$d_1(\delta) = \frac{e^{-\delta-3}}{\delta + e^{\sqrt{\delta}}}, \quad \text{and} \quad d_2(\delta) = e^{\delta-5}.$$

Furthermore,

$$M = \begin{pmatrix} 0,069 & 0 \\ 0 & 0,39 \end{pmatrix},$$

and converges to 0. Hence Theorem 4 implies that the couple system (41)-(42) has a unique solution and is Ulam-Hyers stable.

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