

SOME MORE RESULTS ON $\mathcal{I}$ -CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES*

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Abstract

In this paper, following a very recent and new approach, we further generalize recently introduced $\mathcal{I}$ -convergence of complex uncertain sequences and study the relationship among the existing convergence concept of complex uncertain sequences. We also introduced the notion of $\mathcal{I}$ -convergence in p -distance, completely $\mathcal{I}$ -convergence, and $\mathcal{I}$ -convergence in metric of complex uncertain sequences. Overall this study, mainly presents a complete scenario of interrelationships among all $\mathcal{I}$ -convergence concepts of complex uncertain sequences defined till now and include some observations about the above convergence concept.

Keywords: uncertainty theory, $\mathcal{I}$ -convergence, complex uncertain variable.

MSC: 60B10, 40A35, 40G15.

1 Introduction

The idea of statistical convergence, which is an extension of the usual convergence, was introduced by Fast [12] and Steinhaus [13], individually in

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the year 1951, and since then several generalizations and applications of this notion have been investigated by prominent authors. In particular, one interesting generalization of its, namely $\mathcal{I}$ -convergence was introduced by Kostyrko et al. [21]. Thereafter, lots of developments have occurred in this area like [1, 5, 7, 9–11, 15, 18], and many more.

In the real world, there are different kinds of uncertainty. So, it makes perfect sense to investigate the behavior of uncertain phenomena. To address some aspects of this uncertain phenomenon, Liu [6] introduced initially a theory namely uncertainty theory in the year 2007. In this theory, there are four convergence concepts of uncertain variable sequences namely, convergence almost surely, convergence in measure, convergence in mean, and convergence in distribution were defined by Liu. Then You [8] defined convergence uniformly almost surely which is a new type of convergence concept and slightly different from convergence almost surely. Liu and You mainly examine the relationships of the convergence concept of uncertain sequence with construction of some interesting examples. Thereafter, Peng [29] explored the idea of complex uncertain variables and Chen et al. [27] extended this work to the convergence of complex uncertain sequences.

In an uncertainty theory, the notion of statistical convergence of complex uncertain sequences was introduced by Tripathy and Nath [3]. Then many other researchers like [4, 16, 17, 19, 20, 22, 23, 25, 26] have successfully applied the concept of generalized convergence of sequences on uncertainty theory. Recently Halder et al. [2] introduced the notion of $\mathcal{I}$ -convergence of complex uncertain sequences.

In this article, we have extended the notion of $\mathcal{I}$ -convergence of complex uncertain sequence with some new definitions and results. We defined the concept of $\mathcal{I}$ -convergence in p -distance, completely $\mathcal{I}$ -convergence, and $\mathcal{I}$ -convergence in metric of complex uncertain sequences and study the inter-relationships among all the convergence concepts defined till now in complex uncertainty theory. It has been observed that if (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ , then it is $\mathcal{I}$ -convergent almost surely to ζ but the converse part is not true in general. In this article, we show that the converse part is true when the complex uncertain variables are defined on the same continuous uncertainty space.

2 Definitions and preliminaries

In this section, we provide some basic definitions and results on generalized convergence concepts and the theory of uncertainty which will be used

throughout the article.

Definition 1. [21] Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if

(i) for each $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$;

(ii) for each $A \in \mathcal{I}$ and $B \subset A \implies B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $X \notin \mathcal{I}$.

A non-trivial ideal $\mathcal{I}$ is called an admissible ideal in X if and only if $\{\{x\} : x \in X\} \subset \mathcal{I}$.

Example 1. (i) $\mathcal{I}_f :=$ The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.

(ii) $\mathcal{I}_d :=$ The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.

Definition 2. [14] A sequence (x_n) is said to be statistically convergent to ℓ provided that for each $\varepsilon > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0, \quad n \in \mathbb{N}.$$

Definition 3. [21] A sequence (x_n) is said to be $\mathcal{I}$ -convergent to ℓ , if for every $\varepsilon > 0$, such that $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in \mathcal{I}$.

The usual convergence of sequences is a special case of $\mathcal{I}$ -convergence ($\mathcal{I} = \mathcal{I}_f$ - the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$ -convergence. In this case, $\mathcal{I} = \mathcal{I}_d = \{A \subseteq \mathbb{N} : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n} = 0\}$, where $|A|$ is the cardinality of the set A .

Definition 4. [6] Let L be a σ -algebra on a nonempty set Γ . A set function M on Γ is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (Normality): $M\{\Gamma\} = 1$;

Axiom 2 (Duality): $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in L$;

Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in L$,

$$M\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} M\{\Lambda_j\}.$$

The triplet (Γ, L, M) is called an uncertainty space, and each element Λ in L is called an event. To obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu as:

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M\{\Lambda_k\}.$$

Definition 5. [29] A variable $\zeta = \xi + i\eta$ from an uncertainty space (Γ, L, M) to the set of complex numbers is a complex uncertain variable if and only if ξ and η are uncertain variables, where ξ and η are the real and imaginary parts of ζ , respectively.

Definition 6. [28] An uncertain measure M is called continuous if for any sequence of events Λ_k with $k \rightarrow \infty$, we have

$$M \left\{ \lim_{k \rightarrow \infty} \Lambda_k \right\} = \lim_{k \rightarrow \infty} M \{ \Lambda_k \}.$$

Definition 7. [29] Let $\zeta = \xi + i\eta$ be a complex uncertain variable, where ξ and η are real and imaginary part of ζ , respectively. Then the complex uncertainty distribution of ζ is a function from $\mathbb{C}$ to $[0, 1]$ defined by $\Phi(z) = M\{\xi \leq x, \eta \leq y\}$ for any complex number $z = x + iy$.

Definition 8. [29] Let $\zeta = \xi + i\eta$ be a complex uncertain variable. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exists, then the expected value of ζ is defined by

$$E[\zeta] = E[\xi] + iE[\eta].$$

Definition 9. [24] Let ζ and ζ^* be two complex uncertain variables. Then the p -distance between them is defined as

$$d_p(\zeta, \zeta^*) = (E[\|\zeta - \zeta^*\|^p])^{\frac{1}{p+1}}, p > 0.$$

Definition 10. [4] Let ζ and ζ^* be two complex uncertain variables, then the metric between them is defined as follows

$$D(\zeta, \zeta^*) = \inf \{r : M\{\|\zeta - \zeta^*\| \leq r\} = 1\}.$$

Definition 11. [4] A complex uncertain sequence (ζ_n) is said to be convergent in metric to ζ if

$$\lim_{n \rightarrow \infty} D(\zeta_n, \zeta) = 0.$$

Definition 12. [2] A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$ -convergent almost surely to ζ if for every $\varepsilon > 0$, there exists an event Λ with $M\{\Lambda\} = 1$ such that

$$\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \in \mathcal{I}, \text{ for every } \gamma \in \Lambda.$$

Symbolically we write $\zeta_n \xrightarrow{A_s(\mathcal{I})} \zeta$.

Definition 13. [2] A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$ -convergent in measure to ζ if for every $\varepsilon, \delta > 0$ such that

$$\{n \in \mathbb{N} : M(\|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta) \geq \varepsilon\} \in \mathcal{I}.$$

Definition 14. [2] A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$ -convergent in mean to ζ if for every $\varepsilon > 0$ such that

$$\{n \in \mathbb{N} : E[\|\zeta_n(\gamma) - \zeta(\gamma)\|] \geq \varepsilon\} \in \mathcal{I}.$$

Definition 15. [2] Let $\Phi, \Phi_1, \Phi_2, \dots$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, \dots$, respectively. Then the complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$ -convergent in distribution to ζ if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \|\Phi_n(z) - \Phi(z)\| \geq \varepsilon\} \in \mathcal{I}$$

for all z at which $\Phi(z)$ is continuous.

Definition 16. [2] A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$ -convergent uniformly almost surely to ζ if and only if for every $\varepsilon > 0, \delta > 0$ such that

$$\left\{n \in \mathbb{N} : M\left(\bigcup_{m=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_m(\gamma) - \zeta(\gamma)\| \geq \delta\}\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

Proposition 1. [2] Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables. Then for any $\varepsilon, \delta > 0$ (ζ_n) is $\mathcal{I}$ -convergent a.s. to ζ if and only if

$$\left\{n \in \mathbb{N} : M\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \delta\}\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

Theorem 1. [2] If a complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent in mean to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ .

But the converse is not true in general.

Theorem 2. [2] The complex uncertain sequence (ζ_n) , where $\zeta_n = \xi_n + i\eta_n$ is $\mathcal{I}$ -convergent in measure to $\zeta = \xi + i\eta$ if and only if the uncertain sequence (ξ_n) and (η_n) are $\mathcal{I}$ -convergent in measure to ξ and η , respectively.

Theorem 3. [2] Assume that a complex uncertain sequence (ζ_n) with real part (ξ_n) and imaginary part (η_n) are $\mathcal{I}$ -convergent in measure to ξ and η , respectively. Then the complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent in distribution to $\zeta = \xi + i\eta$.

Throughout the paper, we consider $\mathcal{I}$ to be a non-trivial admissible ideal of $\mathbb{N}$.

3 Main results

Theorem 4. [2] Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ , then it is $\mathcal{I}$ -convergent almost surely to ζ .

Remark 1. But the converse of the above theorem is not true in general.

Example 2. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2, \dots\}$ with power set and $M\{\Gamma\} = 1$, $M\{\phi\} = 0$ and

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1}, & \text{if } \sup_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1}, & \text{if } \sup_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

$$\text{where } \beta_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

and $\zeta \equiv 0$. Take $\mathcal{I} = \mathcal{I}_d$.

For any $\varepsilon > 0$ and there exists an event Λ with $M\{\Lambda\} = 1$, we have

$$\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} = \{n \in \mathbb{N} : \|\zeta_n(\gamma)\| \geq \varepsilon\} \in \mathcal{I}.$$

Also for every $\delta > 0$, we have

$$M\left(\bigcup_{m=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_m(\gamma) - \zeta(\gamma)\| \geq \delta\}\right) = M\left(\bigcup_{m=n}^{\infty} \{\gamma_m\}\right).$$

Then for every $\varepsilon > 0$,

$$\begin{aligned} \left\{n \in \mathbb{N} : M\left(\bigcup_{m=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_m(\gamma) - \zeta(\gamma)\| \geq \delta\}\right) \geq \varepsilon\right\} \\ = \left\{n \in \mathbb{N} : M\left(\bigcup_{m=n}^{\infty} \{\gamma_m\}\right) \geq \varepsilon\right\} \notin \mathcal{I}. \end{aligned}$$

Hence the sequence (ζ_n) is $\mathcal{I}$ -convergent almost surely to $\zeta \equiv 0$ but it is not $\mathcal{I}$ -convergent uniformly almost surely to $\zeta \equiv 0$.

Theorem 5. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on the same continuous uncertainty space. If (ζ_n) is $\mathcal{I}$ -convergent almost surely to ζ , then it is $\mathcal{I}$ -convergent uniformly almost surely to ζ .

Proof. Let the complex uncertain sequence (ζ_n) be $\mathcal{I}$ -convergent almost surely to ζ , then for every $\varepsilon > 0$, $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : M\left(\bigcap_{k=1}^{\infty} \bigcup_{k=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_k - \zeta\| \geq \delta\}\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

Let $p \in \left\{ n \in \mathbb{N} : M \left(\bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \varepsilon \right\}$, then

$$M \left(\bigcup_{m=p}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \varepsilon$$

$$\Rightarrow \lim_{p \rightarrow \infty} M \left(\bigcup_{m=p}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \lim_{p \rightarrow \infty} \varepsilon = \varepsilon$$

$$\Rightarrow M \left(\lim_{p \rightarrow \infty} \bigcup_{m=p}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \varepsilon, \text{ since uncertainty space is continuous}$$

$$\Rightarrow M \left(\bigcap_{p=1}^{\infty} \bigcup_{m=p}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \varepsilon$$

$$\Rightarrow p \in \left\{ n \in \mathbb{N} : M \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ \gamma \in \Gamma : \|\zeta_k - \zeta\| \geq \delta \} \right) \geq \varepsilon \right\}.$$

Thus $\left\{ n \in \mathbb{N} : M \left(\bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \varepsilon \right\}$

$$\subseteq \left\{ n \in \mathbb{N} : M \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ \gamma \in \Gamma : \|\zeta_k - \zeta\| \geq \delta \} \right) \geq \varepsilon \right\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ . $\square$

Theorem 6. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables. If (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ .

Proof. Let the complex uncertain sequence (ζ_n) be $\mathcal{I}$ -convergent uniformly almost surely to ζ , then for every $\varepsilon > 0, \delta > 0$, we have

$$\left\{ n \in \mathbb{N} : M \left(\bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \varepsilon \right\} \in \mathcal{I}.$$

Since, $M \{ \gamma \in \Gamma : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta \} \leq M \left(\bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right)$,

then for any $\varepsilon > 0$,

$$\begin{aligned} & \{ n \in \mathbb{N} : M \{ \gamma \in \Gamma : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta \} \geq \varepsilon \} \\ & \subseteq \left\{ n \in \mathbb{N} : M \left(\bigcup_{m=n}^{\infty} \{ \gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta \} \right) \geq \varepsilon \right\} \in \mathcal{I}. \end{aligned}$$

Thus the sequence (ζ_n) is $\mathcal{I}$ -convergent in measure to ζ . $\square$

Remark 2. But the converse of the above theorem is not true in general.

Example 3. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2, \dots\}$ with power set and $M\{\Gamma\} = 1, M\{\phi\} = 0$ and

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1}, & \text{if } \sup_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1}, & \text{if } \sup_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

$$\text{where } \beta_n = \begin{cases} 1, & \text{if } n = k^2, k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

and $\zeta \equiv 0$. Take $\mathcal{I} = \mathcal{I}_d$.

Clearly the sequence (ζ_n) is $\mathcal{I}$ -convergent in measure to $\zeta \equiv 0$ but it is not $\mathcal{I}$ -convergent uniformly almost surely to $\zeta \equiv 0$.

Theorem 7. Suppose (ζ_n) , where $\zeta_n = \xi_n + i\eta_n$ be a complex uncertain variables sequence and ζ , where $\zeta = \xi + i\eta$ be a complex uncertain variable such that $\zeta_1 \geq \zeta_2 \geq \dots \geq \zeta_n \geq \dots \geq \zeta$ in the sense that $\xi_n \geq \xi_m \geq \xi$ and $\eta_n \geq \eta_m \geq \eta$ for $m \geq n$. Then (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ if it is $\mathcal{I}$ -convergent in measure to ζ .

Proof. Let $\xi_n \geq \xi_m \geq \xi$ and $\eta_n \geq \eta_m \geq \eta$ for $m \geq n$, then

$$\|\zeta_m(\gamma) - \zeta(\gamma)\| \leq \|\zeta_n(\gamma) - \zeta(\gamma)\| \text{ for } m \geq n.$$

Now for every $\delta > 0$, we have

$$\{\gamma \in \Gamma : \|\zeta_m(\gamma) - \zeta(\gamma)\| \geq \delta\} \subseteq \{\gamma \in \Gamma : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta\}.$$

$$\text{Therefore } \bigcup_{m=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_m(\gamma) - \zeta(\gamma)\| \geq \delta\} = \{\gamma \in \Gamma : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta\}.$$

Since the complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent in measure to ζ , then for every $\varepsilon, \delta > 0$ we have

$$\begin{aligned} & \{n \in \mathbb{N} : M(\{\gamma \in \Gamma : \|\zeta_n - \zeta\| \geq \delta\}) \geq \varepsilon\} \in \mathcal{I} \\ \Rightarrow & \left\{ n \in \mathbb{N} : M\left(\bigcup_{m=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}\right) \geq \varepsilon \right\} \in \mathcal{I}. \end{aligned}$$

Hence the sequence (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ . $\square$

Theorem 8. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables. If (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ .

Proof. Let (ζ_n) be $\mathcal{I}$ -convergent uniformly almost surely to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ by the theorem 6, 2 and 3.

Remark 3. But the converse of the above theorem is not true in general.

Example 4. In example 3, the complex uncertainty distributions of (ζ_n) are

$$\Phi_n(z) = \Phi_n(x + iy) = \begin{cases} 0, & \text{if } x < 0, y < \infty \\ 0, & \text{if } x \geq 0, y < 0 \\ 1 - \frac{n\beta_n}{2^{n+1}}, & \text{if } x \geq 0, 0 \leq y < (n+1) \\ 1, & \text{if } x \geq 0, y \geq (n+1) \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

and the complex uncertainty distributions of ζ is

$$\Phi(z) = \Phi(x + iy) = \begin{cases} 0, & \text{if } x < 0, y < \infty \\ 0, & \text{if } x \geq 0, y < 0 \\ 1, & \text{if } x \geq 0, y \geq 0. \end{cases}$$

Then for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \|\Phi_n(z) - \Phi(z)\| \geq \varepsilon\} = \left\{n \in \mathbb{N} : \frac{n\beta_n}{2^{n+1}} \geq \varepsilon\right\} \in \mathcal{I}, \text{ for } x \geq 0, y \geq 0.$$

Thus the sequence (ζ_n) is $\mathcal{I}$ -convergent in distribution to $\zeta \equiv 0$ but it is not $\mathcal{I}$ -convergent uniformly almost surely to $\zeta \equiv 0$.

Theorem 9. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on the same continuous uncertainty space. If (ζ_n) is $\mathcal{I}$ -convergent almost surely to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ .

Proof. Let (ζ_n) be $\mathcal{I}$ -convergent almost surely to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ by the theorem 5 and 6. $\square$

Theorem 10. Suppose (ζ_n) , where $\zeta_n = \xi_n + i\eta_n$ is a complex uncertain variables sequence and ζ , where $\zeta = \xi + i\eta$ be a complex uncertain variable such that $\zeta_1 \geq \zeta_2 \geq \dots \geq \zeta_n \geq \dots \geq \zeta$ in the sense that $\xi_n \geq \xi_m \geq \xi$ and $\eta_n \geq \eta_m \geq \eta$ for $m \geq n$. Then (ζ_n) is $\mathcal{I}$ -convergent almost surely to ζ if it is $\mathcal{I}$ -convergent in measure to ζ .

Proof. Let (ζ_n) be $\mathcal{I}$ -convergent in measure to ζ , then it is $\mathcal{I}$ -convergent almost surely to ζ by the theorem 7 and 4. $\square$

Theorem 11. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on the same continuous uncertainty space. If (ζ_n) is $\mathcal{I}$ -convergent almost surely to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ .

Proof. Let (ζ_n) be $\mathcal{I}$ -convergent almost surely to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ by the theorem 9 and 3. $\square$

Definition 17. A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$ -convergent in p -distance to ζ if for every $\varepsilon > 0$ such that

$$\left\{n \in \mathbb{N} : (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} \geq \varepsilon\right\} \in \mathcal{I}.$$

For $\mathcal{I} = \mathcal{I}_d$, $\mathcal{I}$ -convergence in p -distance of complex uncertain sequences coincide with statistical convergence in p -distance of complex uncertain sequences which was studied by Saha et al. [25].

Theorem 12. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is $\mathcal{I}$ -convergent in p -distance to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ .

Proof. Let the complex uncertain sequence (ζ_n) be $\mathcal{I}$ -convergent in p -distance to ζ , then for every $\varepsilon > 0$, we have $\left\{n \in \mathbb{N} : (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} \geq \varepsilon\right\} \in \mathcal{I}$.

Let $k \in \{n \in \mathbb{N} : M(\|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta) \geq \varepsilon\}$, where $\delta, \varepsilon > 0$.

Then $M(\|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \delta) \geq \varepsilon$

$$\Rightarrow \varepsilon \leq M(\|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \delta) \leq \frac{E[\|\zeta_k - \zeta\|^p]}{\delta^p} \quad (\text{Using Markov Inequality})$$

$$\Rightarrow E[\|\zeta_k - \zeta\|^p] \geq \varepsilon \cdot \delta^p \Rightarrow (E[\|\zeta_k - \zeta\|^p])^{\frac{1}{p+1}} \geq (\varepsilon \cdot \delta^p)^{\frac{1}{p+1}}$$

$$\Rightarrow k \in \left\{n \in \mathbb{N} : (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} \geq \varepsilon'\right\}, \text{ where } \varepsilon' = (\varepsilon \cdot \delta^p)^{\frac{1}{p+1}}.$$

Therefore $\{n \in \mathbb{N} : M(\|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta) \geq \varepsilon\}$

$$\subseteq \left\{n \in \mathbb{N} : (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} \geq \varepsilon'\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$ -convergent in measure to ζ . $\square$

Remark 4. But the converse of the above theorem is not true in general.

Example 5. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, where $D_j = \{2^{j-1}k : 2 \text{ does not divide } k, k \in \mathbb{N}\}$

be the decomposition of $\mathbb{N}$ such that each D_j is infinite and $D_j \cap D_k = \emptyset$, for $j \neq k$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ that can intersect only finite number of D_j 's. Then $\mathcal{I}$ is a nontrivial admissible ideal of $\mathbb{N}$ Kostyrko et al. [21].

Now we consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2, \dots\}$ with power set and $M\{\Gamma\} = 1$, $M\{\emptyset\} = 0$ and

$$M\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \beta_n, & \text{if } \sup_{\gamma_n \in \Lambda} \beta_n < \frac{1}{2} \\ 1 - \sup_{\gamma_n \in \Lambda^c} \beta_n, & \text{if } \sup_{\gamma_n \in \Lambda^c} \beta_n < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

where $\beta_n = \frac{1}{j+1}$, if $n \in D_j$ for $n = 1, 2, 3, \dots$.

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

and $\zeta \equiv 0$.

For any $\varepsilon > 0$ and $n \in \mathbb{N} \setminus D_1$, we have

$$M(\{\gamma \in \Gamma : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\}) = M(\gamma_n) = \beta_n.$$

Then $\{n \in \mathbb{N} : M(\|\zeta_n - \zeta\| \geq \varepsilon) \geq \delta\} = \{n \in \mathbb{N} : \beta_n \geq \delta\} \in \mathcal{I}$.

Therefore the sequence (ζ_n) is $\mathcal{I}$ -convergent in measure to $\zeta \equiv 0$.

$$\text{Now } \|\zeta_n(\gamma) - \zeta(\gamma)\|^p = \begin{cases} (n+1)^p, & \text{if } \gamma = \gamma_n \\ 0, & \text{otherwise} \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

Then for each $n \in \mathbb{N} \setminus D_1$, we have the uncertainty distribution of uncertain variable $\|\zeta_n - \zeta\|^p$ is

$$\Phi_n(r) = \begin{cases} 0, & \text{if } r < 0 \\ 1 - \beta_n, & \text{if } 0 \leq r < (n+1)^p \\ 1, & \text{if } r \geq (n+1)^p \end{cases} \quad \text{for } p > 0.$$

So for $n \in \mathbb{N} \setminus D_1$, we have

$$E[\|\zeta_n - \zeta\|^p] = \int_0^{(n+1)^p} (1 - (1 - \beta_n)) dr = (n+1)^p \beta_n.$$

$$\Rightarrow (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} = ((n+1)^p \beta_n)^{\frac{1}{p+1}}$$

Then for any $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : (E[\|\zeta_n - \zeta\|^p])^{\frac{1}{p+1}} \geq \varepsilon\right\} = \left\{n \in \mathbb{N} : ((n+1)^p \beta_n)^{\frac{1}{p+1}} \geq \varepsilon\right\} \notin \mathcal{I}.$$

Hence the sequence (ζ_n) is not $\mathcal{I}$ -convergent in p -distance to $\zeta \equiv 0$.

Theorem 13. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is $\mathcal{I}$ -convergent in p -distance to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ .

Proof. Let (ζ_n) be $\mathcal{I}$ -convergent in p -distance to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ by the theorem 12 and 3. $\square$

Remark 5. But the converse of the above theorem is not true in general.

Example 6. In Example 5, the complex uncertainty distributions of (ζ_n) are

$$\Phi_n(z) = \Phi_n(x + iy) = \begin{cases} 0, & \text{if } x < 0, y < \infty \\ 0, & \text{if } x \geq 0, y < 0 \\ 1 - \beta_n, & \text{if } x \geq 0, 0 \leq y < (n+1) \\ 1, & \text{if } x \geq 0, y \geq (n+1). \end{cases}$$

and the complex uncertainty distributions of ζ is

$$\Phi(z) = \Phi(x + iy) = \begin{cases} 0, & \text{if } x < 0, y < \infty \\ 0, & \text{if } x \geq 0, y < 0 \\ 1, & \text{if } x \geq 0, y \geq 0. \end{cases}$$

Then for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \|\Phi_n(z) - \Phi(z)\| \geq \varepsilon\} = \{n \in \mathbb{N} : \beta_n \geq \varepsilon\} \in \mathcal{I}, \text{ for } x \geq 0, y \geq 0.$$

Thus the sequence (ζ_n) is $\mathcal{I}$ -convergent in distribution to $\zeta \equiv 0$ but it is not $\mathcal{I}$ -convergent in p -distance to $\zeta \equiv 0$.

Definition 18. A complex uncertain sequence (ζ_n) is said to be completely $\mathcal{I}$ -convergent to ζ if for every $\varepsilon > 0$, $\delta > 0$ such that

$$\left\{ n \in \mathbb{N} : \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}) \geq \varepsilon \right\} \in \mathcal{I}.$$

For $\mathcal{I} = \mathcal{I}_d$, completely $\mathcal{I}$ -convergence of complex uncertain sequences coincide with completely statistical convergence of complex uncertain sequences, which was studied by Saha et al. [25].

Theorem 14. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is completely $\mathcal{I}$ -convergent to ζ , then it is $\mathcal{I}$ -convergent uniformly almost surely to ζ .

Proof. Let the complex uncertain sequence (ζ_n) be completely $\mathcal{I}$ -convergent to ζ , then for every $\varepsilon > 0$, $\delta > 0$ we have

$$\left\{ n \in \mathbb{N} : \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}) \geq \varepsilon \right\} \in \mathcal{I}.$$

It follows from Axiom 3 that,

$$M\left(\bigcup_{m=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}\right) \leq \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}).$$

Then for every $\varepsilon > 0$,

$$\begin{aligned} & \left\{ n \in \mathbb{N} : M\left(\bigcup_{m=n}^{\infty} \{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}\right) \geq \varepsilon \right\} \\ & \subseteq \left\{ n \in \mathbb{N} : \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}) \geq \varepsilon \right\} \in \mathcal{I}. \end{aligned}$$

Hence the sequence (ζ_n) is $\mathcal{I}$ -convergent uniformly almost surely to ζ . $\square$

Theorem 15. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is completely $\mathcal{I}$ -convergent to ζ , then it is $\mathcal{I}$ -convergent almost surely to ζ .

Proof. Let (ζ_n) be completely $\mathcal{I}$ -convergent to ζ , then it is $\mathcal{I}$ -convergent almost surely to ζ by the theorem 14 and 4. $\square$

Remark 6. But the converse of the above theorem is not true in general.

Example 7. From Example 2, we see that the complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent almost surely to $\zeta \equiv 0$.

$$\text{Now } \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}) = \sum_{m=n}^{\infty} M\{\gamma_m\} = \sum_{m=n}^{\infty} \frac{n\beta_n}{2n+1}$$

Then for every $\varepsilon > 0$,

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}) \geq \varepsilon \right\} \\ &= \left\{ n \in \mathbb{N} : \sum_{m=n}^{\infty} \frac{n\beta_n}{2n+1} \geq \varepsilon \right\} \notin \mathcal{I}. \end{aligned}$$

Hence the sequence (ζ_n) is not completely $\mathcal{I}$ -convergent to $\zeta \equiv 0$.

Theorem 16. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is completely $\mathcal{I}$ -convergent to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ .

Proof. Let (ζ_n) be completely $\mathcal{I}$ -convergent to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ by the theorem 14 and 6. $\square$

Remark 7. But the converse of the above theorem is not true in general.

Example 8. From Example 3, we see that the complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent in measure to $\zeta \equiv 0$.

$$\text{Now } \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}) = \sum_{m=n}^{\infty} M\{\gamma_m\} = \sum_{m=n}^{\infty} \frac{n\beta_n}{2n+1}$$

Then for every $\varepsilon > 0$,

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \sum_{m=n}^{\infty} M(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \delta\}) \geq \varepsilon \right\} \\ &= \left\{ n \in \mathbb{N} : \sum_{m=n}^{\infty} \frac{n\beta_n}{2n+1} \geq \varepsilon \right\} \notin \mathcal{I}. \end{aligned}$$

Hence the sequence (ζ_n) is not completely $\mathcal{I}$ -convergent to $\zeta \equiv 0$.

Theorem 17. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is completely $\mathcal{I}$ -convergent to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ .

Proof. Let (ζ_n) be completely $\mathcal{I}$ -convergent to ζ , then it is $\mathcal{I}$ -convergent in distribution to ζ by the theorem 16 and 3. $\square$

Remark 8. But the converse of the above theorem is not true in general.

Example 9. From Example 4 and 8, we see that the complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent in distribution to $\zeta \equiv 0$ but it is not completely $\mathcal{I}$ -convergent to $\zeta \equiv 0$.

Definition 19. A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$ -convergent in metric to ζ if for every $\varepsilon > 0$ such that

$$\{n \in \mathbb{N} : D(\zeta_n, \zeta) \geq \varepsilon\} \in \mathcal{I}.$$

Theorem 18. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is $\mathcal{I}$ -convergent in metric to ζ , then it is $\mathcal{I}$ -convergent in mean to ζ .

Proof. Let the complex uncertain sequence (ζ_n) be $\mathcal{I}$ -convergent in metric to ζ , then for every $\varepsilon > 0$ we have, $\{n \in \mathbb{N} : D(\zeta_n, \zeta) \geq \varepsilon\} \in \mathcal{I}$, where $D(\zeta_n, \zeta) = \inf \{r : M\{\|\zeta_n - \zeta\| \leq r\} = 1\}$.

Let $\Phi_n(r)$ be the complex uncertainty distributions of uncertain variable $\|\zeta_n - \zeta\|$ and $D(\zeta_n, \zeta) = D$, then $D(\zeta_n, \zeta) = \inf \{r : \Phi_n(r) = 1\}$.

Now for any positive number δ ,

$$\begin{aligned} E[\|\zeta_n - \zeta\|] &= \int_0^{+\infty} (1 - \Phi_n(r)) dr = \int_0^{D+\delta} (1 - \Phi_n(r)) dr + \int_{D+\delta}^{+\infty} (1 - \Phi_n(r)) dr \\ &= \int_0^{D+\delta} (1 - \Phi_n(r)) dr \\ &< 1 \cdot (D + \delta) = D + \delta. \end{aligned}$$

$$\Rightarrow E[\|\zeta_n - \zeta\|] \leq D \Rightarrow E[\|\zeta_n - \zeta\|] \leq D(\zeta_n, \zeta).$$

Then for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : E[\|\zeta_n(\gamma) - \zeta(\gamma)\|] \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : D(\zeta_n, \zeta) \geq \varepsilon\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$ -convergent in mean to ζ . $\square$

Remark 9. But the converse of the above theorem is not true in general.

Example 10. From Example 3, we have the complex uncertainty distributions of uncertain variable $\|\zeta_n - \zeta\|$ is

$$\Phi_n(r) = \begin{cases} 0, & \text{if } r < 0 \\ 1 - \frac{n\beta_n}{2n+1}, & \text{if } 0 \leq r < (n+1) \\ 1, & \text{if } r \geq (n+1) \end{cases} \quad \text{for } n = 1, 2, 3, \dots$$

$$\text{Now } E[\|\zeta_n - \zeta\|] = \int_0^{+\infty} (1 - \Phi_n(r)) dr = \int_0^{(n+1)} \frac{n\beta_n}{2n+1} dr = \frac{n(n+1)\beta_n}{2n+1}.$$

Then for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : E[\|\zeta_n(\gamma) - \zeta(\gamma)\|] \geq \varepsilon\} = \left\{n \in \mathbb{N} : \frac{n(n+1)\beta_n}{2n+1} \geq \varepsilon\right\} \in \mathcal{I}.$$

Again the metric between complex uncertain variables ζ_n and ζ is given by $D(\zeta_n, \zeta) = \inf \{r : M\{\|\zeta_n - \zeta\| \leq r\} = 1\} = \inf \{r : \Phi_n(r) = 1\} = n+1$.

Thus for every $\varepsilon > 0$, $\{n \in \mathbb{N} : D(\zeta_n, \zeta) \geq \varepsilon\} = \{n \in \mathbb{N} : (n+1) \geq \varepsilon\} \notin \mathcal{I}$.

Hence the complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent in mean to $\zeta \equiv 0$ but it is not $\mathcal{I}$ -convergent in metric to $\zeta \equiv 0$.

Theorem 19. Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables defined on uncertainty space (Γ, L, M) . If (ζ_n) is $\mathcal{I}$ -convergent in metric to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ .

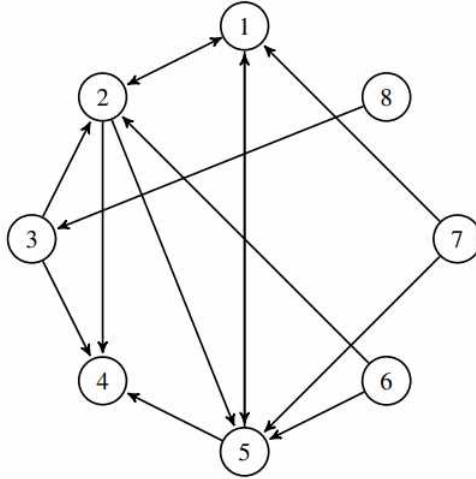
Proof. Let (ζ_n) be $\mathcal{I}$ -convergent in metric to ζ , then it is $\mathcal{I}$ -convergent in measure to ζ by the theorem 18 and 1. $\square$

Remark 10. But the converse of the above theorem is not true in general.

Example 11. From Example 3 and 10, we see that the complex uncertain sequence (ζ_n) is $\mathcal{I}$ -convergent in measure to $\zeta \equiv 0$ but it is not $\mathcal{I}$ -convergent in metric to $\zeta \equiv 0$.

4 Inter-relationships among all convergence concepts

- | | |
|--|---|
| 1. $\mathcal{I}$ -convergence almost surely. | 2. $\mathcal{I}$ -convergence in measure. |
| 3. $\mathcal{I}$ -convergence in mean. | 4. $\mathcal{I}$ -convergence in distribution. |
| 5. $\mathcal{I}$ -convergence uniformly almost surely. | 6. $\mathcal{I}$ -convergence in p -distance. |
| 7. Completely $\mathcal{I}$ -convergence. | 8. $\mathcal{I}$ -convergence in metric. |



5 Conclusion

This paper has been mainly devoted to the discussion of some newly introduced convergence concepts of complex uncertain sequences and established the relationships among them. We initiate the notion of $\mathcal{I}$ -convergence in p -distance, completely $\mathcal{I}$ -convergence and $\mathcal{I}$ -convergence in metric of complex uncertain sequences and include some interesting example related the notion. Also, this paper is a more generalized form of $\mathcal{I}$ -convergence of

complex uncertain sequences. In this paper, we try to establish relationships among all $\mathcal{I}$ -convergence concepts of complex uncertain sequence but we see that some of them are not related to each other. This an open problem for further study and it may attract future researchers in this direction.

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