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PROPERTIES OF A CLASS DRIVEN BY A CONVEX COMBINATION OF BAZILEVIČ AND PSEUDO-STARLIKE FUNCTIONS*

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Abstract

Motivated by the definition of multiplicative derivative, a new class of functions is defined by expressing differential characterizations belonging to a family of Bazilevič and pseudo-starlike functions as a convex combination. Estimates involving the initial coefficients of the functions, which belong to the defined function class, are the main results. Some examples along with graphs have been used to establish the inclusion and closure properties.

Keywords: multiplicative calculus, analytic function, univalent function, Schwarz function, starlike and convex functions.

MSC: 30C45.

1 Introduction

Let $\mathbf{R}$, $\mathbf{C}$ and $\mathcal{N}$ denote the set of real numbers, the set of complex numbers, and the set of natural numbers, respectively. Multiplicative calculus is a calculus that involves exponential functions. Explicitly, for a positive real valued function f, the multiplicative derivative $f^* : \mathbf{R} \longrightarrow \mathbf{R}$ is defined by

$$f^*(x) = \lim_{h\to 0} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}} = e^{\frac{f'(x)}{f(x)}} = e^{[\ln f(x)]'},$$

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where $f'(x)$ is the classical derivative. The $*$-derivative of f at z belonging to a small neighbourhood of a domain in a complex plane, where f is non-vanishing and differentiable, is given by

$$f^{*}(z) = e^{f'(z)/f(z)} \quad \text{and} \quad f^{*(n)}(z) = e^{[f'(z)/f(z)]^{(n)}},\ n \geq 1. \tag{1}$$

From (1), it is clear that $f^{*}(z)$ is not defined if $f(0) = 0$. So, the following question arises: why do we need such a restrictive calculus given that we have a calculus that is much versatile and applicable to most of the physical phenomena. The partial answer to this question is that the multiplicative derivative has been a useful mathematical tool for economics and financial mathematics. Let $\mathcal{A}$ denote the collection containing analytic functions in $\mathcal{U} = \{z : |z| < 1\}$ that have a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

Also, let $\mathcal{S}$ denote the collection of functions in $\mathcal{A}$ that are univalent in $\mathcal{U}$. $\mathcal{P}$ is the class of Carathéodory's function (see [7]), a class of analytic functions with normalization $p(0) = 1$ and which map the unit disc onto a right half-plane. Starlike and convex functions, the well-known geometrically defined subclasses of $\mathcal{A}$, have the following analytic characterizations, respectively

$$\frac{zf'(z)}{f(z)} \in \mathcal{P} \quad \text{and} \quad \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}.$$

We denote the class of starlike and convex functions by $\mathcal{S}^{*}$ and $\mathcal{C}$, respectively. Ma-Minda [16] studied an analytic function ψ satisfying the conditions:

(i) $\operatorname{Re}\psi > 0$, $\mathcal{U}$;

(ii) $\psi(0) = 1,\ \psi'(0) > 0$;

(iii) ψ maps $\mathcal{U}$ onto a starlike region with respect to 1 and symmetric with respect to the real axis.

Also, they assumed that $\psi(z)$ has a series expansion of the form

$$\psi(z) = 1 + M_1 z + M_2 z^2 + M_3 z^3 + \cdots, \qquad (M_1 > 0;\ z \in \mathcal{U}), \tag{2}$$

and introduced and studied the following subclasses of using subordination of analytic functions:

$$\mathcal{S}^{*}(\psi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\}$$

and
$$\mathcal{C}(\psi) := \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec \psi(z) \right\}.$$
By choosing ψ to map the unit disc in some specific regions such as parabolas [2], cardioid [17, 25], lemniscate of Bernoulli [19], booth lemniscate [18] in the right half of the complex plane, various interesting subclasses of starlike and convex functions can be obtained. For a detailed study, refer to [5, 8–10, 14, 22–24, 26–29].

Recently, Wanas et al. [30, Definition 1.] studied convex combinations of two analytic characterizations, namely Bazilevič functions and the pseudo-starlike as follows:
$$\operatorname{Re}\left\{(1-\gamma)\frac{z^{1-\beta}f'(z)}{[f(z)]^{1-\beta}} + \gamma\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}\right\} > 0 \quad (f \in \mathcal{A};\ z \in \mathcal{U}), \tag{3}$$
where $\beta \geq 0$, $\lambda \geq 1$, $0 \leq \gamma \leq 1$. The collection of functions $f \in \mathcal{A}$ that satisfy the condition (3) will be denoted by $\mathcal{R}(\beta;\ \gamma;\ \lambda)$.

Breaz et al. [6] and Karthikeyan and Murugusundaramoorthy [13] recently introduced new classes of meromorphic and analytic functions, respectively, by replacing the ordinary derivative with a multiplicative derivative in the classes of meromorphic star-like and analytic star-like function.

Motivated by the recent study of [6,13] and the definition of the function class $\mathcal{R}(\beta;\ \gamma;\ \lambda)$, we now introduce the following class.

Definition 1. *For $\theta \geq 1$, $0 \leq \gamma \leq 1$, β being any fixed number greater than or equal to zero except 1, let $\mathcal{M}(\beta;\ \gamma;\ \theta;\ \psi)$ denote the class of functions in $f \in \Sigma$ satisfying the condition*
$$(1-\gamma)\frac{z^{1-\beta}e^{\frac{z^2f'(z)}{f(z)}}}{[f(z)]^{1-\beta}} + \gamma\frac{z\left(e^{\frac{z^2f'(z)}{f(z)}}\right)^{\theta}}{f(z)} \prec \psi(z), \tag{4}$$
where $\psi \in \mathcal{P}$ is defined as in (2) and these powers are considered at the main branch, that is $\log 1 = 0$.

By considering $\beta = \gamma = 0$ in (4), the class $\mathcal{M}(\beta;\ \gamma;\ \theta;\ \psi)$ reduces to the family $\mathcal{R}(\psi)$ recently studied by Karthikeyan and Murugusundaramoorthy in [13]. Also, notice that $\mathcal{R}(\psi)$ can be obtained by letting $\theta = \gamma = 1$ in (4).

The class $\mathcal{M}(\beta;\ \gamma;\ \theta;\ \psi)$ is non-empty. Letting $\gamma = \frac{1}{2}$, $\beta = 3$, $\theta = \frac{3}{2}$ and $f(z) = \frac{5z}{5-z}$ in (4), then left hand side of (4) will yield an expression
$$L(z) = \frac{1}{2}\frac{25e^{\frac{5z}{5-z}}}{(5-z)} + \frac{1}{2}\frac{(5-z)\left(e^{\frac{5z}{5-z}}\right)^{3/2}}{5}$$

which maps $\mathcal{U}$ onto a cardioid in the right half plane. Similarly, letting $\gamma = \frac{3}{7}$, $\beta = 0$, $\theta = \frac{3}{2}$ and $f(z) = z\left(1+\frac{z}{7}\right)^3$ in (4), we can easily see that resulting expression

$$K(z) = \frac{3}{7}\, e^{\frac{7z+4z^2}{(7+z)}} \left(\frac{z}{z\left(1+\frac{z}{7}\right)^3}\right) + \frac{1}{7}\left(e^{\frac{7z+4z^2}{(7+z)}}\right)^{\frac{3}{2}} \frac{z}{z\left(1+\frac{z}{7}\right)^3}$$

would maps $\mathcal{U}$ on to the right half plane. However, $K(z)$ will not always map the unit disc on to the right half plane, for all admissible values of γ, β and θ.

2 Initial coefficients and Fekete-Szegő inequality

Now, we will find the solution to the Fekete-Szegő problem for

$$f \in \mathcal{M}(\beta;\, \gamma;\, \theta;\, \psi).$$

Lemma 1. *[21, p. 41] Let $d(z) = 1 + \sum\limits_{k=1}^{\infty} d_k z^k \in \mathcal{P}$. Then*

$$|d_n| \le 2 \qquad (n = 1,\, 2,\, 3, \ldots),$$

and inequality holds if and only if

$$p(z) = \sum_{\nu=1}^{n} \delta_\nu \frac{e^{i\sigma + 2\pi i\nu/n} + z}{e^{i\sigma + 2\pi i\nu/n} - z}$$

for some σ and $\delta_1, \ldots, \delta_n \ge 0$ with $\delta_1 + \delta_2 + \cdots + \delta_n = 1$.

Ma-Minda [16, p. 162] obtained the bounds of $\left|d_2 - \rho d_1^2\right|$ for $d(z) = 1 + \sum\limits_{k=1}^{\infty} d_k z^k \in \mathcal{P}$ when ρ is real. Generalizing the inequality of Livingston [15, Lemma 1], recently Efraimidis [11] obtained the following result.

Lemma 2. *[11, Theorem 1] If $d(z) = 1 + \sum\limits_{k=1}^{\infty} d_k z^k \in \mathcal{P}$, and ρ is a complex number, then*

$$|d_n - \rho d_k d_{n-k}| \le 2\max\{1; |2\rho - 1|\},$$

for all $1 \le k \le n-1$.

Theorem 1. *If $f(z) \in \mathcal{M}(\beta;\, \gamma;\, \theta;\, \psi)$, then for $\beta \neq 1$ we have*

$$|a_2| \leq \frac{1}{|1-\Lambda_{\beta,\gamma}|}\left[|M_1| + 1 + |\Upsilon_{\gamma,\theta}|\right] \tag{5}$$

$$\begin{aligned} |a_3| \quad &\leq \tfrac{|M_1|}{\left|1-\Lambda_{\beta,\gamma}\right|}\left[\max\left\{1,\ \left|\tfrac{M_2}{M_1} - \tfrac{\Omega_{\beta,\gamma}M_1}{2}\right|\right\} + \left|\tfrac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}} + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]\right|\right. \\ &\left. + \tfrac{1}{2|M_1|}\left|[(\theta+1)\Upsilon_{\gamma,\theta}+1] + \tfrac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}] + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2\right|\right] \end{aligned} \tag{6}$$

and for all $\rho \in \mathbf{C}$

$$\begin{aligned} \left|a_3 - \rho a_2^2\right| &\leq \tfrac{|M_1|}{\left|1-\Lambda_{\beta,\gamma}\right|}\left[\max\left\{\tfrac{M_2}{M_1} - \tfrac{\Omega_{\beta,\gamma}M_1}{2} + \tfrac{\rho M_1}{\left[1-\Lambda_{\beta,\gamma}\right]}\right\}\right. \\ + &\left|\tfrac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}} + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}] - \tfrac{2\rho[1+\Upsilon_{\gamma,\theta}]}{\left[1-\Lambda_{\beta,\gamma}\right]}\right| + \tfrac{1}{2|M_1|}\left|[(\theta+1)\Upsilon_{\gamma,\theta}+1]\right. \\ &\left.\left. + \tfrac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}] + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2 - \tfrac{2\rho[1+\Upsilon_{\gamma,\theta}]^2}{\left[1-\Lambda_{\beta,\gamma}\right]}\right|\right], \end{aligned} \tag{7}$$

where $\Lambda_{\beta,\gamma}$, $\Upsilon_{\gamma,\theta}$ and $\Omega_{\beta,\gamma}$ are given by

$$\Lambda_{\beta,\gamma} = \beta(1-\gamma), \quad \Upsilon_{\gamma,\theta} = \gamma(\theta-1) \tag{8}$$

$$\Omega_{\beta,\gamma} = \frac{\beta\Lambda_{\beta,\gamma} - 3\Lambda_{\beta,\gamma} + 2}{[1-\Lambda_{\beta,\gamma}]^2}. \tag{9}$$

The inequality is sharp for each $\rho \in \mathbf{C}$.

Proof. As $f \in \mathcal{M}(\beta;\, \gamma;\, \theta;\, \psi)$, by (4) we have

$$(1-\gamma)\frac{z^{1-\beta}e^{\frac{z^2 f'(z)}{f(z)}}}{[f(z)]^{1-\beta}} + \gamma\frac{z\left(e^{\frac{z^2 f'(z)}{f(z)}}\right)^{\theta}}{f(z)} = \psi\left[w(z)\right]. \tag{10}$$

Thus, let $\vartheta \in \mathcal{P}$ be of the form $\vartheta(z) = 1 + \sum_{k=1}^{\infty}\vartheta_n z^n$ and defined by

$$\vartheta(z) = \frac{1+w(z)}{1-w(z)}, \quad z \in \mathcal{U}.$$

On computation, the right hand side of (10)

$$\psi[w(z)] = 1 + \frac{\vartheta_1 M_1}{2}z + \frac{M_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{M_2}{M_1}\right)\right]z^2 + \cdots. \tag{11}$$

The left hand side of (10) will be of the form

$$(1-\gamma)\frac{z^{1-\beta}e^{\frac{z^2f'(z)}{f(z)}}}{[f(z)]^{1-\beta}}+\gamma\frac{z\left(e^{\frac{z^2f'(z)}{f(z)}}\right)^{\theta}}{f(z)}=1+[1-\gamma(1-\theta)-a_2\left(1-\beta+\beta\gamma\right)]z$$
$$+\tfrac{1}{2}\left[1-\gamma\left(1-\theta^2\right)+2a_2\beta(1-\gamma)+\left(\beta^2(1-\gamma)-3\beta(1-\gamma)+2\right)a_2^2\right.$$
$$\left.-2a_3\left(1-\Lambda_{\beta,\gamma}\right)\right]z^2+\cdots. \tag{12}$$

From (11) and (12), we obtain

$$a_2=-\frac{1}{[1-\Lambda_{\beta,\gamma}]}\left[\frac{\vartheta_1 M_1}{2}-1-\Upsilon_{\gamma,\theta})\right] \tag{13}$$

and

$$a_3=-\tfrac{M_1}{2\left[1-\Lambda_{\beta,\gamma}\right]}\left[\vartheta_2-\tfrac{\vartheta_1^2}{2}\left(1-\tfrac{M_2}{M_1}+\tfrac{\Omega_{\beta,\gamma}M_1}{2}\right)+\vartheta_1\left\{\tfrac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}+\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]\right\}\right.$$
$$\left.-\tfrac{1}{M_1}\left\{[(\theta+1)\Upsilon_{\gamma,\theta}+1]+\tfrac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}]+\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2\right\}\right], \tag{14}$$

where $\Lambda_{\beta,\gamma}$, $\Upsilon_{\gamma,\theta}$ and $\Omega_{\beta,\gamma}$ are given as in (8) and (9). Equation (5) can be obtained by applying Lemma 1 in (13). Applying Lemma 1 and Lemma 2 in (14), we get (6).

Now, to prove the Fekete-Szegő inequality for the class $\mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi)$, we consider

$$\begin{aligned}\left|a_3-\rho a_2^2\right| &= \left|\frac{M_1}{2\left[1-\Lambda_{\beta,\gamma}\right]}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{M_2}{M_1}+\frac{\Omega_{\beta,\gamma}M_1}{2}\right)\right.\right.\\ &+ \vartheta_1\left\{\frac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}+\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]\right\}-\frac{1}{M_1}\left\{[(\theta+1)\Upsilon_{\gamma,\theta}+1]\right.\\ &+ \left.\left.\frac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}]+\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2\right\}\right]\\ &+ \left.\frac{\rho}{[1-\Lambda_{\beta,\gamma}]^2}\left[\frac{\vartheta_1 M_1}{2}-1-\Upsilon_{\gamma,\theta})\right]^2\right|\end{aligned}$$

$$=\left|\tfrac{M_1}{2\left[1-\Lambda_{\beta,\gamma}\right]}\left[\vartheta_2-\tfrac{\vartheta_1^2}{2}\left(1-\tfrac{M_2}{M_1}+\tfrac{\Omega_{\beta,\gamma}M_1}{2}-\tfrac{\rho M_1}{\left[1-\Lambda_{\beta,\gamma}\right]}\right)\right.\right.$$
$$+\vartheta_1\left\{\tfrac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}+\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]-\tfrac{2\rho[1+\Upsilon_{\gamma,\theta}]}{\left[1-\Lambda_{\beta,\gamma}\right]}\right\}-\tfrac{1}{M_1}\left\{[(\theta+1)\Upsilon_{\gamma,\theta}+1]\right.$$
$$\left.\left.\left.+\tfrac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}]+\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2-\tfrac{2\rho[1+\Upsilon_{\gamma,\theta}]^2}{\left[1-\Lambda_{\beta,\gamma}\right]}\right\}\right]\right|.$$

Using the triangle inequality and Lemma 2 in the above equality, we can obtain (7). □

Letting $\gamma = 0$ in Theorem 1, we get:

Corollary 1. *Let $f \in \mathcal{A}$ satisfy the condition*

$$\frac{z^{1-\beta} e^{\frac{z^2 f'(z)}{f(z)}}}{[f(z)]^{1-\beta}} \prec \psi(z).$$

Then for $\beta \neq 1$ we have,

$$|a_2| \leq \frac{1}{|1-\beta|} \left[|M_1| + 1 \right],$$

$$|a_3| \leq \frac{|M_1|}{|1-\beta|} \left[\max\left\{ 1, \left| \frac{M_2}{M_1} - \frac{(\beta-2)M_1}{(\beta-1)} \right| \right\} + \left| \frac{2}{1-\beta} \right| + \frac{1}{2|M_1|} \left| \frac{3}{1-\beta} \right| \right],$$

and for a complex number ρ,

$$\begin{aligned} \left| a_3 - \rho a_2^2 \right| \quad &\leq \tfrac{|M_1|}{|1-\beta|} \left[\max\left\{ \tfrac{M_2}{M_1} - \tfrac{(\beta-2)M_1}{(\beta-1)} + \tfrac{\rho M_1}{[1-\beta]} \right\} + \left| \tfrac{2(1-\rho)}{1-\beta} \right| \right. \\ &\qquad \left. + \tfrac{1}{2|M_1|} \left| \tfrac{3-2\rho}{1-\beta} \right| \right]. \end{aligned}$$

Letting $\gamma = 1$ in Theorem 1, we get:

Corollary 2. *Let $f \in \mathcal{A}$ satisfy the condition*

$$\frac{z \left(e^{\frac{z^2 f'(z)}{f(z)}} \right)^{\theta}}{f(z)} \prec \psi(z).$$

Then,

$$|a_2| \leq \left[|M_1| + |\theta - 1| + 1 \right],$$

$$|a_3| \leq |M_1| \left[\max\left\{ 1, \left| \frac{M_2}{M_1} - M_1 \right| \right\} + |2\theta| + \frac{1}{2|M_1|} \left| 3\theta^2 \right| \right],$$

and for a complex number ρ,

$$\left| a_3 - \rho a_2^2 \right|$$

$$\leq |M_1| \left[\max\left\{ \frac{M_2}{M_1} - (1-\rho) M_1 \right\} + 2 \left| \theta(1-\rho) \right| + \frac{1}{2|M_1|} \left| \theta(3 - 2\rho) \right| \right].$$

Letting $\theta = 1$ and $\psi(z) = \left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}$ in Theorem 2, we get:

Corollary 3. *Let $f \in \mathcal{A}$ satisfy the condition*

$$\frac{z\left(e^{\frac{z^2 f'(z)}{f(z)}}\right)}{f(z)} \prec \left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} \qquad (0 \le \alpha < 1). \tag{15}$$

Then,

$$|a_2| \le 2, \qquad \text{and} \qquad |a_3| \le \frac{9}{2}.$$

Also, for a complex number ρ,

$$\left|a_3 - \rho a_2^2\right| \le \max\left\{\frac{1}{2} - \rho\right\} + 2\,|1-\rho| + \frac{1}{2}\,|3 - 2\rho|\,.$$

Proof. The function $F_\alpha(z) = \left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}$ is convex univalent in $\mathcal{U}$ (see [12, Theorem 2.4]) and has a power series expansion of the form

$$\psi(z) = \left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}} = 1 + z + \frac{1}{2}z^2 + \frac{1}{3}\left(\alpha + \frac{1}{2}\right)z^3 + \cdots.$$

So replacing $M_1 = 1$, $M_2 = \frac{1}{2}$ and $\theta = 1$ in Corollary 2, we get the desired result. $\square$

Taking $\beta = 0$ in Corollary 1, we have the following:

Corollary 4. *[13, Corollary 1] Let $f \in \mathcal{R}(\psi)$. Then,*

$$|a_2| \le 1 + |M_1|$$

$$|a_3| \le |M_1|\left[\max\left\{1; \left|\frac{M_2}{M_1} - M_1\right|\right\} + \frac{3}{2\,|M_1|} + 2\right]$$

and for a complex number ρ,

$$\left|a_3 - \rho a_2^2\right| \le |M_1|\left[\max\left\{1; \left|\frac{M_2}{M_1} - M_1(1-\rho)\right|\right\} + 2\,|1-\rho| + \frac{1}{2\,|M_1|}\,|3-2\rho|\right].$$

Remark 1. *Alternatively, Corollary 4 can be obtained if we let $\theta = \gamma = 1$ in Theorem 1.*

Letting $\psi(z) = (1+z)/(1-z)$ in Corollary 4, we get:

Corollary 5. *Let $f \in \mathcal{A}$ satisfy the condition*

$$Re\left(\frac{ze^{\frac{z^2 f'(z)}{f(z)}}}{f(z)}\right) > 0.$$

Then,

$$|a_2| \leq 3, \qquad |a_3| \leq \frac{15}{2}$$

and for a complex number ρ,

$$\left|a_3 - \rho a_2^2\right| \leq |M_1| \left[\max\{1;\ |2\rho - 1|\} + 2\,|1-\rho| + \frac{1}{4}\,|3-2\rho|\right].$$

3 Coefficient estimates of $f^{-1}(z)$

It is well-known from Koebe 1/4-quarter theorem that every function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$, $z \in \mathcal{U}$ and $f(f^{-1}(w)) = w$, $(|w| < r;\ r \geq 1/4)$ where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^2 - \left(5a_2^2 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{16}$$

The functions in $\mathcal{M}(\beta;\ \gamma;\ \theta;\ \psi)$ need not be univalent, but since $f'(0) = 1 \neq 0$ for all $f \in \mathcal{M}(\beta;\ \gamma;\ \theta;\ \psi)$ and $f(0) = 0$, there exists an inverse function in some small disk with center at $w = 0$. The next result is valid only for the functions in $\mathcal{M}(\beta;\ \gamma;\ \theta;\ \psi)$ which are univalent.

Theorem 2. *Let $f \in \mathcal{M}(\beta;\ \gamma;\ \theta;\ \psi)$ and let f^{-1} be the inverse of f defined by*

$$f^{-1}(w) = w + \sum_{k=2}^{\infty} b_k w^k, \quad (|w| < r;\ r \geq 1/4).$$

Then, for $\beta \neq 1$, we have

$$|b_2| \leq \frac{1}{|1 - \Lambda_{\beta,\gamma}|}\left[|M_1| + 1 + |\Upsilon_{\gamma,\theta}|\right]$$

and

$$\begin{aligned} |b_3| \leq \quad & \tfrac{|M_1|}{\left|1-\Lambda_{\beta,\gamma}\right|}\left[\max\left\{\tfrac{M_2}{M_1} - \tfrac{\Omega_{\beta,\gamma}M_1}{2} + \tfrac{2M_1}{\left[1-\Lambda_{\beta,\gamma}\right]}\right\}\right. \\ & + \left|\tfrac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}} + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}] - \tfrac{4[1+\Upsilon_{\gamma,\theta}]}{\left[1-\Lambda_{\beta,\gamma}\right]}\right| + \tfrac{1}{2|M_1|}\left|[(\theta+1)\Upsilon_{\gamma,\theta}+1]\right. \\ & \left.\left. + \tfrac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}] + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2 - \tfrac{4[1+\Upsilon_{\gamma,\theta}]^2}{\left[1-\Lambda_{\beta,\gamma}\right]}\right|\right]. \end{aligned}$$

Also, for all $\tau \in \mathbf{C}$

$$\left|b_3 - \tau b_2^2\right| \leq \frac{|M_1|}{\left|1-\Lambda_{\beta,\gamma}\right|}\left[\max\left\{\frac{M_2}{M_1} - \frac{\Omega_{\beta,\gamma}M_1}{2} + \frac{(\tau-2)M_1}{\left[1-\Lambda_{\beta,\gamma}\right]}\right\} + \left|\frac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}} + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}] - \frac{2(\tau-2)[1+\Upsilon_{\gamma,\theta}]}{\left[1-\Lambda_{\beta,\gamma}\right]}\right| + \frac{1}{2|M_1|}\left|[(\theta+1)\Upsilon_{\gamma,\theta}+1] + \frac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}] + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2 - \frac{2(\tau-2)[1+\Upsilon_{\gamma,\theta}]^2}{\left[1-\Lambda_{\beta,\gamma}\right]}\right|\right], \quad (17)$$

where $\Lambda_{\beta,\gamma}$, $\Upsilon_{\gamma,\theta}$ *and* $\Omega_{\beta,\gamma}$ *is defined as in (8) and (9). The inequality is sharp for each* $\tau \in \mathbf{C}$.

Proof. From $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and (16), we have

$$b_2 = -a_2 \quad \text{and} \quad b_3 = 2a_2^2 - a_3.$$

The estimate for $|b_2| = |a_2|$ follows immediately from (13). Letting $\rho = 2$ in (7), we get the estimate $|b_3|$. To find the Fekete-Szegő inequality for the inverse function, consider

$$\left|b_3 - \tau b_2^2\right| = \left|2a_2^2 - a_3 - \tau a_2^2\right| = \left|a_3 - (\tau-2)a_2^2\right|.$$

Changing $\rho = (\tau - 2)$ in the (7), we get the desired result. □

Remark 2. *Letting* $\gamma = \beta = 0$ *in (2), we get the result obtained Karthikeyan and Murugusundaramoorthy [13, Theorem 2].*

4 Logarithmic coefficients for functions belonging to $\mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi)$

Logarithmic coefficients took the spotlight when Milin in [20] studied the properties which would imply the bounds of the Taylor coefficients of univalent functions. The Milin conjecture about the inequalities of the logarithmic coefficients garnered the attention of several researchers because proving Milin conjecture would imply proving Robertson conjecture and the Bieberbach conjecture. Refer to [1,3,4] for the detailed study on properties and significance of the logarithmic coefficients.

If the function f is analytic in $\mathcal{U}$, such that $\frac{f(z)}{z} \neq 0$ for all $z \in \mathcal{U}$, then the well-known logarithmic coefficients $c_n := c_n(f), n \in \mathcal{N}$, of f are given by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} c_n z^n, z \in \mathcal{U}, \quad \log 1 = 0. \quad (18)$$

Now, we will include additional criterion to the class $\mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi)$, so that logarithmic coefficients of $\mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi)$ is well-defined. That is, we let $\mathcal{BM}(\beta;\,\gamma;\,\theta;\,\psi) = \mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi) \cap \left\{ f \text{ is analytic in } \mathcal{U} : \frac{f(z)}{z} \neq 0,\, z \in \mathcal{U} \right\}$. Note that for all functions $\mathcal{BM}(\beta;\,\gamma;\,\theta;\,\psi)$, the relation (18) is well-defined.

Theorem 3. *If $f(z) \in \mathcal{BM}(\beta;\,\gamma;\,\theta;\,\psi)$ with the logarithmic coefficients given by (18), then for $\beta \neq 1$ we have*

$$|c_1| \leq \frac{1}{2\,|1-\Lambda_{\beta,\gamma}|}\left[|M_1| + 1 + |\Upsilon_{\gamma,\theta}|\right], \tag{19}$$

$$\begin{aligned} |c_2| \leq \quad & \tfrac{|M_1|}{2\left|1-\Lambda_{\beta,\gamma}\right|}\left[\max\left\{\tfrac{M_2}{M_1} - \tfrac{\Omega_{\beta,\gamma}M_1}{2} + \tfrac{M_1}{2\left[1-\Lambda_{\beta,\gamma}\right]}\right\} + \left|\tfrac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}\right.\right. \\ & \left. +\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}] - \tfrac{[1+\Upsilon_{\gamma,\theta}]}{\left[1-\Lambda_{\beta,\gamma}\right]}\right| + \tfrac{1}{2|M_1|}\Big|[(\theta+1)\Upsilon_{\gamma,\theta}+1] \\ & \left.\left. + \tfrac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}] + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2 - \tfrac{[1+\Upsilon_{\gamma,\theta}]^2}{\left[1-\Lambda_{\beta,\gamma}\right]}\right|\right], \end{aligned} \tag{20}$$

and for each $\mu \in \mathbf{C}$

$$\begin{aligned} \left|c_2 - \mu c_1^2\right| \leq \quad & \tfrac{|M_1|}{2\left|1-\Lambda_{\beta,\gamma}\right|}\left[\max\left\{\tfrac{M_2}{M_1} - \tfrac{\Omega_{\beta,\gamma}M_1}{2} + \tfrac{(1+\mu)M_1}{2\left[1-\Lambda_{\beta,\gamma}\right]}\right\} + \left|\tfrac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}\right.\right. \\ & \left. +\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}] - \tfrac{(1+\mu)[1+\Upsilon_{\gamma,\theta}]}{\left[1-\Lambda_{\beta,\gamma}\right]}\right| + \tfrac{1}{2|M_1|}\Big|[(\theta+1)\Upsilon_{\gamma,\theta}+1] \\ & \left.\left. + \tfrac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}] + \Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2 - \tfrac{(1+\mu)[1+\Upsilon_{\gamma,\theta}]^2}{\left[1-\Lambda_{\beta,\gamma}\right]}\right|\right], \end{aligned} \tag{21}$$

where $\Lambda_{\beta,\gamma}$, $\Upsilon_{\gamma,\theta}$ and $\Omega_{\beta,\gamma}$ are given as in (8) and (9).

Proof. From $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and equating the first two coefficients of relation (18), we get

$$c_1 = \frac{a_2}{2},\ c_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right).$$

Using (13) and (14), we obtain

$$\begin{aligned}
c_1 &= -\frac{1}{2\left[1-\Lambda_{\beta,\gamma}\right]}\left[\frac{\vartheta_1 M_1}{2}-1-\Upsilon_{\gamma,\theta})\right],\\
c_2 &= \frac{1}{2}\left(a_3-\frac{a_2^2}{2}\right)\\
&= \frac{M_1}{4\left[1-\Lambda_{\beta,\gamma}\right]}\left[\vartheta_2-\frac{\vartheta_1^2}{2}\left(1-\frac{M_2}{M_1}+\frac{\Omega_{\beta,\gamma}M_1}{2}-\frac{M_1}{4\left[1-\Lambda_{\beta,\gamma}\right]}\right)\right.\\
&+ \vartheta_1\left\{\frac{\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}+\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]-\frac{[1+\Upsilon_{\gamma,\theta}]}{2\left[1-\Lambda_{\beta,\gamma}\right]}\right\}\\
&- \frac{1}{M_1}\left\{[(\theta+1)\Upsilon_{\gamma,\theta}+1]+\frac{2\Lambda_{\beta,\gamma}}{1-\Lambda_{\beta,\gamma}}[1+\Upsilon_{\gamma,\theta}]\right.\\
&+ \left.\left.\Omega_{\beta,\gamma}[1+\Upsilon_{\gamma,\theta}]^2-\frac{[1+\Upsilon_{\gamma,\theta}]^2}{[1-\Lambda_{\beta,\gamma}]}\right\}\right].
\end{aligned}$$

We obtain (19) and (20) by using Lemma 2 and taking the modulus on both sides. To obtain (21), we consider

$$\left|c_2-\mu c_1^2\right|=\frac{1}{2}\left[a_3-\frac{(1+\mu)}{2}a_2^2\right].$$

Changing $\rho=\frac{1+\mu}{2}$ in (7), we get the desired result. □

Remark 3. *Letting $\gamma=\beta=0$ in 3, we get the result obtained Karthikeyan and Murugusundaramoorthy [13, Theorem 3].*

5 Conclusion

In this paper, we have introduced and studied a new family which was expressed as a convex combination of differential characterizations belonging to well-known subclasses. We have obtained bounds of the initial coefficients of functions belonging to the defined function classes. Since the defined function class is subordinate to a very general functions and involves lots of parameters, our main results have lots of applications. Now, the following question arises: (1) what are necessary and sufficient conditions for functions to be in $\mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi)$? (2) the functions in $\mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi)$ need not be univalent, so for what radius of the disc $|z|<r$, the functions in $\mathcal{M}(\beta;\,\gamma;\,\theta;\,\psi)$ would be univalent?

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