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A NEW FAST CONVERGENCE TO THE CONSTANT e

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Abstract

The aim of this paper is to introduce a new sequence convergent to the constant e.

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1 Introduction

The constant e, defined as the limit of the sequence

$$e_n = \left(1 + \frac{1}{n}\right)^n, \quad n \geq 1,$$

is of great importance in mathematics. Numerically, we have $e = 2.71828....$ It arises naturally in many areas, including calculus, complex analysis, probability, and financial mathematics. Known as the base of the natural logarithm, e plays a central role in describing exponential growth and decay processes. The number e is essential to solving some differential equations, including differential equations that model real world phenomena.

Consequently, several researchers have been concerned with finding new sequences that converge to e, even sacrificing simplicity.

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2 An auxiliary result

A main aspect in the theory of convergent sequences is the speed of convergence. The faster a sequence converges to its limit, the more useful it is; for example, in problems involving the estimation of its limit.

An important tool for measuring the speed of convergence is the following consequence of Stolz-Cesàro lemma, first time used in this form by Mortici [5]:

Lemma 1. *Let x_n be a sequence convergent to zero, such that*

$$\lim_{n\to\infty} n^k (x_n - x_{n+1}) = l,$$

for some real $k > 1$ and $l \neq 0$. Then

$$\lim_{n\to\infty} n^{k-1} x_n = \frac{l}{k-1}.$$

This lemma shows us that if $x_n - x_{n+1}$ converges to zero like n^{-k}, then x_n converges to zero like $n^{-(k-1)}$.

Lemma 1 was proven to be a useful tool for accelerating some convergencies, establishing new fast sequences related to factorial function, gamma function, or classical constants such as e or π. For details, see [1]- [4].

Note that the sequence e_n converges increasingly to e like n^{-1}, as

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^n - e\right) = -\frac{1}{2}e.$$

Analogously, the sequence $e'_n = \left(1+\frac{1}{n}\right)^{n+1}$ converges decreasingly to e like n^{-1}, as

$$\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+1} - e\right) = \frac{1}{2}e.$$

In fact, for all $\alpha \in [0,1] \setminus \{1/2\}$, the sequence $\left(1+\frac{1}{n}\right)^{n+\alpha}$, $n \geq 1$, converges to e like n^{-1}. It is only the sequence

$$e''_n = \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}, \ n \geq 1$$

that converges to e like n^{-2}.

3 The main results

In the theory of convergent sequences, many researchers are concerned with modifying them in order to obtain new sequences that converge more rapidly. This is often done at the expense of simplicity.

In this paper, we introduce a modified form of the sequence e_n that converges more rapidly to e and keeps a simple form.

More exactly, we consider the family of sequences

$$e_n(a,b) = \left(1 + \frac{1}{\sqrt{n^2+a}}\right)^{n+b},$$

depending on real parameters a, b. Note that

$$e_n = e_n(0,0), \quad e''_n = e_n\left(0, \frac{1}{2}\right), \quad e'_n = e_n(0,1).$$

In our attempt to determine the parameters a and b for which the resulting sequence $e_n(a,b)$ converges most rapidly to e, we will discover and discuss the sequence

$$f_n = \left(1 + \frac{1}{\sqrt{n^2+\frac{1}{6}}}\right)^{n+\frac{1}{2}}, \quad n \geq 1.$$

This new sequence f_n remains of a simple form, and converges to e like n^{-3}.

To do this, we apply Lemma 1 to the sequence

$$x_n = \ln e_n(a,b).$$

We have

$$x_n - x_{n+1} = (n+b)\ln\left(1 + \frac{1}{\sqrt{n^2+a}}\right) - (n+1+b)\ln\left(1 + \frac{1}{\sqrt{(n+1)^2+a}}\right).$$

Using Maple software for symbolic computation, we deduce that

$$\begin{aligned} x_n - x_{n+1} &= \frac{6b-3}{6n^2} - \frac{6a+12b-7}{6n^3} \\ &\quad + \frac{18a+21b-9ab-18}{6n^4} - \frac{42a+42b-30ab-9a^2-29}{6n^5} \\ &\quad + O\left(\frac{1}{n^6}\right). \end{aligned}$$

We are in a position to use Lemma 1 in order to give the following:

Theorem 1. *a) If $b \neq 1/2$, then*

$$\lim_{n\to\infty} n^2 (x_n - x_{n+1}) = \frac{6b-3}{6} \quad \textit{and} \quad \lim_{n\to\infty} n x_n = \frac{6b-3}{6}.$$

b) If $b = 1/2$ and $a \neq 1/6$, then

$$\lim_{n\to\infty} n^3 (x_n - x_{n+1}) = \frac{1-6a}{6} \quad \textit{and} \quad \lim_{n\to\infty} n^2 x_n = \frac{1-6a}{12} \neq 0.$$

Consequently, for all $a \neq 1/6$, the sequence

$$\left(1 + \frac{1}{\sqrt{n^2+a}}\right)^{n+\frac{1}{2}}, \quad n \geq 1$$

converges to e like n^{-2}.
c) If $b = 1/2$ and $a = 1/6$, then

$$x_n - x_{n+1} = -\frac{7}{8n^4} + \frac{5}{8n^5} + O\left(\frac{1}{n^6}\right).$$

Moreover,

$$\lim_{n\to\infty} n^4 (x_n - x_{n+1}) = -\frac{7}{8} \quad \textit{and} \quad \lim_{n\to\infty} n^3 x_n = -\frac{7}{24} \neq 0.$$

Consequently, the sequence

$$f_n = \left(1 + \frac{1}{\sqrt{n^2+\frac{1}{6}}}\right)^{n+\frac{1}{2}}, \quad n \geq 1$$

converges to e like n^{-3}.

4 An asymptotic expansion and some estimates

We give the following:

Theorem 2. *The following asymptotic expansion holds true, as $n \to \infty$:*

$$\left(1 + \frac{1}{\sqrt{n^2+\frac{1}{6}}}\right)^{n+\frac{1}{2}} = \exp\left\{1 - \frac{1}{24n^3} + \frac{7}{160n^4} - \frac{97}{2880n^5} + \frac{649}{24\,192n^6} - \frac{1087}{48\,384n^7} + O\left(\frac{1}{n^8}\right)\right\}.$$

Proof. By taking the logarithm, we have to expand the function

$$\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{\sqrt{n^2+\frac{1}{6}}}\right).$$

This can be done by using the standard series of the logarithm function:

$$\ln(1+x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^k$$

and the generalized binomial series $(1+x)^r$ for $r=-\frac{1}{2}$:

$$\frac{1}{\sqrt{1+x}}=(1+x)^{-\frac{1}{2}}=\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}x^k.$$

A direct way is to use again the Maple software. The proof is complete. $\square$

Let us now state the following:

Theorem 3. *The following inequalities hold true, for all integers $n\geq 1$:*

$$\exp\left\{1-\frac{1}{24n^3}\right\}<\left(1+\frac{1}{\sqrt{n^2+\frac{1}{6}}}\right)^{n+\frac{1}{2}}<\exp\left\{1-\frac{1}{24n^3}+\frac{7}{160n^4}\right\}.$$

Proof. The left-hand inequality can be equivalently written as

$$\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{\sqrt{n^2+\frac{1}{6}}}\right)>1-\frac{1}{24n^3},$$

or

$$\ln\left(1+\frac{1}{\sqrt{n^2+\frac{1}{6}}}\right)>\frac{1-\frac{1}{24n^3}}{n+\frac{1}{2}}.$$

As

$$\ln(1+x)>x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{6}x^6,\quad x>0,$$

it suffices to observe that

$$\begin{aligned} & \frac{1}{\sqrt{n^2+\frac{1}{6}}}-\frac{1}{2\left(n^2+\frac{1}{6}\right)}+\frac{1}{3\left(n^2+\frac{1}{6}\right)\sqrt{n^2+\frac{1}{6}}} \\ & -\frac{1}{4\left(n^2+\frac{1}{6}\right)^2}+\frac{1}{5\left(n^2+\frac{1}{6}\right)^2\sqrt{n^2+\frac{1}{6}}}-\frac{1}{6\left(n^2+\frac{1}{6}\right)^3} \\ > \; & \frac{1-\frac{1}{24n^3}}{n+\frac{1}{2}}. \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \left(1+\frac{1}{3\left(n^2+\frac{1}{6}\right)}+\frac{1}{5\left(n^2+\frac{1}{6}\right)^2}\right)\frac{1}{\sqrt{n^2+\frac{1}{6}}} \\ > \; & \frac{1-\frac{1}{24n^3}}{n+\frac{1}{2}}+\frac{1}{2\left(n^2+\frac{1}{6}\right)}+\frac{1}{4\left(n^2+\frac{1}{6}\right)^2}+\frac{1}{6\left(n^2+\frac{1}{6}\right)^3}, \end{aligned}$$

so we have to prove that $u>0$, where

$$\begin{aligned} u(n) \;=\; & \left(1+\frac{1}{3\left(n^2+\frac{1}{6}\right)}+\frac{1}{5\left(n^2+\frac{1}{6}\right)^2}\right)^2\frac{1}{n^2+\frac{1}{6}} \\ & -\left(\frac{1-\frac{1}{24n^3}}{n+\frac{1}{2}}+\frac{1}{2\left(n^2+\frac{1}{6}\right)}+\frac{1}{4\left(n^2+\frac{1}{6}\right)^2}+\frac{1}{6\left(n^2+\frac{1}{6}\right)^3}\right)^2. \end{aligned}$$

We have

$$u(n)=\frac{P(n-2)}{3600n^6(2n+1)^2(6n^2+1)^6}>0, \quad n\geq 2,$$

where

$$\begin{aligned} P(n) \;=\; & 58\,786\,560n^{14}+1630\,160\,640n^{13}+20\,880\,815\,040n^{12} \\ & +163\,659\,916\,800n^{11}+876\,180\,053\,376n^{10} \\ & +3385\,681\,047\,936n^{9}+9723\,075\,891\,408n^{8} \\ & +21\,036\,928\,029\,120n^{7}+34\,353\,801\,586\,080n^{6} \\ & +41\,947\,056\,386\,352n^{5}+37\,432\,962\,400\,932n^{4} \\ & +23\,394\,794\,060\,496n^{3}+9468\,256\,624\,380n^{2} \\ & +2115\,309\,639\,600n+169\,539\,284\,375 \\ > \; & 0. \end{aligned}$$

It follows that $u(n) > 0$, for all integers $n \geq 2$.
The right-hand side inequality is equivalent to

$$\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{\sqrt{n^2+\frac{1}{6}}}\right) < 1-\frac{1}{24n^3}+\frac{7}{160n^4},$$

or

$$\ln\left(1+\frac{1}{\sqrt{n^2+\frac{1}{6}}}\right) < \frac{1-\frac{1}{24n^3}+\frac{7}{160n^4}}{n+\frac{1}{2}}.$$

As

$$\ln(1+x) < x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{6}x^6+\frac{1}{7}x^7, \quad x>0,$$

it suffices to

$$\begin{aligned}
&\frac{1}{\sqrt{n^2+\frac{1}{6}}}-\frac{1}{2\left(n^2+\frac{1}{6}\right)}+\frac{1}{3\left(n^2+\frac{1}{6}\right)\sqrt{n^2+\frac{1}{6}}}\\
&-\frac{1}{4\left(n^2+\frac{1}{6}\right)^2}+\frac{1}{5\left(n^2+\frac{1}{6}\right)^2\sqrt{n^2+\frac{1}{6}}}-\frac{1}{6\left(n^2+\frac{1}{6}\right)^3}\\
&+\frac{1}{7\left(n^2+\frac{1}{6}\right)^3\sqrt{n^2+\frac{1}{6}}}\\
&<\frac{1-\frac{1}{24n^3}+\frac{7}{160n^4}}{n+\frac{1}{2}}.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\left(1+\frac{1}{3\left(n^2+\frac{1}{6}\right)}+\frac{1}{5\left(n^2+\frac{1}{6}\right)^2}+\frac{1}{7\left(n^2+\frac{1}{6}\right)^3}\right)\frac{1}{\sqrt{n^2+\frac{1}{6}}}\\
<\ &\frac{1-\frac{1}{24n^3}+\frac{7}{160n^4}}{n+\frac{1}{2}}+\frac{1}{2\left(n^2+\frac{1}{6}\right)}+\frac{1}{4\left(n^2+\frac{1}{6}\right)^2}+\frac{0}{6\left(n^2+\frac{1}{6}\right)^3},
\end{aligned}$$

so we have to prove that $v < 0$, where

$$\begin{aligned}
v(n) \quad = \quad &\left(1+\frac{1}{3\left(n^2+\frac{1}{6}\right)}+\frac{1}{5\left(n^2+\frac{1}{6}\right)^2}+\frac{1}{7\left(n^2+\frac{1}{6}\right)^3}\right)^2\frac{1}{n^2+\frac{1}{6}}\\
&-\left(\frac{1-\frac{1}{24n^3}+\frac{7}{160n^4}}{n+\frac{1}{2}}+\frac{1}{2\left(n^2+\frac{1}{6}\right)}+\frac{1}{4\left(n^2+\frac{1}{6}\right)^2}+\frac{1}{6\left(n^2+\frac{1}{6}\right)^3}\right)^2.
\end{aligned}$$

But

$$v(n) = -\frac{Q(n-2)}{2822\,400n^8\,(2n+1)^2\,(6n^2+1)^7} < 0, \quad n \geq 2,$$

with

$$\begin{aligned} Q(n) &= 45\,320\,333\,731\,077\,449\,318\,400n^{34} \\ &\quad +3054\,906\,423\,145\,204\,914\,585\,600n^{33} + ..., \end{aligned}$$

a 34-th degree polynomial with all coefficients positive. The required inequality is also true for $n = 1$ (by direct computation), so the proof is complete. □

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