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AN APPROACH TO GÂTEAUX AND FRÉCHET DIFFERENTIABILITY BASED ON DELTA-CONVEX FUNCTIONS*

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Abstract

The aim of this paper is to present an approach to Gâteaux and Fréchet differentiability based on delta-convex functions. We extend some results by Ivan and Raşa and by Marumo and Takeda.

Keywords: convexity, Gâteaux differentiability, Fréchet differentiability, delta-convex functions.

MSC: 49J50.

1 Introduction

Ivan and Raşa [1] extended the notion of divided difference as follows.

Let X be a real Hilbert space, $C \subset X$ convex. For a function $f : X \to \mathbb{R}$, $x, y \in C$, $x \neq y$, $a \in (0,1)$, we denote:

$$(x, a, y; f) = (1-a) f(x) + a f(y) - f((1-a)x + ay).$$

The number

$$\frac{(x, a, y; f)}{\left(x, a, y; \|\cdot\|^2\right)}$$

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is called the divided difference with knots x, $(1-a)\,x+ay$, y ($||\cdot||$ is the norm of X).

It is proven in [1] that

$$\left|\frac{(x,a,y;f)}{\left(x,a,y;\|\cdot\|^2\right)}\right|\leq\frac{1}{2}M,$$

for every function $f:C\to\mathbb{R}$ twice Fréchet differentiable, with $\|f''(y)\|\leq M$, $y\in C$. Here, $C\subset X$ is open and convex.

Marumo and Takeda [2, Lemma 3.1] proved that if $f:\mathbb{R}^n\to\mathbb{R}$ is twice differentiable on $\mathbb{R}^n$ such that

$$\left\|\bigtriangledown^2 f(x)-\bigtriangledown^2 f(y)\right\|\leq M_f\,\|x-y\|\,,\quad x,y\in\mathbb{R}^n,$$

for some $M_f>0$, then

$$\left\|\bigtriangledown f\left(\sum_{i=1}^{k}\lambda_i x_i\right)-\sum_{i=1}^{k}\lambda_i\bigtriangledown f(x_i)\right\|\leq\frac{M_f}{2}\sum_{1\leq i<j\leq k}\lambda_i\lambda_j\left\|x_i-x_j\right\|^2,$$

for all $\lambda_1,...,\lambda_k\geq 0$, with $\lambda_1+...+\lambda_k=1$ and $x_1,...,x_k\in\mathbb{R}^n$. This result was applied for solving some minimum nonconvex problems.

Boţ et al. [3] proved the next

Theorem 1. *Let X be a real Hilbert space, Y a reflexive Banach space, $F:X\to Y$ continuous and $L>0$. The following assertions are equivalent:*
i) F is Fréchet differentiable on X and the differential $F':X\to(X,Y)^*$ is L-Lipschitz.
ii) the following inequality holds true:

$$\left\|F\left(\sum_{i=1}^{n}\lambda_i x_i\right)-\sum_{i=1}^{n}\lambda_i F(x_i)\right\|\leq\frac{L}{2}\sum_{1\leq i<j\leq n}\lambda_i\lambda_j\left\|x_i-x_j\right\|^2,$$

for all $\lambda_1,...,\lambda_n\geq 0$, with $\lambda_1+...+\lambda_n=1$ and $x_1,...,x_n\in X$.

In this paper, we present a new approach based on delta-convex functions and the results from [8], or on the results stated by Marinescu and Mortici [9, Theorem 7]. We refer to the following

Theorem 2 (Marinescu and Mortici)**.** *Let X,Y be normed spaces and $C\subset X$ convex. Let $F:C\to Y$ be delta-convex with the control-function $f:C\to\mathbb{R}$. Then*

$$\|p_F(s)-p_F(t)\|\leq|p_f(s)-p_f(t)|\,,$$

for all $s,t\in[0,1]$.

2 The results

The delta-convex functions were first introduced by Busemann and Feller [4]. These functions play an important role in non-smooth optimization, especially in situations where standard convexity conditions are too limiting. Since their introduction, delta-convex functions have been widely explored and applied in various areas, including optimization, control theory, and economics.

Let X, Y be normed spaces and $C \subset X$ non-empty and convex. Then a function $F : C \to Y$ is delta-convex if and only if there exists a so called *control-function* $f : C \to Y$ such that, for every $x, y \in C$ and $a \in [0,1]$, the following inequality holds:

$$\begin{aligned}&\left\|(1-a)F(x)+aF(y)-F((1-a)x+ay)\right\|\\ &\leq (1-a)f(x)+af(y)-f((1-a)x+ay).\end{aligned}$$

For details and further properties, see, e.g., [5]- [7] and all references therein.

For every $n \in \mathbb{N}$, $n \geq 2$, x_1, x_2, ..., $x_n \in C$, a_1, a_2, ..., $a_n \geq 0$, with $a_1 + a_2 + ... + a_n = 1$, denote by

$$\overline{x} = \sum_{i=1}^{n} a_i x_i.$$

Let $f : C \to Y$ be a function. The function $p_f : [0,1] \to Y$ defined by:

$$p_f(t) = \sum_{i=1}^{n} a_i f((1-t)x_i + t\overline{x}), \quad \forall\ t \in [0,1],$$

is called *the Pečarić function associated to* f, *and systems* x_1, x_2, ..., x_n and a_1, a_2, ..., a_n.

We are in a position to give the following:

Theorem 3. *Let* $(X, \|\cdot\|)$ *be a real prehilbertian space,* $C \subset X$ *open, convex, be a continuous function* $f : C \to \mathbb{R}$ *and* $L > 0$*. The following assertions are equivalent:*
i) f *is Gâteaux differentiable on* C *and*

$$\left|f(x) - f(y) - < f'(y), x-y >\right| \leq L\|x-y\|^2,$$

for all $x, y \in C$*.*
ii) the following inequality holds true:

$$|p_f(s) - p_f(t)| \leq L\left(\sum_{1\leq i<j\leq n} \lambda_i\lambda_j \|x_i - x_j\|^2\right)|s-t|(2-s-t),$$

for all $\lambda_1, ..., \lambda_n \geq 0$, *with* $\lambda_1 + ... + \lambda_n = 1$ *and* $x_1, ..., x_n \in C$ *and* $s, t \in [0, 1]$.

Proof. i)⇒ii). Let us consider the functions $F_1,\ F_2 : C \to \mathbb{R}$,

$$F_1(x) = L\|x\|^2 - f(x) \quad , \quad F_2(x) = L\|x\|^2 + f(x), \quad x \in C. \tag{1}$$

We have

$$-L\|x - y\|^2 \leq \left| f(x) - f(y) - < f'(y), x - y > \right| \leq L\|x - y\|^2. \tag{2}$$

For arbitrarily fixed y, we deduce from the left-hand side inequality (2) that

$$f(x) + L\|x\|^2 - f(y) - L\|y\|^2 \geq < f'(y) + 2Ly, x - y >,$$

for all $x \in C$. By taking $y^* = f'(y) + 2Ly$, we get

$$F_1(x) - F_1(y) \geq < y^*, x - y >,$$

for all $x \in C$, $y \in Y^*$. Thus F_1 is convex.
By a similar argument, F_2 is also convex, i.e., f is delta-convex with the control-function $g : C \to \mathbb{R}$, $g(x) = L\|x\|^2$.
Further, for all $s, t \in [0, 1]$, we have

$$|p_f(s) - p_f(t)| \leq |p_g(s) - p_g(t)|.$$

By using [10, Theorem 3.1], we get

$$p_g(t) = L\left\|\sum_{i=1}^{n} \lambda_i x_i\right\|^2 + (1 - t)^2 L \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \|x_i - x_j\|^2.$$

In consequence,

$$\begin{aligned} |p_g(s) - p_g(t)| &= L\left|(1 - s)^2 - (1 - t)^2\right| \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \|x_i - x_j\|^2 \\ &= L|s - t|(2 - s - t) \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \|x_i - x_j\|^2. \end{aligned}$$

The implication i)⇒ii) is completely proved.
ii)⇒i). We have

$$\begin{aligned} |(1 - t) f(x) + tf(y) - f((1 - t)x + ty)| &\leq Lt(1 - t)\|x - y\|^2 \\ &= L(1 - t)\|x\|^2 + Lt\|y\|^2 - Lt\|(1 - t)x + ty\|^2, \end{aligned} \tag{3}$$

which means that f is delta-convex with the control-function $g(x) = L\|x\|^2$. As g is Gâteaux differentiable on C, it follows that f is Gâteaux differentiable on C (see [8, Proposition 3.9]). From (3), we deduce that

$$\begin{aligned} -Lt(1-t)\|x-y\|^2 \\ \leq (1-t)[f(x)-f(y)] - [f(y+(1-y)(x-y)) - f(y)] \\ \leq Lt(1-t)\|x-y\|^2, \end{aligned}$$

for all $t \in [0,1]$ and $x, y \in C$. Thus

$$Lt\|x-y\|^2 \leq f(x) - f(y) - \frac{f(y+(1-y)(x-y)) - f(y)}{1-t} \leq Lt\|x-y\|^2.$$

By taking the limit as $t \nearrow 1$, we get:

$$-L\|x-y\|^2 \leq f(x) - f(y) - < f'(y), x-y > \leq L\|x-y\|^2.$$

The proof is completed. □

Theorem 4. *Let $(X, \|\cdot\|)$ be a real prehilbertian space, $C \subset X$ open, convex, $f : C \to \mathbb{R}$ and $L > 0$. The following assertions are equivalent:*
i) f is Fréchet differentiable on C and

$$\|f'(x) - f'(y)\| \leq L\|x-y\|,$$

for all $x, y \in C$.
ii) The following inequality holds true:

$$|p_f(s) - p_f(t)| \leq \frac{L}{2} \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \|x_i - x_j\|^2 |s-t| (2-s-t)$$

for all $\lambda_1, ..., \lambda_n \geq 0$, with $\lambda_1 + ... + \lambda_n = 1$ and $x_1, ..., x_n \in C$ and $s, t \in [0,1]$.

Proof. i)⇒ii). The function f is Gâteaux differentiable and continuous. By [11, Lemma 2.1], we have

$$|f(x) - f(y) - < f'(y), x-y >| \leq \frac{L}{2}\|x-y\|^2.$$

The assertion ii) follows now by Theorem 3.
ii)⇒i). Proceeding like in the proof of Theorem 3, we obtain that f is delta-convex with control-function $h : C \to \mathbb{R}$, $h(x) = \frac{L}{2}\|x\|^2$, $x \in C$.
The function h is Fréchet differentiable, and by using a result stated in [8,

Proposition 3.9], it follows that f is Fréchet differentiable. Further, f is Gâteaux differentiable and according to Theorem 3, we have:

$$\left|f(x) - f(y) - < f'(y), x - y >\right| \leq \frac{L}{2}\|x - y\|^2, \quad x, y \in C.$$

By a result presented in [11, Lemma 2.1], we deduce that the Gâteaux differential f' is L-Lipschitz. As f is Fréchet differentiable, the Gâteaux differential and the Fréchet differentiable coincides. The proof is completed. □

3 Applications

Theorem 5. *Let $(X, \|\cdot\|)$ be a real prehilbertian space, $C \subset X$ open, convex, be a continuous function $f : C \to \mathbb{R}$ and $L > 0$. The following assertions are equivalent:*
i) f is Gâteaux differentiable on C and

$$\left\|f'(x) - f'(y)\right\| \leq L\|x - y\|,$$

for all $x, y \in C$.
ii)

$$\left|f(x) - f(y) - < f'(y), x - y >\right| \leq \frac{L}{2}\|x - y\|^2,$$

for all $x, y \in C$.
iii)

$$\left|\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)\right| \leq \frac{L}{8}\|x - y\|^2, \tag{4}$$

for all $x, y \in C$.

Proof. i)⇒ii) is a result presented in [11, Lemma 2.1].
ii)⇒iii) is a consequence of Theorem 3.
iii)⇒i) The inequality (4) shows that the functions F_1 and F_2 defined in (1) are semiconvex. They are also continuous, so they are convex. Now, by proceeding in a similar way as in the proof of Theorem 3 (implication ii)⇒i)), we deduce that f is Gâteaux differentiable and f' is L-Lipschitz. □

Theorem 6. *Let $(X, \|\cdot\|)$ be a real prehilbertian space, $C \subset X$ open, convex, be a continuous function $f : C \to \mathbb{R}$ and $L > 0$. The following assertions are equivalent:*
i) f is Fréchet differentiable on C and

$$\left\|f'(x) - f'(y)\right\| \leq L\|x - y\|,$$

for all $x, y \in C$.
ii)

$$\left| f(x) - f(y) - < f'(y), x - y > \right| \leq \frac{L}{2} \|x - y\|^2,$$

for all $x, y \in C$.
iii)

$$\left| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right| \leq \frac{L}{8} \|x - y\|^2,$$

for all $x, y \in C$.

Proof. The conclusion follows in a similar way as the proof of Theorem 5, using in this case the results stated in [12, Lemma 2.64] and Theorem 4. □

A consequence of Theorems 5-6 is the following

Theorem 7. *Let* $(X, \|\cdot\|)$ *be a real prehilbertian space,* $C \subset X$ *open, convex, be a continuous function* $f : C \to \mathbb{R}$ *and* $L > 0$. *The following assertions are equivalent:*
i)

$$\left| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right| \leq \frac{L}{8} \|x - y\|^2,$$

for all $x, y \in C$.
iii) f *is Gâteaux differentiable on* C *and*

$$\left\| f'(x) - f'(y) \right\| \leq L \|x - y\|,$$

for all $x, y \in C$.
iii) f *is Fréchet differentiable on* C *and*

$$\left\| f'(x) - f'(y) \right\| \leq L \|x - y\|,$$

for all $x, y \in C$.

Finally, remark that the results stated by Ivan and Raşa [1] and Marumo and Takeda [2] are particular cases of Theorem 6.

Theorem 3 is a generalization of Applications 2.2.-2.3 stated in [6].

Proposition 2.2 presented in [13] is a consequence of Theorem 3.

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