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NONLINEAR ERROR BOUNDS FOR MAPS ON PREORDERED PSEUDOMETRIC SPACES*

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Abstract

We establish sufficient conditions for the existence of nonlinear error bounds for submonotone maps defined on a pseudometric space endowed with a preorder. This covers the case of submonotone maps (thus a fortiori of lower semicontinuous maps) on a metric space (endowed with the trivial preorder). In particular our results generalize the existing results for this case. Our arguments are based on an appropriate version of Ekeland's variational principle.

Keywords: nonlinear error bound, pseudometric space, preorder, submonotone map, variational principle.
MSC: 49J52, 58E30, 06A75.

1 Introduction

In [1] the next theorem is shown:

Theorem 1 ([1, Theorem 4.3]). *Let (M, d) be a complete metric space and let $f : M \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. Let $a \in \mathbb{R}$ and*

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$b \in (a, +\infty]$ with $[f \leq a] \neq \emptyset$, and let $\beta : (0, +\infty) \to (0, +\infty)$ be continuous and nondecreasing. Assume that

$$a < f(x) < b \quad \Rightarrow \quad |\nabla f|(x) \geq \beta(d(x, [f \leq a])).$$

Then,

$$a < f(x) < b \quad \Rightarrow \quad f(x) - a \geq \int_0^{d(x,[f \leq a])} \beta(s)\, ds.$$

In this theorem, $d(x, [f \leq a]) := \inf\{d(x, y) : y \in [f \leq a]\}$ stands for the distance from x to the sublevel set $[f \leq a] := \{x \in M : f(x) \leq a\}$ and $|\nabla f| : \mathrm{dom}(f) \to [0, +\infty]$ is the *strong slope* of f defined by

$$|\nabla f|(x) = \begin{cases} 0 & \text{if } x \text{ is a local minimizer of } f, \\ \limsup\limits_{y \to x} \frac{f(x)-f(y)}{d(x,y)} & \text{otherwise.} \end{cases}$$

One says that f has a *(global) nonlinear error bound* between the levels $a \in \mathbb{R}$ and $b \in (a, +\infty]$ if there is a nondecreasing function $\gamma : (0, +\infty) \to (0, +\infty)$ such that

$$a < f(x) < b \quad \Rightarrow \quad f(x) - a \geq \gamma(d(x, [f \leq a])).$$

Therefore, Theorem 1 provides a sufficient condition for the existence of a global nonlinear error bound.

In the present paper, we obtain (in particular) the following generalization of the above theorem:

Theorem 2. *Let $(P, \leq, d)$ be a preordered pseudometric space such that $\leq$ is self-closed and P is $\leq$-complete. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be $\leq$-submonotone. Let $a \in \mathbb{R}$ and $b \in (a, +\infty]$. Let $\beta : [0, +\infty) \to [0, +\infty)$ be nondecreasing and set $\beta(+\infty) = \lim_{s \to +\infty} \beta(s)$. Assume that*

$$a < f(x) < b \quad \Rightarrow \quad |\nabla_{\leq} f|(x) \geq \beta(d(\geq x, [f \leq a])).$$

Then,

$$a < f(x) < b \quad \Rightarrow \quad f(x) - a \geq \int_0^{d(\geq x,[f \leq a])} \beta(s)\, ds.$$

Moreover, if $\beta \not\equiv 0$ and $\inf_P f < b$, then $[f \leq a] \neq \emptyset$.

See Theorem 6 (combined with Corollary 3) for an in fact more general statement. The setting of preordered pseudometric space and the notions involved in Theorem 2 are introduced in Section 2. In particular a metric space (M, d) endowed with the trivial preorder $\leq$ (i.e., such that $x \leq y$ for all $x, y \in M$) is an example of preordered pseudometric space, and in this case the distance $d(\geq x, [f \leq a])$ and the slope $|\nabla_{\leq} f|$ coincide with $d(x, [f \leq a])$ and $|\nabla f|$, respectively. However, even in this case, the assumption that f is $\leq$-submonotone is weaker than lower semicontinuity (see, e.g., Remark 3 (d)). Therefore, even for metric spaces endowed with trivial preorder, Theorem 2 generalizes Theorem 1. Note also that the condition on β is weaker in Theorem 2 (where it may be discontinuous and may vanish) and the conclusion incorporates the fact that $[f \leq a] \neq \emptyset$ while it is an assumption in Theorem 1.

In [1], there are also local versions of Theorem 1 (namely, [1, Theorems 4.1 and 4.2]) and a linear version ([1, Theorem 2.2]) which are themselves generalized in Theorems 4–5 and Corollary 4 below. The formulation of Theorems 4–5 in fact incorporates both the local and global settings (the global setting is recovered when the "radius" ρ in the statements is set to $+\infty$).

Our arguments are also more elementary in the sense that we do not rely on a change-of-metric principle (which is the basic tool in [1]). This is precisely what allows us to go beyond the setting of metric spaces and the case of a continuous, positive β.

The paper is organized as follows. In Section 2, we present the setting of preordered pseudometric spaces and the relevant notions involved in Theorem 2 and needed throughout the paper. This setting has been mostly introduced in [2], but Section 2 also provides further developments. A basic result in our arguments is a version of Ekeland's variational principle which is given in Section 3 (Theorem 3). This result is already shown in [2] but we provide a full proof for making this paper as self contained as possible.

In Section 4, by relying on Theorem 3, we obtain a key technical result (Proposition 1) which yields a lower estimate of a $\leq$-submonotone map f on a subset U by a quantity which combines the infimum of the map and of its slope $|\nabla_{\leq} f|$ on U. As a byproduct, we show that the slope is generically finite on the domain of f (Corollaries 1–2).

The main results of this paper are shown in Section 5, and provide general local and global criteria of existence of nonlinear or linear error bounds.

2 Preliminaries

Throughout this paper, we consider a (nonempty) set P and the following structure on P:

(a) We assume that P is endowed with a *preorder*, i.e., a binary relation $\leq$ which is reflexive and transitive (but not necessarily antisymmetric).

(b) We also assume that P is endowed with a *pseudometric*, i.e., a map $d : P \times P \to [0, +\infty)$ such that $d(x, x) = 0$ for all $x \in P$. Moreover, we assume that

- d is *symmetric*, i.e., $d(x, y) = d(y, x)$ for all $x, y \in P$;
- d is $\leq$-*triangular*, i.e., $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in P$ such that $x \leq y \leq z$.

Then we say that $(P, \leq, d)$ is a *preordered pseudometric space.*

We consider the following terminology related to sequences in P.

Definition 1. (a) A sequence $(x_n) \subset P$ is said to be $\leq$-*ascending* if $x_n \leq x_{n+1}$ for all n.
(b) Given $x \in P$ and a sequence $(x_n) \subset P$, we say that (x_n) *converges to* x if $\lim_{n\to\infty} d(x_n, x) = 0$. Then we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$. We say that (x_n) is a *Cauchy sequence* if for every $\varepsilon > 0$ there is a rank n_0 such that $d(x_n, x_m) < \varepsilon$ whenever $n_0 \leq n \leq m$. Then we say that P is $\leq$-*complete* if every $\leq$-ascending Cauchy sequence in P is convergent in P.
(c) We say that the preorder $\leq$ is *self-closed* if, whenever $(x_n) \subset P$ is $\leq$-ascending and such that $x_n \to x$, we have $x_n \leq x$ for all n.

Remark 1. (a) A sequence (x_n) can have several limits. When there are $x, y \in P$ with $x \neq y$ such that $d(x, y) = 0$, the constant sequence defined by $x_n := x$ converges to both x and y. In fact, even if d is nondegenerate (i.e., $d(x, y) = 0 \Rightarrow x = y$), the limit is a priori not unique: let $P = [0, 1] \cup \{2\}$ be endowed with the standard order and the pseudometric d whose restriction to $[0, 1]$ is the standard metric and such that $d(1, 2) = d(2, 1) = 1$ and $d(x, 2) = d(2, x) = 1 - x$ for all $x \in [0, 1)$. With this definition, $(P, \leq, d)$ is a preordered pseudometric space, with nondegenerate pseudometric d, note also that P is $\leq$-complete and $\leq$ is self-closed. However, the sequence $(1 - \frac{1}{n})$ has two limits: 1 and 2. In this example, d is not a metric as it is not triangular: $d(1, 2) > d(1, \frac{3}{4}) + d(\frac{3}{4}, 2)$.

Assuming that d is nondegenerate is nevertheless sufficient for guaranteeing that every constant sequence has a unique limit.
(b) In what follows, it will be often useful to consider also the reversed preorder $\geq$. Note that $(P, \geq, d)$ is also a preordered pseudometric space (in particular the $\geq$-triangularity of d is deduced from the $\leq$-triangularity thanks to the fact that d has been supposed symmetric). However, the

properties of $(P, \leq, d)$ are not necessarily preserved. For instance $\leq$ may be self-closed whereas $\geq$ is not: take $P = [0, 1]$ endowed with the standard metric and with the order $\leq$ whose restriction to $(0, 1]$ is the standard order and such that $x \leq 0$ for all x. Also P may be $\leq$-complete without being $\geq$-complete: take for instance $P = (0, 1]$ equipped with the standard metric and order.

Our next task is to endow P with a topology. In fact we define various topologies on P.

Definition 2. Let $A \subset P$ be a subset.
(a) Let $L_{\leq}(A) = \{x \in P : \exists (x_n) \subset A \ \leq\text{-ascending},\ x_n \to x\}$. We say that A is $\leq$-*closed* if $A = L_{\leq}(A)$.
(b) Let $S_{\leq}(A) = \{x \in P : \forall \varepsilon > 0,\ \exists x' \in A,\ x' \leq x \text{ and } d(x', x) < \varepsilon\}$. We say that A is $\leq$-*saturated* if $A = S_{\leq}(A)$.
(c) Given $x \in P$, we set $d(\leq x, A) = \inf\{d(x', x) : x' \in A,\ x' \leq x\} \in [0, +\infty]$.

Lemma 1. (a) *$\leq$-closed subsets and $\leq$-saturated subsets are the respective closed sets of two topologies on P, which we will respectively call* $\leq$-topology *and* $S_{\leq}$-topology.
(b) *For every subset $A \subset P$, letting $\overline{A}^{\leq}$ be the closure of A in the $\leq$-topology, we have the inclusions*

$$A \subset L_{\leq}(A) \subset \overline{A}^{\leq}.$$

Moreover, $A = L_{\leq}(A) \Leftrightarrow A = \overline{A}^{\leq}$.
(c) *For every subset $A \subset P$, we have that $S_{\leq}(A)$ is the closure of A in the $S_{\leq}$-topology. Moreover, for all $x \in P$, we have $d(\leq x, A) = d(\leq x, S_{\leq}(A))$, and*

$$S_{\leq}(A) = \{x \in P : d(\leq x, A) = 0\}.$$

(d) *Assume that $\leq$ is self-closed. Then for every $A \subset P$, we have the inclusions*

$$A \subset L_{\leq}(A) \subset \overline{A}^{\leq} \subset S_{\leq}(A).$$

In particular, if A is $\leq$-saturated then A is $\leq$-closed, so that the $\leq$-topology is finer than the $S_{\leq}$-topology.

Proof. (a) Evidently, $\emptyset$ and P are both $\leq$-closed and $\leq$-saturated. Let A_i, $i \in I$, be a collection of $\leq$-closed subsets. Since the map $L_{\leq}$ is clearly nondecreasing with respect to inclusion, we get $\bigcap_{i \in I} A_i \subset L_{\leq}(\bigcap_{i \in I} A_i) \subset \bigcap_{i \in I} L_{\leq}(A_i) = \bigcap_{i \in I} A_i$ hence $\bigcap_{i \in I} A_i$ is $\leq$-closed. The argument for showing that an intersection of $\leq$-saturated subsets is $\leq$-saturated is similar.

Let $A, B \subset P$ be $\leq$-closed subsets and let $(x_n) \subset A \cup B$ be a $\leq$-ascending sequence such that $x_n \to x$ with $x \in P$. The sequence (x_n) has either a subsequence in A or in B which implies that $x \in A$ or $x \in B$, hence $x \in A \cup B$. This shows that $A \cup B$ is $\leq$-closed.

Now assume that A, B are $\leq$-saturated and let $x \in S_{\leq}(A \cup B)$. If $x \in S_{\leq}(A) = A$ then we get in particular $x \in A \cup B$. So assume that $x \notin S_{\leq}(A)$, i.e., there is $\varepsilon_0 > 0$ such that $\{x' \in A : x' \leq x,\ d(x', x) < \varepsilon_0\} = \emptyset$. For every $\varepsilon \in (0, \varepsilon_0)$ there is $x' \in A \cup B$ with $x' \leq x$ and $d(x', x) < \varepsilon$, and then necessarily $x' \in B$; hence $x \in S_{\leq}(B) = B \subset A \cup B$. Finally we get that $A \cup B$ is $\leq$-saturated.

(b) The inclusion $A \subset L_{\leq}(A)$ is immediate. As for the second inclusion, we have $L_{\leq}(A) \subset L_{\leq}(\overline{A}^{\leq}) = \overline{A}^{\leq}$. The equivalence $A = L_{\leq}(A) \Leftrightarrow A = \overline{A}^{\leq}$ follows from the fact that each one of the two equalities means that A is $\leq$-closed.

(c) The inequality $d(\leq x, A) < \varepsilon$ means that there is $x' \in A$ with $x' \leq x$ and $d(x', x) < \varepsilon$, hence

$$S_{\leq}(A) = \{x \in P : \forall \varepsilon > 0,\ d(\leq x, A) < \varepsilon\} = \{x \in P : d(\leq x, A) = 0\}.$$

For every $x \in P$, since $A \subset S_{\leq}(A)$, we have the inequality $d(\leq x, S_{\leq}(A)) \leq d(\leq x, A)$. For every $y \in S_{\leq}(A)$ with $y \leq x$, using that d is $\leq$-triangular, it is easy to see that $d(\leq x, A) \leq d(\leq y, A) + d(y, x) = d(y, x)$ (in view of the above description of $S_{\leq}(A)$); whence, finally, $d(\leq x, A) = d(\leq x, S_{\leq}(A))$.

The last equality, combined with the above description of $S_{\leq}(A)$, yields $S_{\leq}(S_{\leq}(A)) = S_{\leq}(A)$, hence $S_{\leq}(A)$ is $\leq$-saturated. If $A \subset B$ where B is $\leq$-saturated, then we get $S_{\leq}(A) \subset S_{\leq}(B) = B$. Hence $S_{\leq}(A)$ is the closure of A in the $S_{\leq}$-topology.

(d) Assume that $\leq$ is self-closed. It remains to show the inclusion $\overline{A}^{\leq} \subset S_{\leq}(A)$. To do this, it suffices to check that $S_{\leq}(A)$ is $\leq$-closed. So let $(x_n) \subset S_{\leq}(A)$ be a $\leq$-ascending sequence such that $x_n \to x$ with $x \in P$. The fact that $\leq$ is self-closed yields $x_n \leq x$ for all n. Let $\varepsilon > 0$. There is n such that $d(x_n, x) < \frac{\varepsilon}{2}$. Moreover, since $x_n \in S_{\leq}(A)$, we can find $x' \in A$ with $x' \leq x_n$ and $d(x', x_n) < \frac{\varepsilon}{2}$. Whence $x' \leq x_n \leq x$ and $d(x', x) < \varepsilon$. This shows that $x \in S_{\leq}(A)$, and the proof of the lemma is complete. □

Remark 2. (a) In addition to the $\leq$- and $S_{\leq}$-topologies on P, we get two additional topologies simply by switching the preorder $\leq$ to the opposite preorder $\geq$ (see Remark 1 (b)). Also we define

$$d(\geq x, A) = \inf\{d(x, y) : y \in A,\ y \geq x\}.$$

Then, in view of Lemma 1 (c), the closure of A in the $S_{\geq}$-topology is given by $S_{\geq}(A) = \{x \in P : d(\geq x, A) = 0\}$.

Of course the $\leq$- and $\geq$-topologies (as well as the $S_{\leq}$- and $S_{\geq}$-topologies) are different in general. For example, for $P = [0,1]$ equipped with its standard metric and order, the subset $A := (0,1]$ is $\leq$-closed and $\leq$-saturated (since $d(\leq 0, A) = +\infty$) but it is not $\geq$-closed and not $\geq$-saturated (since $d(\geq 0, A) = 0$). Symmetrically, $B := [0,1)$ is closed in the $\geq$- and $S_{\geq}$-topologies but not in the $\leq$- and $S_{\leq}$-topologies.
(b) The inclusion $\overline{A}^{\leq} \subset S_{\leq}(A)$ does not hold in general without the assumption that $\leq$ is self-closed: take for instance $P = [0,1]$ equipped with its standard metric and with the order $\leq$ whose restriction to $[0,1)$ is the standard order and such that $1 \leq s$ for all s (the so-obtained order on $[0,1]$ is not self-closed). Then $A := [0,1)$ is $\leq$-saturated but $\overline{A}^{\leq} = [0,1]$.

Also the reversed inclusion may not hold even when $\leq$ is self-closed: take $P = [0,1]$ equipped with its standard metric and with the order $\leq$ whose restriction to $(0,1]$ is the standard order and such that $s \leq 0$ for all s. Then $\leq$ is self-closed, $A := (0,1]$ is $\leq$-closed, but $S_{\leq}(A) = [0,1]$. This example also shows that, when $\leq$ is self-closed, the $\leq$-topology is in general strictly finer than the $S_{\leq}$-topology.
(c) The inclusions $A \subset L_{\leq}(A) \subset \overline{A}^{\leq}$ can be strict. Take for example $P = [0,1] \times [0,1]$ equipped with its standard metric and with the (self-closed) partial order defined by

$$(x,y) \leq (x',y') \quad \text{if} \quad (x = x' \text{ and } y \leq y') \text{ or } (y = y' = 1 \text{ and } x \leq x').$$

Then $A := [0,1) \times [0,1) \subsetneq L_{\leq}(A) = [0,1) \times [0,1] \subsetneq \overline{A}^{\leq} = L_{\leq}(A) \cup \{(1,1)\}$.
(d) Say that a subset $A \subset P$ is *closed* if for every $x \in P$ and $(x_n) \subset A$ with $x_n \to x$, we have $x \in A$. It is straightforward to check that this defines a topology on P, which we call *d-topology*, and which is coarser than both the $\leq$-topology and the $S_{\leq}$-topology. However, the d-topology is less relevant and thus not involved in this paper.

In the case where we are given a map $f : P \to \mathbb{R} \cup \{+\infty\}$, it is sometimes useful to modify the preorder $\leq$ to a new preorder $\preceq$ according to Lemma 2 below. First we present the following notions (already considered in [2]).

Definition 3. (a) The $\leq$-*strong slope* of f is the operator

$$|\nabla_{\leq} f| : \mathrm{dom}(f) := f^{-1}(\mathbb{R}) \to [0, +\infty]$$

defined as follows. For $\eta > 0$ let $P^0_\eta(x \leq) = \{y \in P : x \leq y,\ 0 < d(x,y) < \eta\}$.

- If x is a global minimizer of $f|_{P^0_\eta(x\leq)\cup\{x\}}$ for some $\eta > 0$, we set $|\nabla_{\leq} f|(x) = 0$.
- Otherwise, we set

$$|\nabla_{\leq} f|(x) = \lim_{\eta\to 0^+} \sup\Big\{\frac{f(x)-f(y)}{d(x,y)} : y \in P^0_\eta(x \leq)\Big\}.$$

(b) We say that $f : P \to \mathbb{R}\cup\{+\infty\}$ is $\leq$-*lower semicontinuous* if, for every $\leq$-ascending sequence (x_n) such that $x_n \to x$ with $x \in P$, we have $\liminf_{n\to\infty} f(x_n) \geq f(x)$.
(c) We say that $f : P \to \mathbb{R}\cup\{+\infty\}$ is $\leq$-*submonotone* if, for every $\leq$-ascending sequence (x_n) such that $(f(x_n))$ is nonincreasing and $x_n \to x$ with $x \in P$, we have $f(x_n) \geq f(x)$ for all n.

Remark 3. (a) If $x \in P$ is maximal with respect to $\leq$, then $|\nabla_{\leq} f|(x) = 0$.
(b) In the case where $P \subset \mathbb{R}$ is an open interval endowed with its standard metric and order and $f : P \to \mathbb{R}$ is derivable on the right at $x \in \operatorname{dom}(f)$, then $|\nabla_{\leq} f|(x) = \max\{0, -f'_r(x)\}$ where $f'_r(x)$ stands for the right derivative at x. Symmetrically, if f is derivable on the left at x with left derivative $f'_l(x)$, then $|\nabla_{\geq} f|(x) = \max\{0, f'_l(x)\}$.

In general, thanks to the preorder $\leq$ and the strong slope $|\nabla_{\leq} f|$ (or $|\nabla_{\geq} f|$), we get a notion of "right (or left) derivative" for functions on a preordered pseudometric space P.
(c) Given $f : P \to \mathbb{R}\cup\{+\infty\}$, it is easy to see that f is $\leq$-lower semicontinuous if and only if, for every $a \in \mathbb{R}$, the sublevel set $[f \leq a]$ is $\leq$-closed (which equivalently means that f is lower semicontinuous with respect to the $\leq$-topology).
(d) Also it is immediate that the following implication holds

$$f \text{ is } \leq\text{-lower semicontinuous} \quad \Rightarrow \quad f \text{ is } \leq\text{-submonotone}.$$

However, the converse is not true, even if $\leq$ is the trivial preorder on P (i.e., such that $x \leq y$ for all x, y): take for example $f : [0,1] \to \mathbb{R}$ such that $f(x) = x$ for all $x \in [0,1)$ and $f(1) = 2$, and endow $P := [0,1]$ with its standard metric; then f is $\leq$-submonotone with respect to any preorder on P, but f is not $\leq$-lower semicontinuous when $\leq$ is such that $1 \in L_{\leq}([0,1))$.

In a general preordered pseudometric space $(P, \leq, d)$, the class of $\leq$-lower semicontinuous functions of course includes the class of lower semicontinuous functions (i.e., which are lower semicontinuous with respect to the trivial preorder). It also includes the class of $\leq$-nonincreasing functions (i.e., such that $x \leq y \Rightarrow f(x) \geq f(y)$) provided that the preorder $\leq$ is self-closed.

When d is nondegenerate (i.e., $d(x,y) = 0 \Rightarrow x = y$) the class of $\leq$-submonotone maps includes in addition the class of $\leq$-increasing functions (i.e., such that $(x \leq y,\ x \neq y) \Rightarrow f(x) < f(y)$). Indeed, for such an f, every $\leq$-ascending sequence (x_n) such that $(f(x_n))$ is nonincreasing is necessarily constant; and if $x_n \to x$, then the fact that d is nondegenerate forces $x = x_n$ for all n.

Lemma 2. *Let $f : P \to \mathbb{R} \cup \{+\infty\}$. We define*

$$x \preceq y \quad \text{if} \quad (x \leq y \quad \text{and} \quad f(x) \geq f(y)).$$

Then:
(a) *$(P, \preceq, d)$ is a preordered pseudometric space.*
(b) *If P is $\leq$-complete, then P is $\preceq$-complete.*
(c) *$|\nabla_{\leq} f|(x) = |\nabla_{\preceq} f|(x)$ for all $x \in \operatorname{dom}(f)$.*
(d) *The following conditions are equivalent:*
  (i) *f is $\leq$-submonotone;*
  (ii) *f is $\preceq$-submonotone;*
  (iii) *f is $\preceq$-lower semicontinuous;*
  (iv) *for all $a \in \mathbb{R}$, $[f \leq a]$ is $\preceq$-closed;*
  (v) *for all $a \in \mathbb{R} \cup \{+\infty\}$, $[f < a] := \{x \in P : f(x) < a\}$ is $\preceq$-closed.*
(e) *If $\leq$ is self-closed and f is $\leq$-submonotone, then $\preceq$ is self-closed.*
(f) *Let $A \subset P$ and let $x \in P$. If $f(x) \leq \inf\{f(y) : y \in A,\ y \leq x\}$, then $d(\leq x, A) = d(\preceq x, A)$. Similarly, if $f(x) \geq \sup\{f(y) : y \in A,\ y \geq x\}$, then $d(\geq x, A) = d(\succeq x, A)$.*

Proof. (a) is straightforward.

(b) Let $(x_n) \subset P$ be a Cauchy sequence which is $\preceq$-ascending. Then a fortiori (x_n) is $\leq$-ascending, which implies that (x_n) is convergent.

(c) Assume that x is a global minimizer of $f|_{P^0_\eta(x\leq)\cup\{x\}}$ for some $\eta > 0$. Since $P^0_\eta(x \preceq) \subset P^0_\eta(x \leq)$, this implies that x is also a global minimizer of the restriction of f to $P^0_\eta(x \preceq) \cup \{x\}$. Whence $|\nabla_{\leq} f|(x) = |\nabla_{\preceq} f|(x) = 0$ in this case. Now assume that for every $\eta > 0$, x is not a global minimizer of the restriction of f to $P^0_\eta(x \leq) \cup \{x\}$. This yields $y_\eta \in P^0_\eta(x \leq)$ with $f(x) > f(y_\eta)$. Then $x \preceq y_\eta$. Hence x is not a global minimizer of the restriction of f to $P_\eta^0(x \preceq) \cup \{x\}$. In this case, we have

$$\begin{aligned} |\nabla_{\leq} f|(x) &= \lim_{\eta\to 0^+} \sup\left\{\frac{f(x)-f(y)}{d(x,y)} : y \in P_\eta^0(x \leq)\right\} \\ &= \lim_{\eta\to 0^+} \sup\left\{\frac{f(x)-f(y)}{d(x,y)} : y \in P_\eta^0(x \leq),\ f(x) \geq f(y)\right\} \\ &= \lim_{\eta\to 0^+} \sup\left\{\frac{f(x)-f(y)}{d(x,y)} : y \in P_\eta^0(x \preceq)\right\} = |\nabla_{\preceq} f|(x). \end{aligned}$$

(d) By definition of the preorder $\preceq$, a sequence $(x_n) \subset P$ is $\preceq$-ascending if and only if (x_n) is $\leq$-ascending and $(f(x_n))$ is nonincreasing. The equivalence (i)$\Leftrightarrow$(ii) easily follows from this observation. By Remark 3 (c)–(d), one has (iv)$\Leftrightarrow$(iii)$\Rightarrow$(ii). The implication (v)$\Rightarrow$(iv) comes from the fact that we can write $[f \leq a] = \bigcap_{b>a} [f < b]$. It remains to show that (i)$\Rightarrow$(v). To do this, let (x_n) in $[f < a]$ be a $\preceq$-ascending sequence such that $x_n \to x$ with $x \in P$. Thus (x_n) is $\leq$-ascending and $(f(x_n))$ is nonincreasing. Since f is $\leq$-submonotone, this implies that $f(x) \leq f(x_n)$ for all n, thus $x \in [f < a]$.

(e) Let $(x_n) \subset P$ be a $\preceq$-ascending sequence such that $x_n \to x$. This means that (x_n) is $\leq$-ascending and $(f(x_n))$ is nonincreasing. Since $\leq$ is self-closed, we deduce that $x_n \leq x$ for all n, and since f is $\leq$-submonotone, we also deduce that $f(x_n) \geq f(x)$ for all n. Therefore, $x_n \preceq x$ for all n.

(f) The assumption implies that

$$\{y \in A : y \leq x\} = \{y \in A : y \leq x \text{ and } f(y) \geq f(x)\} = \{y \in A : y \preceq x\}.$$

Whence $d(\leq x, A) = d(\preceq x, A)$. The proof of the second part of (f) is similar. □

Lemma 3. *Given a $\leq$-submonotone map $f : P \to \mathbb{R} \cup \{+\infty\}$, we consider the preorder $\preceq$ defined in Lemma 2. We consider the map $f^+ : P \to [0, +\infty]$, $x \mapsto f(x)^+ := \max\{0, f(x)\}$. Then:*
(a) *f^+ is $\preceq$-submonotone.*
(b) *For all $x \in \operatorname{dom}(f)$, we have*

$$|\nabla_{\preceq}(f^+)|(x) = |\nabla_{\leq}(f^+)|(x) = \begin{cases} 0 & \text{if } f(x) \leq 0, \\ |\nabla_{\leq} f|(x) & \text{if } f(x) > 0. \end{cases}$$

Proof. (a) Let (x_n) be a $\preceq$-ascending sequence such that $(f^+(x_n))$ is nonincreasing and $x_n \to x$ with $x \in P$. Thus (x_n) is $\leq$-ascending and $(f(x_n))$ is nonincreasing. Since f is $\leq$-submonotone, we deduce that $f(x) \leq f(x_n)$ for all n. Whence $f^+(x) \leq f^+(x_n)$ for all n. This shows (a).

(b) Let $x \in \mathrm{dom}(f)$ be such that $f(x) \leq 0$. Then $f^+(x) = 0$ so that x is a global minimizer of f^+ on P, thus a fortiori on $P^0_\eta(x \preceq) \cup \{x\}$ and $P^0_\eta(x \leq) \cup \{x\}$ for all $\eta > 0$, and this yields $|\nabla_{\preceq}(f^+)|(x) = |\nabla_{\leq}(f^+)|(x) = 0$. Next let $x \in \mathrm{dom}(f)$ be such that $f(x) > 0$. We distinguish two cases.

- First case: for every $\eta > 0$, $P^0_\eta(x \leq) \cap [f \leq 0] \neq \emptyset$. Then take $z_\eta \in P^0_\eta(x \leq)$ such that $f(z_\eta) \leq 0$. In particular $f(z_\eta) < f(x)$ hence $x \preceq z_\eta$, whence $z_\eta \in P^0_\eta(x \preceq)$. Also x is not a global minimizer of f neither of f^+ on $P^0_\eta(x \leq) \cup \{x\}$ or on $P^0_\eta(x \preceq) \cup \{x\}$. We have
$$\frac{f(x) - f(z_\eta)}{d(x, z_\eta)} \geq \frac{f^+(x) - f^+(z_\eta)}{d(x, z_\eta)} = \frac{f(x)}{d(x, z_\eta)} \geq \frac{f(x)}{\eta}.$$
Whence $|\nabla_{\leq} f|(x) = |\nabla_{\leq}(f^+)|(x) = |\nabla_{\preceq}(f^+)|(x) = +\infty$.
- Second case: $P^0_\eta(x \leq) \cap [f \leq 0] = \emptyset$ whenever $\eta > 0$ is small enough. This implies that the restrictions of f and f^+ to $P^0_\eta(x \leq)$ thus a fortiori to $P^0_\eta(x \preceq)$ coincide. Whence $|\nabla_{\leq}(f^+)|(x) = |\nabla_{\leq} f|(x) = |\nabla_{\preceq} f|(x) = |\nabla_{\preceq}(f^+)|(x)$, where we use also Lemma 2 (c).

The proof of the lemma is complete. □

3 Ekeland's variational principle

We will use the following version of Ekeland's variational principle:

Theorem 3 ([2, §3]). *Let $(P, \leq, d)$ be a preordered pseudometric space such that $\leq$ is self-closed and P is $\leq$-complete. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be a $\leq$-submonotone map such that $\inf_P f \in \mathbb{R}$. Let $x \in \mathrm{dom}(f)$ and $\eta \in (0, +\infty)$. Then, there exists $y \in \mathrm{dom}(f)$ satisfying the following conditions:*

(a) *$x \leq y$;*

(b) *$\eta d(x, y) \leq f(x) - f(y)$;*

(c) *for all $(z, z') \in \mathrm{dom}(f) \times P$ such that $y \leq z \leq z'$ and $d(z, z') \neq 0$, we have*
$$\eta d(y, z) > f(y) - f(z) \quad \text{or} \quad \eta d(z, z') > f(z) - f(z').$$

Proof. The result is shown in [2, §3]. We give also a proof here for the sake of completeness. We consider the subset $P' := \{z \in P : f(z) \leq f(x)\}$ equipped with the relation $\leq'$ defined by
$$z \leq' w \quad \text{if} \quad z \leq w \ \text{ and } \ \eta d(z, w) \leq f(z) - f(w).$$

It is easy to see that $(P', \leq', d)$ is a preordered pseudometric space. Whenever $(z_n) \subset P'$ is $\leq'$-ascending, we claim that

$$(z_n) \text{ is a Cauchy sequence} \tag{1}$$

and

$$\text{there is } z \in P' \text{ such that } z_n \leq' z \text{ for all } n. \tag{2}$$

Based on (1) and (2), we can invoke [3, Theorem 2] which asserts that P' contains an element y with $x \leq' y$ and y is *d-maximal* in the sense that whenever $y \leq' z \leq' w$ for some $z, w \in P'$, we have $d(z, w) = 0$. It is easy to check that y then fulfills conditions (a)–(c) of the present theorem. Therefore, it remains to show (1) and (2).

The fact that (z_n) is $\leq'$-ascending implies that

$$\eta d(z_n, z_m) \leq f(z_n) - f(z_m) \quad \text{whenever } n \leq m, \tag{3}$$

so in particular $(f(z_n)) \subset \mathbb{R}$ is nonincreasing. Since $\inf_P f \in \mathbb{R}$, this implies that $(f(z_n))$ is convergent, thus a Cauchy sequence in $\mathbb{R}$. Then (1) is implied by (3).

Since (z_n) is $\leq'$-ascending, it is a fortiori $\leq$-ascending. Since P is $\leq$-complete, due to (1), there is an element $z \in P$ such that $z_n \to z$. Using that f is $\leq$-submonotone, we get also $f(z) \leq f(z_n)$ for all n, thus $f(z) \leq f(x)$ and so $z \in P'$.

Moreover, knowing that $\leq$ is self-closed, we have $z_n \leq z$ for all n. Using (3) and the fact that d is $\leq$-triangular, whenever $n \leq m$ we see that

$$\eta d(z_n, z) - \eta d(z_m, z) \leq \eta d(z_n, z_m) \leq f(z_n) - f(z_m) \leq f(z_n) - f(z).$$

Letting $m \to \infty$, we derive $\eta d(z_n, z) \leq f(z_n) - f(z)$ and finally $z_n \leq' z$ for all n. This shows (2) and the proof of the theorem is complete. □

4 Density of points with finite strong slope

We start this section with a proposition which is a key result towards the proof of our main theorems in Section 5. As a side consequence of the proposition, in Corollaries 1–2 we show the density of points with finite strong slope, for the $S_{\geq}$-topology (thus a fortiori for the d-topology; see Remark 2 (d)).

Proposition 1. *Let $(P, \leq, d)$ be a preordered pseudometric space such that P is $\leq$-complete and $\leq$ is self-closed. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be $\leq$-submonotone. Let $U \subset P$ be a subset such that*

$$U \cap \operatorname{dom}(f) \neq \emptyset \quad \textit{and} \quad \forall x \in U \cap \operatorname{dom}(f),\ d(\geq x, P \setminus U) > 0.$$

Then,

$$\forall x \in U \cap \operatorname{dom}(f), \quad f(x) - \inf_U f \geq \Big(\inf_{U \cap \operatorname{dom}(f)} |\nabla_{\leq} f| \Big) d(\geq x, P \setminus U)$$

with the convention that the right-hand side is zero if $\inf_{U \cap \operatorname{dom}(f)} |\nabla_{\leq} f| = 0$ *and* $d(\geq x, P \setminus U) = +\infty$.

Proof. Let $x \in U \cap \operatorname{dom}(f)$. Set $\mu = \inf_U f$. We can assume $\mu > -\infty$. Let $r \in (0, +\infty)$ be such that $r < d(\geq x, P \setminus U)$. Let $\sigma \in \mathbb{R}$ be such that $\sigma > \frac{f(x) - \mu}{r} (\geq 0)$. Let $g = \mu + (f - \mu)^+ : P \to \mathbb{R} \cup \{+\infty\}$ so that

$$g(z) = \begin{cases} f(z) & \text{if } z \in U, \\ \max\{\mu, f(z)\} & \text{if } z \in P \setminus U. \end{cases}$$

In this way

$$\inf_P g = \mu. \tag{4}$$

We consider the preorder $\preceq$ defined in Lemma 2. It follows from Lemma 3 that

$$g \text{ is } \preceq\text{-submonotone} \tag{5}$$

and

$$\forall z \in \operatorname{dom}(g) = \operatorname{dom}(f),\ |\nabla_{\preceq} g|(z) = |\nabla_{\leq} g|(z) = \begin{cases} 0 & \text{if } f(z) \leq \mu, \\ |\nabla_{\leq} f|(z) & \text{if } f(z) > \mu. \end{cases}$$

We apply Theorem 3 to the space $(P, \preceq, d)$ (which is possible due to Lemma 2 (a), (b), (e)), the map g, and with $\eta = \sigma$, and this yields $y \in \operatorname{dom}(g) = \operatorname{dom}(f)$ such that

- $x \preceq y$, i.e., $x \leq y$ and $f(x) \geq f(y)$;
- $\sigma d(x, y) \leq g(x) - g(y)$;
- for all $z \in P$ such that $y \preceq z$ and $d(y, z) \neq 0$, we have $\sigma d(y, z) > g(y) - g(z)$.

The last point implies

$$\forall z \in P, \quad (y \leq z \quad \text{and} \quad d(y,z) \neq 0) \quad \Rightarrow \quad \sigma d(y,z) > g(y) - g(z) \tag{6}$$

(noting that, if $y \leq z$ but $y \not\preceq z$ then we have $f(y) < f(z)$, in which case the last inequality of (6) is immediate). Since $x \in U \cap \operatorname{dom}(f)$, we have $g(x) = f(x)$, hence $g(x) < \mu + \sigma r$ due to the choice of σ. Hence

$$d(x,y) \leq \frac{g(x) - g(y)}{\sigma} < \frac{(\mu + \sigma r) - \mu}{\sigma} = r.$$

Since $r < d(\geq x, P \setminus U)$ and $x \leq y$, we must have $y \in U$, hence $g(y) = f(y)$.

Next we claim that

$$|\nabla_{\leq} f|(y) \leq \sigma. \tag{7}$$

Indeed, if the restriction of f to $P^0_\eta(y \leq) \cup \{y\}$ has its minimum at y for some $\eta > 0$, then $|\nabla_{\leq} f|(y) = 0$, and the inequality is clear. Now assume that y is not a point of minimum of the restriction of f to $P^0_\eta(y \leq) \cup \{y\}$ for any $\eta > 0$, so that $P^0_\eta(y \leq)$ is nonempty for all $\eta > 0$ and we have

$$|\nabla_{\leq} f|(y) = \lim_{\eta \to 0^+} \sup_{z \in P^0_\eta(y \leq)} \frac{f(y) - f(z)}{d(y,z)}.$$

By the assumption made in the proposition, we have $\delta := d(\geq y, P \setminus U) > 0$. Let $\eta \in (0, \delta)$. For every $z \in P^0_\eta(y \leq)$, we have $y \leq z$ and $0 < d(y,z) < \eta < d(\geq y, P \setminus U)$, which ensures that $z \in U$ and so $g(z) = f(z)$. By (6), we deduce that

$$\forall \eta \in (0, \delta), \quad \forall z \in P^0_\eta(y \leq), \quad \frac{f(y) - f(z)}{d(y,z)} = \frac{g(y) - g(z)}{d(y,z)} \leq \sigma.$$

Whence $|\nabla_{\leq} f|(y) \leq \sigma$. This establishes (7).

Using that $y \in U \cap \operatorname{dom}(f)$, we get $\inf_{U \cap \operatorname{dom}(f)} |\nabla_{\leq} f| \leq \sigma$. Since $\sigma \in (\frac{f(x) - \mu}{r}, +\infty)$ is arbitrary, we conclude that

$$\inf_{U \cap \operatorname{dom}(f)} |\nabla_{\leq} f| \leq \frac{f(x) - \mu}{r} = \frac{f(x) - \inf_U f}{r}.$$

Since $r \in (0, d(\geq x, P \setminus U))$ is arbitrary, we deduce the desired formula. □

Corollary 1. *Assume that* $(P, \leq, d)$ *is* $\leq$*-complete and* $\leq$ *is self-closed. Let* $f : P \to \mathbb{R} \cup \{+\infty\}$ *be a* $\leq$*-submonotone map. For every subset* $U \subset P$ *such that*

$$U \cap \operatorname{dom}(f) \neq \emptyset \quad \textit{and} \quad \forall x \in U \cap \operatorname{dom}(f), \quad d(\geq x, P \setminus U) > 0, \tag{8}$$

we have

$$\inf_{U\cap \mathrm{dom}(f)} |\nabla_{\leq} f| \in [0,+\infty).$$

Proof. We claim that

$$\exists a\in\mathbb{R},\quad \{x\in U\cap\mathrm{dom}(f): d(\geq x,[f<a])>0\}\neq\emptyset. \tag{9}$$

Arguing by contradiction, assume that (9) is not true. This implies that

$$\forall x\in U\cap\mathrm{dom}(f),\ \ \forall a\in\mathbb{R},\ \ \forall\varepsilon>0,\ \ \exists z\in[f<a],\ \ x\leq z\ \text{ and }\ d(x,z)<\varepsilon. \tag{10}$$

We construct a sequence $(x_n)\subset U\cap\mathrm{dom}(f)$ by induction:

- Choose $x_0\in U\cap\mathrm{dom}(f)$ (see (8)).
- Assuming that x_n has been defined, by applying (10), we obtain an element x_{n+1} such that

  $$f(x_{n+1})<\min\{f(x_n),-n\},\qquad x_n\leq x_{n+1},$$
  $$d(x_n,x_{n+1})<\min\{d(\geq x_n,P\setminus U),2^{-n}\}.$$

  These inequalities imply in particular that $x_{n+1}\in U\cap\mathrm{dom}(f)$.

In this way, the sequence (x_n) satisfies that

$$(x_n)\text{ is }\leq\text{-ascending},\quad (f(x_n))\text{ is decreasing},\quad \lim_{n\to\infty} f(x_n)=-\infty, \tag{11}$$

and moreover $d(x_n,x_{n+1})<2^{-n}$ for all n. The last inequality implies that (x_n) is a Cauchy sequence. Since P is $\leq$-complete (and (x_n) is $\leq$-ascending), there is $x\in P$ such that $x_n\to x$. The first two assertions in (11) combined with the assumption that f is $\leq$-submonotone imply that $f(x)\leq f(x_n)$ for all n, but this is impossible in view of the last part of (11). We have shown (9).

With $a\in\mathbb{R}$ provided by (9), we consider the subset

$$V:=\{x\in U: d(\geq x,[f<a])>0\}=U\setminus S_{\geq}([f<a])$$

(see Lemma 1 (c) or Remark 2 (a)). By (9), we have in particular that $V\cap\mathrm{dom}(f)\neq\emptyset$. Moreover, for every $x\in V\cap\mathrm{dom}(f)$, using that $P\setminus V=(P\setminus U)\cup S_{\geq}([f<a])$, we have

$$d(\geq x,P\setminus V)=\min\{d(\geq x,P\setminus U),d(\geq x,S_{\geq}([f<a]))\}>0$$

since we know that $d(\geq x, P \setminus U) > 0$ (see (8)) and $d(\geq x, S_{\geq}([f < a])) = d(\geq x, [f < a]) > 0$ (where we use Lemma 1 (c), applied with the preorder $\geq$ instead of $\leq$, and the fact that $x \in V$). Note that $\inf_V f \geq a$. By applying Proposition 1 to the subset V (instead of U), for a chosen $x \in V \cap \mathrm{dom}(f)$, we obtain the inequalities

$$\inf_{U \cap \mathrm{dom}(f)} |\nabla_{\leq} f| \leq \inf_{V \cap \mathrm{dom}(f)} |\nabla_{\leq} f| \leq \frac{f(x) - a}{d(\geq x, P \setminus V)} < +\infty$$

which yield the conclusion. □

Corollary 2. *Assume that $(P, \leq, d)$ is $\leq$-complete and $\leq$ is self-closed. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be a $\leq$-submonotone map. Then*

$$\{x \in \mathrm{dom}(f) : |\nabla_{\leq} f|(x) < +\infty\}$$

is a dense subset of $\mathrm{dom}(f)$ *for the topology induced by the $S_{\geq}$-topology on P.*

Remark 4. (a) In the general setting considered in Corollaries 1–2, we cannot guarantee that every $x \in \mathrm{dom}(f)$ has a neighborhood V_x with respect to $S_{\geq}$-topology, such that $\inf_{V_x} f \in \mathbb{R}$. This is the reason why the particular construction of the subset V (which is not a priori a neighborhood of an x fixed beforehand) made in the proof of Corollary 1 was needed.

Take for instance $P = [0, 1]$ endowed with the standard metric and order and let $f : [0, 1] \to \mathbb{R}$ be given by $f(0) = 0$ and $f(x) = -\frac{1}{x}$ if $x \in (0, 1]$. This map is $\leq$-submonotone (since every $\leq$-ascending sequence (x_n) such that $(f(x_n))$ is nonincreasing must be stationary). If V is a neighborhood of 0 with respect to $S_{\geq}$-topology, then $\delta := \min\{d(\geq 0, P \setminus V), 1\} \in (0, 1]$, which implies that $[0, \delta) \subset V$ hence $\inf_V f = \inf_{[0,\delta)} f = -\infty$.

(b) In the case where $\leq$ is the trivial preorder on P (i.e., $x \leq y$ for all x, y), the fact that f is submonotone guarantees that every $x \in \mathrm{dom}(f)$ has an open neighborhood V with $\inf_V f \in \mathbb{R}$. (For otherwise, there would be a sequence $(x_n) \subset \mathrm{dom}(f)$ such that $x_n \to x$ and $(f(x_n))$ decreases to $-\infty$, but then the assumption that f is submonotone yields $f(x) \leq f(x_n)$ for all n, which is impossible.) Thus in this case, Corollaries 1–2 become immediate consequences of Proposition 1.

5 Existence of nonlinear error bounds

This section contains our main results. The first theorem can be viewed as a general integration result which provides a lower estimate of the considered map f as a nondecreasing function of the distance to some fixed subset C, from an analogous lower estimate of the slope $|\nabla_{\leq} f|$.

Theorem 4. *Let $(P, \leq, d)$ be a preordered pseudometric space such that $\leq$ is self-closed and P is $\leq$-complete. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be $\leq$-submonotone. Let $C \subset P$ be $\geq$-saturated. Let $\pi : P \to [0, +\infty]$ be a map such that*

$$\forall x \in P, \quad \pi(x) \leq d(\geq x, C), \tag{12}$$

$$\forall x, y \in P, \quad x \leq y \quad \Rightarrow \quad \pi(y) \leq \pi(x) + d(x, y). \tag{13}$$

Let $\beta : [0, +\infty) \to [0, +\infty)$ be a nondecreasing map and set $\beta(+\infty) = \lim_{s \to +\infty} \beta(s)$. Let $\rho \in (0, +\infty]$. Assume that

$$\forall x \in \mathrm{dom}(f), \quad \pi(x) < 2\rho \quad \Rightarrow \quad |\nabla_{\leq} f|(x) \geq \beta(d(\geq x, C)).$$

Then, for every $x \in \mathrm{dom}(f) \cap \mathrm{dom}(\pi) \setminus C$ with $d(\geq x, C) \leq \rho$, we have

$$f(x) - \inf\{f(y) : y \in P \setminus C,\ x \leq y,\ \pi(y) < 2\rho\} \geq \int_0^{d(\geq x, C)} \beta(s)\, ds.$$

Remark 5. (a) The map $\pi \equiv 0$ clearly satisfies (12) and (13). When $\pi \equiv 0$ and $\rho = +\infty$, the above theorem is a global result in the sense that the assumption on the slope concerns all elements x in $\mathrm{dom}(f)$ and the conclusion is valid for all elements x in $\mathrm{dom}(f) \setminus C$.

Note also that, in that case, if we assume in addition that $\beta \not\equiv 0$ and f is bounded below on $P \setminus C$, then the theorem implies that every $x \in P$ such that $d(\geq x, C) = +\infty$ must satisfy $x \notin \mathrm{dom}(f)$, i.e., $f(x) = +\infty$.
(b) Assume that the pseudometric d satisfies $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in P$ with $y \leq z$. Let $\pi : x \mapsto d(x, C) := \inf\{d(x, y) : y \in C\}$. Then (12) is immediate. Moreover, letting $y \in P$ be such that $x \leq y$, we have

$$\forall z \in C, \quad d(y, C) \leq d(z, y) \leq d(z, x) + d(x, y)$$

whence (13). Applying the theorem with $\pi(x) = d(x, C)$ and $\rho \in (0, +\infty)$, the theorem becomes a local result, and it generalizes [1, Theorem 4.1].

Proof. Let $x \in \mathrm{dom}(f) \cap \mathrm{dom}(\pi) \setminus C$ be such that $d(\geq x, C) \leq \rho$. Moreover, we have $d(\geq x, C) > 0$ since $x \notin C$ and C is $\geq$-saturated. We define the sets

$$P' = \{y \in P : x \leq y\} \quad \text{and} \quad C' = C \cap P' = \{y \in C : x \leq y\}.$$

Equipped with the restrictions of the preorder $\leq$ and the pseudometric d, we get that $(P', \leq, d)$ is a preordered pseudometric space. Since, by assumption, $\leq$ is self-closed on P, it is also self-closed on P', and P' is $\leq$-closed in P thus $\leq$-complete. Whenever $y \in P'$, we get $\{z \in P : y \leq z\} \subset P'$ hence

$$\forall y \in P', \quad d(\geq y, C) = d(\geq y, C') \quad \text{and} \quad |\nabla_{\leq} f|(y) = |\nabla_{\leq}(f|_{P'})|(y).$$

Based on these observations, up to considering $P', C', f|_{P'}$ instead of P, C, f, we can assume that $x \leq y$ for all $y \in P$, so that

$$\inf\{f(y) : y \in P \setminus C,\ x \leq y,\ \pi(y) < 2\rho\} = \inf_{\pi^{-1}([0,2\rho))\setminus C} f. \tag{14}$$

Note that, if $\inf_{\pi^{-1}([0,2\rho))\setminus C} f = -\infty$, then the formula claimed in the theorem is immediate. Hence we can assume that f is bounded below on $\pi^{-1}([0, 2\rho)) \setminus C$.

First we note that

$$\text{for all } \tau \in (0, +\infty], \quad \pi^{-1}([\tau, +\infty]) \text{ is } \geq\text{-saturated}. \tag{15}$$

For showing this, let $y \in P$ be such that $d(\geq y, \pi^{-1}([\tau, +\infty])) = 0$ and we have to show that $y \in \pi^{-1}([\tau, +\infty])$ (see Lemma 1 (c)). To do this, let $\varepsilon > 0$. There is $z \in \pi^{-1}([\tau, +\infty])$ with $y \leq z$ and $d(y, z) \leq \varepsilon$. In view of (13) we get $\tau \leq \pi(z) \leq \pi(y) + d(y, z) \leq \pi(y) + \varepsilon$, hence $\pi(y) \geq \tau - \varepsilon$. Letting $\varepsilon \to 0$, we deduce that $y \in \pi^{-1}([\tau, +\infty])$. This shows (15).

Let $\sigma \in (0, d(\geq x, C))$. Let $n \geq 1$ be an integer and, for all $i \in \{0, \ldots, n\}$, we set $t_i = \frac{i}{n}\sigma$, so that

$$0 = t_0 < t_1 < \ldots < t_n = \sigma.$$

For every $i \in \{0, \ldots, n\}$, let

$$C_i = \{y \in P : d(\geq y, C) \leq t_i\}$$

and

$$U_i = \pi^{-1}([0, 2\rho - t_i)) \setminus C_i = \{y \in P : d(\geq y, C) > t_i,\ \pi(y) < 2\rho - t_i\}.$$

Since C is $\geq$-saturated, we get $C_0 = C$ and $U_0 = \pi^{-1}([0, 2\rho)) \setminus C$. Moreover, if $\rho < +\infty$ then by (12) we have $\pi(x) \leq d(\geq x, C) \leq \rho < 2\rho - t_n$, while if $\rho = +\infty$ then the fact that $x \in \mathrm{dom}(\pi)$ implies $\pi(x) < 2\rho - t_n$. In each case, we get $x \in \pi^{-1}([0, 2\rho - t_n))$. Thus

$$C = C_0 \subset C_1 \subset \ldots \subset C_n, \quad x \in U_n \subset \ldots \subset U_1 \subset U_0 = \pi^{-1}([0, 2\rho)) \setminus C.$$

We claim that

$$\text{for every } i \in \{0, \ldots, n\},\ C_i \text{ is } \geq\text{-saturated}. \tag{16}$$

For showing this, in view of Lemma 1 (c) (or Remark 2 (a)), it suffices to show that every $y \in P$ such that $d(\geq y, C_i) = 0$ must belong to C_i. Letting

$\varepsilon > 0$, there is $y' \in C_i$ with $y \leq y'$ and $d(y, y') \leq \frac{\varepsilon}{2}$. Also since $y' \in C_i$, there is $y'' \in C$ with $y' \leq y''$ and $d(y', y'') \leq t_i + \frac{\varepsilon}{2}$. We then have $y \leq y' \leq y''$ and, using that d is $\leq$-triangular, we get $d(y, y'') \leq t_i + \varepsilon$, whence $d(\geq y, C) \leq t_i + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $d(\geq y, C) \leq t_i$, whence $y \in C_i$, and the verification of (16) is complete.

Our next claim is:

$$\forall i \in \{0, \ldots, n-1\}, \quad \inf_{U_{i+1}} f \geq \inf_{U_i} f + \beta(t_i)(t_{i+1} - t_i). \tag{17}$$

For showing this, we aim to apply Proposition 1 with $U = U_i$. Note that $U_i \cap \mathrm{dom}(f) \neq \emptyset$ since $x \in U_i \cap \mathrm{dom}(f)$. Also, since $P \setminus U_i = C_i \cup \pi^{-1}([2\rho - t_i, +\infty])$ is $\geq$-saturated (due to (15) and (16)), we have

$$\forall y \in U_i, \quad d(\geq y, P \setminus U_i) > 0.$$

This allows us to apply the proposition, and we get

$$\forall y \in U_i \cap \mathrm{dom}(f),\ f(y) - \inf_{U_i} f \geq \Big(\inf_{U_i \cap \mathrm{dom}(f)} |\nabla_{\leq} f| \Big) d(\geq y, P \setminus U_i). \tag{18}$$

The assumption made in the theorem combined with the definition of U_i and the fact that β is nondecreasing yields

$$\inf_{U_i \cap \mathrm{dom}(f)} |\nabla_{\leq} f| \geq \beta(t_i). \tag{19}$$

Moreover, we have

$$\forall y \in U_{i+1}, \quad d(\geq y, P \setminus U_i) \geq t_{i+1} - t_i. \tag{20}$$

Indeed, fix an element $y \in U_{i+1}$ and let $z \in P \setminus U_i$ be such that $y \leq z$. Thus $d(\geq z, C) \leq t_i$ or $\pi(z) \geq 2\rho - t_i$. In the latter case, by (13), we get

$$2\rho - t_i \leq \pi(z) \leq \pi(y) + d(y, z) < 2\rho - t_{i+1} + d(y, z)$$

(note that we must have $\rho < +\infty$ in these circumstances), hence $d(y, z) \geq t_{i+1} - t_i$. In the former case, for every $\varepsilon > 0$, we find $z' \in C$ with $z \leq z'$ and $d(z, z') \leq t_i + \varepsilon$. Thus $y \leq z \leq z'$, and we have

$$t_{i+1} < d(\geq y, C) \leq d(y, z') \leq d(y, z) + d(z, z') \leq d(y, z) + t_i + \varepsilon.$$

Since ε is arbitrary, we get $d(y, z) \geq t_{i+1} - t_i$. Finally we have shown

$$\forall z \in P \setminus U_i, \quad y \leq z \quad \Rightarrow \quad d(y, z) \geq t_{i+1} - t_i,$$

whence (20).

Combining (18) with (19) and (20), for every $y \in U_{i+1} \cap \mathrm{dom}(f)$ (thus $y \in U_i \cap \mathrm{dom}(f)$) we get

$$f(y) \geq \inf_{U_i} f + \beta(t_i)(t_{i+1} - t_i),$$

and the same formula is immediate if $y \in U_{i+1} \setminus \mathrm{dom}(f)$. This shows (17).

From (17), since $x \in U_n$ and $U_0 = \pi^{-1}([0, 2\rho)) \setminus C$, we obtain

$$f(x) \geq \inf_{U_n} f \geq \inf_{\pi^{-1}([0,2\rho))\setminus C} f + \sum_{i=0}^{n-1} \beta(t_i)(t_{i+1} - t_i).$$

Passing to the limit as $n \to +\infty$, we derive

$$f(x) \geq \inf_{\pi^{-1}([0,2\rho))\setminus C} f + \int_0^\sigma \beta(s)\, ds.$$

Finally, letting $\sigma \to d(\geq x, C)$ and remembering (14), we get the formula stated in the theorem. □

As an application of Theorem 4 to sublevel sets, we obtain a criterion of existence of nonlinear error bounds, whose formulation unifies both local and global situations.

Theorem 5. *Let $(P, \leq, d)$ be a preordered pseudometric space such that $\leq$ is self-closed and P is $\leq$-complete. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be $\leq$-submonotone. Let $\beta : [0, +\infty) \to [0, +\infty)$ be nondecreasing. Let $a \in \mathbb{R}$ and $b \in (a, +\infty]$. Let A be a subset of $[f \leq a]$ and let B be either of the subsets $[f < b]$ or $[f \leq b]$. Let $\pi : P \to [0, +\infty]$ be a map satisfying (12) with $C = A$ and (13). Let $\rho \in (0, +\infty]$. Assume that $[f < a] \cap \pi^{-1}([0, 2\rho)) \subset A$ and*

$$\forall x \in B \cap \mathrm{dom}(f), \quad \pi(x) < 2\rho \quad \Rightarrow \quad |\nabla_{\leq} f|(x) \geq \beta(d(\geq x, A))$$

with $\beta(+\infty) := \lim_{s \to +\infty} \beta(s)$. Then,

$$\forall x \in B \cap \mathrm{dom}(\pi) \setminus A, \quad d(\geq x, A) \leq \rho \quad \Rightarrow \quad f(x) - a \geq \int_0^{d(\geq x, A)} \beta(s)\, ds.$$

Proof. We consider the preorder $\preceq$ defined in Lemma 2. By Lemma 2, $(P, \preceq, d)$ is a preordered pseudometric space such that $\preceq$ is self-closed and P is $\preceq$-complete; moreover, f is $\preceq$-submonotone.

By Lemma 2 (d), B is a $\preceq$-closed subset of P, hence it is $\preceq$-complete.

Let $C = S_{\succeq}(A) \cap B$ so that C is the closure of A with respect to the $S_{\succeq}$-topology of B (see Lemma 1 (c)). We have

$$\forall x \in P, \quad d(\geq x, A) \leq d(\succeq x, A) = d(\succeq x, C) = d(\succeq x, S_{\succeq}(A)), \tag{21}$$

where the last two equalities come from Lemma 1 (c). This guarantees that the map π satisfies $\pi(x) \leq d(\succeq x, C)$ for all $x \in P$, so that π satisfies conditions (12) and (13) with respect to the preorder $\preceq$ and the subset C.

Let $x \in P$. We claim that

$$\pi(x) < 2\rho \quad \Rightarrow \quad d(\geq x, A) = d(\succeq x, C), \tag{22}$$

$$d(\geq x, A) \leq \rho \quad \Leftrightarrow \quad d(\succeq x, C) \leq \rho \quad \Rightarrow \quad d(\geq x, A) = d(\succeq x, C). \tag{23}$$

First note that, in the case where $x \in A$, we have $d(\geq x, A) = d(\succeq x, C) = 0$, so that (22) and (23) are trivially true in this case. So assume that $x \notin A$. For showing (22), assume that $\pi(x) < 2\rho$. Since $[f < a] \cap \pi^{-1}([0, 2\rho)) \subset A$, we get $f(x) \geq a \geq \sup_A f$, whence $d(\geq x, A) = d(\succeq x, A) = d(\succeq x, C)$ in view of Lemma 2 (f) and (21). This establishes (22), and we turn our attention to (23). The implication $d(\succeq x, C) \leq \rho \Rightarrow d(\geq x, A) \leq \rho$ follows from (21) and we focus on the other implications. In the case where $d(\geq x, A) = \rho = +\infty$ then (21) yields $d(\succeq x, C) = +\infty = d(\geq x, A)$. If $d(\geq x, A) < \rho$ or $d(\geq x, A) \leq \rho < +\infty$, then we get $\pi(x) \leq d(\geq x, A) < 2\rho$ and the conclusion follows from (22). The verification of (23) is complete.

It easily follows from Definition 3 (a) that $|\nabla_{\preceq}(f|_B)|(x) = |\nabla_{\preceq} f|(x)$ for all $x \in B \cap \mathrm{dom}(f) = \mathrm{dom}(f|_B)$. Invoking also Lemma 2 (c), we then obtain

$$\forall x \in B \cap \mathrm{dom}(f), \quad |\nabla_{\preceq}(f|_B)|(x) = |\nabla_{\leq} f|(x).$$

Finally, we observe that

$$\{y \in B \setminus C : \pi(y) < 2\rho\} \subset (B \setminus A) \cap \pi^{-1}([0, 2\rho)) \subset [f \geq a].$$

Based on all these observations, we can apply Theorem 4 to the preordered pseudometric space $(B, \preceq, d)$, the $\preceq$-submonotone map $f|_B : B \to \mathbb{R} \cup \{+\infty\}$, and the $\succeq$-saturated subset $C \subset B$, and this gives the formula

$$f(x) - a \geq \int_0^{d(\geq x, A)} \beta(s)\, ds,$$

for all $x \in B \cap \mathrm{dom}(f) \cap \mathrm{dom}(\pi) \setminus C$ such that $d(\geq x, A) \leq \rho$. The above formula remains valid for all $x \in C \setminus A$: indeed, we then have $\pi(x) = d(\geq x, A) = d(\succeq x, C) = 0$ (by (21)) and $f(x) \geq a$ (since $[f < a] \cap \pi^{-1}([0, 2\rho)) \subset A$ by assumption). The formula is also valid, of course, if $x \notin \mathrm{dom}(f)$. This completes the proof of the theorem. $\square$

By specializing Theorem 5 to the case where $\pi \equiv 0$ and $\rho = +\infty$, we obtain the following global result.

Theorem 6. *Let $(P, \leq, d)$ be a preordered pseudometric space such that $\leq$ is self-closed and P is $\leq$-complete. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be $\leq$-submonotone. Let $\beta : [0, +\infty) \to [0, +\infty)$ be nondecreasing. Let $a \in \mathbb{R}$ and $b \in (a, +\infty]$. Let A be such that $[f < a] \subset A \subset [f \leq a]$ and let B be either of the subsets $[f < b]$ or $[f \leq b]$. Assume that*

$$\forall x \in B \cap \operatorname{dom}(f), \quad |\nabla_{\leq} f|(x) \geq \beta(d(\geq x, A))$$

with $\beta(+\infty) := \lim_{s \to +\infty} \beta(s)$. Then,

$$\forall x \in B \setminus A, \quad f(x) - a \geq \int_0^{d(\geq x, A)} \beta(s)\, ds.$$

Corollary 3. *Under the assumptions of Theorem 6, assuming in addition that $\beta \not\equiv 0$, we have*

$$\forall x \in B \cap \operatorname{dom}(f), \quad d(\geq x, A) < +\infty. \tag{24}$$

Moreover,

$$B \cap \operatorname{dom}(f) \neq \emptyset \quad \Leftrightarrow \quad A \neq \emptyset \quad \Leftrightarrow \quad [f \leq a] \neq \emptyset.$$

Proof. The assumption that $\beta \not\equiv 0$ combined with the fact that β is nondecreasing yields $\int_0^{+\infty} \beta(s)\, ds = +\infty$. For all $x \in B$ such that $d(\geq x, A) = +\infty$ (in particular $x \notin A$), we then have $f(x) = +\infty$ by Theorem 6. This shows (24).

Since $A \subset [f \leq a] \subset B \cap \operatorname{dom}(f)$, we have already the implications

$$A \neq \emptyset \quad \Rightarrow \quad [f \leq a] \neq \emptyset \quad \Rightarrow \quad B \cap \operatorname{dom}(f) \neq \emptyset,$$

whereas (24) yields the remaining implication $B \cap \operatorname{dom}(f) \neq \emptyset \Rightarrow A \neq \emptyset$. □

As a byproduct of Theorem 6 and Corollary 3, we obtain the following criterion of existence of *linear* error bound.

Corollary 4. *Let $(P, \leq, d)$ be a preordered pseudometric space such that $\leq$ is self-closed and P is $\leq$-complete. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be $\leq$-submonotone. Let $a \in \mathbb{R}$ and $b \in (a, +\infty]$. Let A be such that $[f < a] \subset A \subset [f \leq a]$ and let B be either of the subsets $[f < b]$ or $[f \leq b]$. Assume that $B \cap \operatorname{dom}(f) \neq \emptyset$ and*

$$\forall x \in B \cap \operatorname{dom}(f), \quad d(\geq x, A) > 0 \quad \Rightarrow \quad |\nabla_{\leq} f|(x) \geq \sigma$$

for some $\sigma > 0$. Then $A \neq \emptyset$ and

$$\forall x \in B \setminus A, \quad f(x) - a \geq \sigma d(\geq x, A).$$

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