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CONVERGENCE CRITERION FOR A HISTORY-DEPENDENT VARIATIONAL INEQUALITY*

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This paper is dedicated to the memory of Haïm Brezis, a prominent mathematician whose work has impacted the research of thousands of scientists all over the world. I have had the privilege to be in contact with him concerning the organization of the 6^{th} *French-Romanian Colloquium on Applied Mathematics held in Perpignan, in August 2002.*

Abstract

We consider a variational inequality in a reflexive Banach space X, governed by a history-dependent operator. The existence of a unique solution to the inequality can be proven by using a fixed point argument. Based on this fixed point property, we provide necessary and sufficient conditions which guarantee the uniform convergence of a sequence of functions to the solution of the variational inequality. We then exploit this result in the study of both a penalty method and the well-posedness analysis of the problem. Moreover, we present an example which arises in Contact Mechanics. It concerns the study of a mathematical model which describes the contact of a viscoelastic membrane with a foundation.

Keywords: variational inequality, history-dependent operator, convergence criterion, penalty method, well-posedness result, viscoelastic membrane, contact problem.

MSC: 49J40, 47J20, 47G10, 74K15, 74M15.

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1 Introduction

The weak formulation of a large number of boundary value problems is expressed in terms of variational inequalities. Such kind of inequalities abound in Mechanics, Physics and Engineering Sciences, as shows in [3, 6, 13, 18, 22] and the references therein. Started in early sixties, the theory of variational inequalities deals with the study of various classes of elliptic, time-dependent and evolutionary inequalities for which it provides existence, uniqueness and optimal control results. It uses arguments on nonlinear and convex analysis, including the properties of monotone and pseudomonotone opertors, lower semicontinuous functions and the subdifferential of convex functions. Comprehensive references in the field are [1, 2, 13, 18], for instance. Results on the numerical analysis of different types of variational inequality problems, including error estimates and algorithms to approximate the solution, can be found in [7, 10].

History-dependent operators represent a special class of nonlinear operators defined on spaces of continuous functions. Such kind of operators arise in Nonlinear Analysis, the theory of Differential and Integral Equations, and Mechanics, as well. Two elementary exemples in Nonlinear Analysis are provided by the integral operator and the Volterra operator. In Classical Mechanics, the curent position of a material point is determined by the initial position and the history of the velocity function and, therefore, it is expressed in terms of a history-dependent operator. In Contact Mechanics it is usual to consider that the coefficient of friction depends on the total slip or the total slip rate which, again, leads to history-dependent operators. History-dependent operators have been introduced in [21] and then intensively used in the literature. References can be found the books [20, 23], for instance.

Convergence results represent an important topic in Nonlinear and Numerical Analysis and, in particular, in the study of variational inequalities. The convergence of the solution of a penalty problem to the solution of the original problem as the penalty parameter converges and the convergence of the discrete solution to the solution of the continuous problem as the time step or the discretization parameter converges to zero are only two simple examples, among others. Motivated by important applications, a considerable effort was done to obtain convergence results in the study of various inequality problems.

In this paper we deal with a convergence criterion for a special type of variational inequalities, governed by a history-dependent operator. The functional framework we consider is the following: X is a real reflexive

Banach space with dual X^*, $\langle \cdot, \cdot \rangle$ denotes the duality paring between X^* and X and $T > 0$ represents the time interval of interest. The problem data are the set K, the operators A and $\mathcal{S}$, as well as the functions j and f, which will be described in the next section. With these data, the inequality problem we consider in this paper is the following.

Problem $\mathcal{P}$. *Find a function $u : [0,T] \to X$ such that*

$$\begin{aligned} u(t) \in K, \quad & \langle Au(t), v - u(t) \rangle + j(\mathcal{S}u(t), u(t), v) - j(\mathcal{S}u(t), u(t), u(t)) \\ & \geq \langle f(t), v - u(t) \rangle \quad \forall\, v \in K,\ t \in [0,T]. \end{aligned} \tag{1}$$

The unique solvability of Problem $\mathcal{P}$ was proved in [23], based on a fixed point argument. Inequality problems of the form (1) arise in Mechanics and describe the contact between an elastic or viscoelastic body with a foundation. There, the operator A models the elastic properties of the material, $\mathcal{S}$ describes its viscoelastic properties, the function f is related to the applied forces and, finally, the set K and the function j describe the frictional or frictionless contact boundary conditions. Details can be found in the books [22, 23], for instance.

Our current paper has two aims. The first one is to state and prove a convergence criterion to the solution of Problem $\mathcal{P}$. The second one is to illustrate the use of this criterion in various applications. Convergence results for history-dependent inequalities of the form (1) have been obtained in [22–24], for instance. Nevertheless, there, only sufficient conditions for convergence have been provided. Moreover, in part of these references, the functional j had a particular structure and, in addition, only the Hilbertian framework was considered. Working in the framework of a reflexive Banach space, with a general form for the function j, and obtaining necessary and sufficient conditions which guarantee the convergence of an arbitrary sequence of continuous functions to the solution u of problem $\mathcal{P}$ represents the main traits of novelty of the current paper.

The rest of the manuscript is structured as follows. In Section 2 we list the assumptions on the data and recall an existence and uniqueness result in the study of Problem $\mathcal{P}$, Theorem 2. Then, in Section 3 we state and prove our main result, Theorem 3. The proof of the theorem is carried out in several steps and is based on the fixed point structure of Problem $\mathcal{P}$. We apply Theorem 3 in Sections 4 and 5, where we introduce a penalty method and two well-posedness concepts, respectively, in the study of the variational inequality (1). Finally, in Section 6 we provide an example of boundary value problem which leads to an inequality of the form (1). The problem models the contact of a viscoelastic membrane with an obstacle, the unknown u being the vertical displacement field.

2 Preliminaries

We denote by $\|\cdot\|_X$ the norm on the space X and, without losing the generality, we assume that $(X, \|\cdot\|_X)$ is strictly convex. Details can be found in [27, Proposition 32.22]. We use notation 2^{X^*} for the set of parts of X^*. Besides the space X and its dual X^*, we also consider a real normed space $(Y, \|\cdot\|_Y)$. Moreover, unless it is specified otherwise, we use n to denote a given positive integer. All the limits, lower limits and upper limits below are considered as $n \to \infty$, even if we do not mention it explicitly. The symbols "$\to$" and "$\rightharpoonup$" denote the strong and the weak convergence in various spaces which will be specified, except in the case when these convergences take place in $\mathbb{R}$. For a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ which converges to zero we use the short hand notation $0 \leq \varepsilon_n \to 0$. In addition, we denote by $d(u, K)$ the distance between an element $u \in X$ and the set $K \subset X$, that is

$$d(u,K) = \inf_{v \in K} \|u - v\|_X.$$

If $(Z, \|\cdot\|_Z)$ is a normed space, we denote by $C([0,T];Z)$ the space of continuous functions defined on $[0,T]$ with values in Z. The space $C([0,T];Z)$ will be endowed with the norm of the uniform convergence, that is,

$$\|u\|_{C([0,T];Z)} = \max_{t \in [0,T]} \|u(t)\|_Z \qquad \forall\, u \in C([0,T];Z).$$

Moreover, for an operator $\mathcal{S}$ defined on the space of $C([0,T];X)$ with values in the space $C([0,T];X)$, $C([0,T];Y)$ or $C([0,T];X^*)$ we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point t, i.e., $\mathcal{S}u(t) = (\mathcal{S}u)(t)$, for all $u \in C([0,T];X)$ and $t \in [0,T]$. Finally, we shall use the short hand notation $C([0,T])$ for the space $C([0,T], \mathbb{R})$.

In the study of Problem $\mathcal{P}$ we consider the following assumptions.

$$K \;\; \text{is nonempty closed convex subset of } X. \tag{2}$$

$$\left\{\begin{array}{l} A\colon X \to X^* \text{ is pseudomonotone and strongly monotone, i.e.:} \\ \text{(a) } A \text{ is bounded and } u_n \rightharpoonup u \text{ in } X \text{ with } \limsup \langle Au_n, u_n - u\rangle \leq 0 \\ \qquad \text{implies that } \liminf \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle \quad \forall\, v \in X. \\ \text{(b) There exists } m_A > 0 \text{ such that} \\ \qquad \langle Au - Av, u - v\rangle \geq m_A \|u - v\|_X^2 \quad \forall\, u,\, v \in X. \end{array}\right. \tag{3}$$

$$\begin{cases} \mathcal{S} : C([0,T];X) \to C([0,T];Y) \text{ is a history-dependent} \\ \text{operator, i.e., there exists } L_S > 0 \text{ such that} \\ \quad \|\mathcal{S}u(t) - \mathcal{S}v(t)\|_Y \le L_S \displaystyle\int_0^t \|u(s) - v(s)\|_X ds \\ \qquad \forall\, u,\ v \in C([0,T];X),\ \ t \in [0,T]. \end{cases} \tag{4}$$

$$\begin{cases} j\colon Y \times X \times X \to \mathbb{R} \text{ is a function such that:} \\ \quad \text{(a) } j(y,u,\cdot)\colon X \to \mathbb{R} \text{ is convex and lower semicontinuous on } X, \\ \qquad \text{for all } y \in Y,\ u \in X. \\ \quad \text{(b) There exists } \alpha_j \ge 0 \text{ and } \beta_j \ge 0 \text{ such that} \\ \qquad\quad j(y_1,u_1,v_2) - j(y_1,u_1,v_1) + j(y_2,u_2,v_1) - j(y_2,u_2,v_2) \\ \qquad\qquad \le \alpha_j\|u_1 - u_2\|_X\, \|v_1 - v_2\|_X + \beta_j\|y_1 - y_2\|_Y\, \|v_1 - v_2\|_X \\ \qquad\qquad\quad \forall\, y_1,\, y_2 \in Y,\ \ u_1,\, u_2,\, v_1,\, v_2 \in X. \end{cases} \tag{5}$$

$$\alpha_j < m_A. \tag{6}$$

$$f \in C([0,T];X^*). \tag{7}$$

The unique solvability of the variational inequality (1) is given by the following existence and uniqueness result.

Theorem 1. *Assume* (2)–(7)*. Then, inequality* (1) *has a unique solution with regularity* $u \in C([0,T];X)$*.*

A proof of Theorem 2 can be found in [23, p.160], based on a fixed point argument. For the convenience of the reader, as well as for the needs of the next section, we present in what follows its sketch, carried out in several steps.

Proof. Step 1. We fix $\eta \in C([0,T];X)$ and denote by $y_\eta \in C([0,T];Y)$ the function given by

$$y_\eta(t) = \mathcal{S}\eta(t) \qquad \forall\, t \in [0,T]. \tag{8}$$

Then, using standard arguments on elliptic variational inequalities, we prove that there exists a unique function $u_\eta \in C([0,T];X)$ such that, for all $t \in [0,T]$, the following inequality holds:

$$\begin{aligned} u_\eta(t) \in K, \quad &\langle Au_\eta(t), v - u_\eta(t)\rangle + j(y_\eta(t), u_\eta(t), v) \\ &-j(y_\eta(t), u_\eta(t), u_\eta(t)) \ge \langle f(t), v - u(t)\rangle \qquad \forall\, v \in K. \end{aligned} \tag{9}$$

Step 2. We define the operator $\Lambda\colon C([0,T];X) \to C([0,T];X)$ by equality

$$\Lambda\eta = u_\eta \quad \forall\, \eta \in C([0,T];X) \tag{10}$$

and prove that this operator is history-dependent, i.e., there exists $L_\Lambda > 0$ such that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X \le L_\Lambda \int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds, \tag{11}$$

for all $\eta_1,\, \eta_2 \in C([0,T];X)$ and $t \in [0,T]$. Based on this inequality we show that a power p of the operator Λ is a contraction on the space $C([0,T];X)$, which implies that Λ has a unique fixed point $\eta^* \in C([0,T];X)$.

Step 3. Let $\eta^* \in C([0,T];X)$ be the fixed point of the operator Λ. It follows from (8) and (10) that, for all $t \in [0,T]$, the following equalities hold:

$$y_{\eta^*}(t) = \mathcal{S}\eta^*(t) \quad \text{and} \quad u_{\eta^*}(t) = \eta^*(t). \tag{12}$$

We now write the inequality (9) for $\eta = \eta^*$ and then use the equalities (12) to conclude that the function $\eta^* \in C([0,T];X)$ is a solution to the variational-variational inequality (1). This proves the existence part of the theorem. The uniqueness part is a consequence of the uniqueness of the fixed point of the operator Λ defined by (10). □

3 A convergence criterion

In this section we provide necessary and sufficient conditions which guarantee the uniform convergence of an arbitrary sequence $\{u_n\} \subset C([0,T];X)$ to the solution of Problem $\mathcal{P}$. To this end we assume that (2)–(7) hold and we denote by $u \in C([0,T];X)$ the solution of inequality (1) obtained in Theorem 2. Moreover, we consider the following additional assumptions on the operator A and function j.

$$\left\{ \begin{array}{l} A : X \to X^* \text{ is a Lipschitz continuous operator, i.e.,} \\ \text{there exists } L_A > 0 \text{ such that} \\ \|Au - Av\|_{X^*} \le L_A \|u - v\|_X \quad \forall\, u,\, v \in X. \end{array} \right. \tag{13}$$

$$\left\{ \begin{array}{l} \text{There exists a function } c_j : \mathbb{R}_+ \to \mathbb{R}_+ \\ \text{which maps bounded sets into bounded sets such that} \\ j(y,u,v) - j(y,u,w) \le c_j(\|y\|_Y, \|u\|_X)\|v - w\|_X \\ \forall\, y \in Y,\ u,\ v,\ w \in X. \end{array} \right. \tag{14}$$

Note that assumptions (13) and (3)(b) imply that condition (3)(a) is safisfied.

Next, given a sequence $\{u_n\} \subset C([0,T];X)$ we consider the following statements.

(S_1) $\qquad u_n \to u \quad \text{in} \;\; C([0,T];X), \quad \text{as} \;\; n \to \infty.$

(S_2) $\qquad$ There exists $0 \le \varepsilon_n \to 0$ such that

$$\left\{\begin{array}{l} \text{(a)} \quad d(u_n(t), K) \le \varepsilon_n, \\[2ex] \text{(b)} \quad \langle A u_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v) \\ \qquad\quad -j(\mathcal{S}u_n(t), u_n(t), u_n(t)) + \varepsilon_n(1 + \|v - u_n(t)\|_X) \\ \qquad\qquad \ge \langle f(t), v - u_n(t)\rangle \quad \forall\, v \in K(t), \\[2ex] \text{for all } n \in \mathbb{N} \text{ and } t \in [0,T]. \end{array}\right. \tag{15}$$

Our result in this section is the following.

Theorem 2. *Assume (2)–(7) and (14). Then, the statement (S_2) implies the statement (S_1). The converse is true if, moreover, (13) holds.*

The proof of Theorem 3 is based on some preliminary results. To provide it we use the operator Λ defined by equality (10) and consider the following intermediate statement.

(S_3) $\qquad u_n - \Lambda u_n \to 0 \quad \text{in} \;\; C([0,T];X), \quad \text{as} \;\; n \to \infty.$

The first preliminary result is the following.

Lemma 1. *Assume (2)–(7), (14) and let $\{u_n\} \subset C([0,T];X)$ be a sequence which satisfies condition (15)(b). Then, there exists a constant $M > 0$ which does not depend on n and t such that*

$$\|u_n(t)\|_X \le M, \quad \|\mathcal{S}u_n(t)\|_Y \le M, \quad \|\Lambda u_n(t)\|_X \le M, \tag{16}$$

for all $n \in \mathbb{N}$ and $t \in [0,T]$.

Proof. Let $n \in \mathbb{N}$, $t \in [0,T]$ and note that, below in this proof we use the notation C_i, $i = 1, 2, \ldots$, for various positive constants which do not depend on n ant t. Fix an element $v_0 \in K$. We use inequality (15)(b) with $v = v_0$ to find that

$$\begin{aligned} &\langle A u_n(t) - A v_0, u_n(t) - v_0\rangle \le \langle A v_0, v_0 - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v_0) \\ &\qquad -j(\mathcal{S}u_n(t), u_n(t), u_n(t)) + \varepsilon_n(1 + \|u_n(t) - v_0\|_X) + \langle f(t), u_n(t) - v_0\rangle. \end{aligned}$$

Then, using assumption (3)(a) yields

$$m_A\|u_n(t)-v_0\|_X^2 \le (\|Av_0\|_{X^*}+\|f(t)\|_{X^*}+\varepsilon_n)\|u_n(t)-v_0\|_X \qquad (17)$$
$$+\varepsilon_n+\Big[j(\mathcal{S}u_n(t),u_n(t),v_0)-j(\mathcal{S}u_n(t),u_n(t),u_n(t))\Big].$$

We now write

$$\begin{aligned} &j(\mathcal{S}u_n(t),u_n(t),v_0)-j(\mathcal{S}u_n(t),u_n(t),u_n(t))\\ &=\Big[j(\mathcal{S}u_n(t),u_n(t),v_0)-j(\mathcal{S}u_n(t),u_n(t),u_n(t))+\\ &\quad +j(\mathcal{S}v_0(t),v_0,u_n(t))-j(\mathcal{S}v_0(t),v_0,v_0)\Big]\\ &\quad +\Big[j(\mathcal{S}v_0(t),v_0,v_0)-j(\mathcal{S}v_0(t),v_0,u_n(t))\Big], \end{aligned}$$

then we use assumptions (5)(b) and (14) to deduce that

$$\begin{aligned} &j(\mathcal{S}u_n(t),u_n(t),v_0)-j(\mathcal{S}u_n(t),u_n(t),u_n(t)) \qquad (18)\\ &\le \alpha_j\|u_n(t)-v_0\|_X^2+\beta_j\|\mathcal{S}u_n(t)-\mathcal{S}v_0(t)\|_Y\|u_n(t)-v_0\|_X\\ &\quad +c_j(\|\mathcal{S}v_0(t)\|_Y,\|v_0\|_X)\|u_n(t)-v_0\|_X. \end{aligned}$$

Next, we combine inequalities (17) and (18) to see that

$$\begin{aligned} &(m_A-\alpha_j)\,\|u_n(t)-v_0\|_X^2 \qquad (19)\\ &\le (\|Av_0\|_{X^*}+\|f(t)\|_{X^*}+\varepsilon_n)\|u_n(t)-v_0\|_X\\ &\quad +\varepsilon_n+\beta_j\|\mathcal{S}u_n(t)-\mathcal{S}v_0(t)\|_Y\|u_n(t)-v_0\|_X\\ &\quad +c_j(\|\mathcal{S}v_0(t)\|_Y,\|v_0\|_X)\|u_n(t)-v_0\|_X \end{aligned}$$

and use assumptions (4), (7) and (14) to find a constant $C_1>0$ such that

$$\|Av_0\|_X+\|f(t)\|_{X^*}+c_j(\|\mathcal{S}v_0(t)\|_Y,\|v_0\|_X)\le C_1. \qquad (20)$$

Note that inequalities (19), (20) and assumption (4) yield

$$\begin{aligned} &(m_A-\alpha_j)\,\|u_n(t)-v_0\|_X^2\\ &\le \Big(C_1+\varepsilon_n+\beta_jL_S\int_0^t\|u_n(t)-v_0\|_X\Big)\|u_n(t)-v_0\|_X+\varepsilon_n. \end{aligned}$$

We now use the smallness assumption (6) and the boundedness of the sequence $\{\varepsilon_n\}$ to see that

$$\|u_n(t) - v_0\|_X^2 \le C_2\Big(1 + \int_0^t \|u_n(t) - v_0\|_X\Big)\|u_n(t) - v_0\|_X + C_3$$

Then, using the elementary inequality

$$x^2 \le ax + b \quad \Longrightarrow \quad x \le a + \sqrt{b} \qquad \forall\, x,\ a,\ b \ge 0 \tag{21}$$

we deduce that

$$\|u_n(t) - v_0\|_X \le C_4 \int_0^t \|u_n(t) - v_0\|_X + C_5. \tag{22}$$

Finally, we use (22) and the Gronwall inequality to find that

$$\|u_n(t) - v_0\|_X \le C_6, \tag{23}$$

which implies that

$$\|u_n(t)\|_X \le C_6 + \|v_0\|_X. \tag{24}$$

We now write

$$\|\mathcal{S}u_n(t)\|_Y \le \|\mathcal{S}u_n(t) - \mathcal{S}v_0(t)\|_Y + \|\mathcal{S}v_0(t)\|_Y,$$

then we use assumption (4) and inequality (23) to deduce that

$$\|\mathcal{S}u_n(t)\|_Y \le L_S T C_6 + \max_{t\in[0,T]} \|\mathcal{S}v_0(t)\|_Y. \tag{25}$$

Finally, using similar arguments combined with inequality (11) we see that

$$\|\Lambda u_n(t)\|_X \le L_\Lambda T C_6 + \max_{t\in[0,T]} \|\Lambda v_0(t)\|_X. \tag{26}$$

Lemma 3 is now a consequence of inequalities (24)–(26). □

The second preliminary result is the following.

Lemma 2. *Assume (2)–(7). Then, the statements (S_1) and (S_3) are equivalent.*

Proof. Assume that the statement (S_1) holds. We use inequality (11) to see that the operator $\Lambda : C([0,T];X) \to C([0,T];X)$ is continuous and, therefore, (S_1) implies that $u_n - \Lambda u_n \to u - \Lambda u$ in $C([0,T];X)$. On the other hand, the Step 3 in the proof of Theorem 2 shows that $u = \Lambda u$. We conclude from above that the statement (S_3) holds.

Conversely, assume that the statement (S_3) holds. Denote by Λ^k the powers of the operator Λ, for $k = 1, 2, \ldots$ Then, inequality (11) implies that $\Lambda^k : C([0,T];X) \to C([0,T];X)$ is a Lipschitz continuous operator with some constant $L_k > 0$, that is

$$\|\Lambda^k u - \Lambda^k v\|_{C([0,T];X)} \leq L_k \|u - v\|_{C([0,T];X)} \quad \forall\, u,\, v \in C([0,T];X). \tag{27}$$

Moreover, Step 2) in the proof of Theorem 2 guarantees that there exists $p \in \mathbb{N}$ such that Λ^p is a contraction on the space $C([0,T];X)$, i.e.,

$$L_p < 1. \tag{28}$$

Next, we use equality $\Lambda^p u = u$ to see that

$$\begin{aligned}
&\|u_n - u\|_{C([0,T];X)} \\
&\quad \leq \|u_n - \Lambda u_n\|_{C([0,T];X)} + \|\Lambda u_n - \Lambda^2 u_n\|_{C([0,T];X)} \\
&\qquad + \ldots + \|\Lambda^{p-1} u_n - \Lambda^p u_n\|_{C([0,T];X)} + \|\Lambda^p u_n - \Lambda^p u\|_{C([0,T];X)}
\end{aligned}$$

and, therefore, (27) we implies that

$$\begin{aligned}
&\|u_n - u\|_{C([0,T];X)} \\
&\quad \leq \big(1 + L_1 + L_2 + \ldots + L_{p-1}\big) \|u_n - \Lambda u_n\|_{C([0,T];X)} \\
&\qquad + L_p \|u_n - u\|_{C([0,T];X)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1 - L_p)\|u_n - u\|_{C([0,T];X)} \\
&\quad \leq \big(1 + L_1 + L_2 + \ldots + L_{p-1}\big) \|u_n - \Lambda u_n\|_{C([0,T];X)}
\end{aligned}$$

and, using inequality (28), we find that

$$\begin{aligned}
&\|u_n - u\|_{C([0,T];X)} \\
&\quad \leq \frac{1}{1 - L_p}\big(1 + L_1 + L_2 + \ldots + L_{p-1}\big) \|u_n - \Lambda u_n\|_{C([0,T];X)}.
\end{aligned}$$

We now use assumption (S_3) to see that the statement (S_1) holds, which concludes the proof. $\square$

The third preliminary result we need is as follows.

Lemma 3. *Assume (2)–(7) and (14). Then, the statements (S_2) implies the statement (S_3). The converse is true if, moreover, (13) holds.*

Proof. Assume the statement (S_2). Let $n \in \mathbb{N}$, $t \in [0,T]$, and let $\widetilde{u}_n$ be the function defined by

$$\widetilde{u}_n = \Lambda u_n. \tag{29}$$

Then, using the definition of the operator Λ in the proof of Theorem 2, it follows that

$$\begin{aligned} &\widetilde{u}_n(t) \in K, \quad \langle A\widetilde{u}_n(t), v - \widetilde{u}_n(t)\rangle + j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v) \\ &\qquad -j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \geq \langle f(t), v - \widetilde{u}_n(t)\rangle \quad \forall\, v \in K. \end{aligned} \tag{30}$$

Recall that the space $(X, \|\cdot\|_X)$ is a assumed to be strictly convex and K is a nonempty closed subset of X. Then, we are in a position to define the projection operator $P_K : X \to K$ by equality

$$v = P_K\xi \iff v \in K \ \text{ and } \ \|v - \xi\|_X = \min_{w\in K} \|w - \xi\|_X = d(\xi, K), \tag{31}$$

for any $\xi \in X$. Details can be find in [9, p.52], for instance. We denote $v_n(t) = P_K u_n(t)$ and $w_n(t) = u_n(t) - P_K u_n(t)$. Then,

$$u_n(t) = v_n(t) + w_n(t), \quad v_n(t) \in K(t) \tag{32}$$

and, since $d(u_n(t), K) = \|w_n(t)\|_X$, condition (S_2) (a) implies that

$$\|w_n(t)\|_X \leq \varepsilon_n. \tag{33}$$

We now use condition (S_2)(b) with $v = \widetilde{u}_n(t) \in K$ to see that

$$\begin{aligned} &\langle Au_n(t), \widetilde{u}_n(t) - u_n(t)\rangle \\ &\quad + j(\mathcal{S}u_n(t), u_n(t), \widetilde{u}_n(t)) - j(\mathcal{S}u_n(t), u_n(t), u_n(t)) \\ &\qquad + \varepsilon_n(1 + \|\widetilde{u}_n(t) - u_n(t)\|_X) \geq \langle f(t), \widetilde{u}_n(t) - u_n(t))\rangle. \end{aligned} \tag{34}$$

On the other hand, we use the regularity $v_n(t) \in K$ in (32) and test with $v = v_n(t)$ in (30) to find that

$$\begin{aligned} &\langle A\widetilde{u}_n(t), v_n(t) - \widetilde{u}_n(t)\rangle + j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v_n(t)) \\ &\qquad -j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \geq \langle f(t), v_n(t) - \widetilde{u}_n(t)\rangle. \end{aligned} \tag{35}$$

We now add inequalities (34), (35) to obtain that

$$\begin{aligned}&\langle Au_n(t), u_n(t) - \widetilde{u}_n(t)\rangle + (A\widetilde{u}_n(t), \widetilde{u}_n(t) - v_n(t)\rangle\\&\quad\le j(\mathcal{S}u_n(t), u_n(t), \widetilde{u}_n(t)) - j(\mathcal{S}u_n(t), u_n(t), u_n(t))\\&\qquad+j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v_n(t)) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t))\\&\qquad\quad+\varepsilon_n(1 + \|\widetilde{u}_n(t) - u_n(t)\|_X) + \langle f(t), u_n(t) - v_n(t)\rangle.\end{aligned}$$

and, therefore,

$$\begin{aligned}&\langle Au_n(t), u_n(t) - \widetilde{u}_n(t)\rangle + (A\widetilde{u}_n(t), \widetilde{u}_n(t) - u_n(t)\rangle\\&\quad\le (A\widetilde{u}_n(t), \widetilde{u}_n(t) - u_n(t)\rangle + (A\widetilde{u}_n(t), v_n(t) - \widetilde{u}_n(t))\\&\qquad+\Big[j(\mathcal{S}u_n(t), u_n(t), \widetilde{u}_n(t)) - j(\mathcal{S}u_n(t), u_n(t), u_n(t))\\&\qquad\quad+j(\mathcal{S}u_n(t), \widetilde{u}_n(t), u_n(t)) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t))\Big]\\&\qquad\qquad+\Big[j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v_n(t)) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), u_n(t))\Big]\\&\qquad\qquad\quad+\varepsilon_n(1 + \|\widetilde{u}_n(t) - u_n(t)\|_X) + \langle f(t), u_n(t) - v_n(t)\rangle.\end{aligned}$$

We now use assumptions (3)(a), (5)(b), (14) and equality $u_n(t) = v_n(t) + w_n(t)$, to find that

$$\begin{aligned}&(m_A - \alpha_j)\|u_n(t) - \widetilde{u}_n(t)\|_X^2 &(36)\\&\quad\le \|A\widetilde{u}_n(t)\|_{X^*}\|w_n(t)\|_X + c_j(\|\mathcal{S}u_n(t)\|_Y, \|\widetilde{u}_n(t)\|_X)\|w_n(t)\|_X\\&\qquad+\varepsilon_n + \varepsilon_n\|u_n(t) - \widetilde{u}_n(t)\|_X + \|f(t)\|_{X^*}\|w_n(t)\|_X.\end{aligned}$$

Next, (29), (16) combined with the properties of the operators A, $\mathcal{S}$ and the functions c_j, f allows us to find a constant D which does not depend on n and t such that

$$\|A\widetilde{u}_n(t)\|_{X^*} + c_j(\|\mathcal{S}u_n(t)\|_Y, \|\widetilde{u}_n(t)\|_X) + \|f(t)\|_{X^*} \le D. \tag{37}$$

Therefore, inequality (36) combined with (37) and (33) implies that

$$(m_A - \alpha_j)\|u_n(t) - \widetilde{u}_n(t)\|_X^2 \le \varepsilon_n\|u_n(t) - \widetilde{u}_n(t)\|_X + (D + 1)\varepsilon_n$$

and, moreover, the smallness assumption (6) guarantees that

$$\|u_n(t) - \widetilde{u}_n(t)\|_X^2 \le \frac{\varepsilon_n}{m_A - \alpha_j}\|u_n(t) - \widetilde{u}_n(t)\|_X + \frac{D + 1}{m_A - \alpha_j}\,\varepsilon_n. \tag{38}$$

We now use (38), the elementary inequality (21) and the convergence $\varepsilon_n \to 0$ to find that

$$\max_{t\in[0,T]} \|u_n(t) - \widetilde{u}_n(t)\|_X \to 0.$$

This implies that that $u_n - \widetilde{u}_n \to 0$ in $C([0,T];X)$ and, using equality (29), we conclude that the statement (S_3) holds.

Assume now that, in addition, condition (13) is satisfied. Also, assume that the statement (S_3) holds, let $n \in \mathbb{N}$, $t \in [0,T]$, $v \in K$, and keep the notation (29). Then, since $\widetilde{u}_n(t) \in K$, it follows that

$$d(u_n(t), K) \le \|u_n(t) - \widetilde{u}_n(t)\|_X \le \delta_n \tag{39}$$

with δ_n been given by

$$\delta_n = \max_{t\in[0,T]} \|u_n(t) - \widetilde{u}_n(t)\|_X. \tag{40}$$

Next, we use (30) to deduce that

$$\begin{aligned}
&\Big[\langle Au_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v)\\
&\quad -j(\mathcal{S}u_n(t), u_n(t), u_n(t)) - \langle f(t), v - u_n(t)\rangle\Big]\\
&\quad\quad +\langle A\widetilde{u}_n(t), v - \widetilde{u}_n(t)\rangle + j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v)\\
&\quad\quad\quad -j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \ge \langle f(t), v - \widetilde{u}_n(t)\rangle\\
&\quad\quad\quad\quad +\Big[\langle Au_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v)\\
&\quad\quad\quad\quad\quad -j(\mathcal{S}u_n(t), u_n(t), u_n(t)) - \langle f(t), v - u_n(t)\rangle\Big]
\end{aligned}$$

and, after some algebra, we obtain that

$$\begin{aligned}
&\langle Au_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v) - j(\mathcal{S}u_n(t), u_n(t), u_n(t))\\
&\quad +\Big[\langle A\widetilde{u}_n(t), v - \widetilde{u}_n(t)\rangle + \langle Au_n(t), u_n(t) - v\rangle\Big]\\
&\quad +\Big[j(\mathcal{S}u_n(t), u_n(t), u_n(t)) - j(\mathcal{S}u_n(t), u_n(t), v)\\
&\quad\quad +j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t))\Big]\\
&\quad\quad +\Big[\langle f(t), v - u_n(t)\rangle - \langle f(t), v - \widetilde{u}_n(t)\rangle\Big] \ge \langle f(t), v - u_n(t)\rangle.
\end{aligned} \tag{41}$$

We now estimate the three terms between brackets in inequality (41). First, we write

$$\begin{aligned}
&\langle A\widetilde{u}_n(t), v-\widetilde{u}_n(t)\rangle + \langle Au_n(t), u_n(t)-v\rangle \\
&\ = \langle A\widetilde{u}_n(t), u_n(t)-\widetilde{u}_n(t)\rangle + \langle A\widetilde{u}_n(t)-Au_n(t), v-u_n(t)\rangle \\
&\ \ \le \|A\widetilde{u}_n(t)\|_{X^*}\|u_n(t)-\widetilde{u}_n(t)\|_X + \|A\widetilde{u}_n(t)-Au_n(t)\|_X\|v-u_n(t)\|_X,
\end{aligned}$$

then we use assumptions (13), Lemma 3 and notation (40) to see that

$$\langle A\widetilde{u}_n(t), v-\widetilde{u}_n(t)\rangle + \langle Au_n(t), v-u_n(t)\rangle \le N\delta_n + L_A\delta_n\|v-u_n(t)\|_X, \tag{42}$$

with N being a positive constant which does not depend on n and t.

Second, we use assumptions (5)(b), (14) to see that

$$\begin{aligned}
&j(\mathcal{S}u_n(t), u_n(t), u_n(t)) - j(\mathcal{S}u_n(t), u_n(t), v) \\
&\quad + j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \\
&\quad = \Big[j(\mathcal{S}u_n(t), u_n(t), u_n(t)) - j(\mathcal{S}u_n(t), u_n(t), v) \\
&\qquad + j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), u_n(t)) \Big] \\
&\qquad + \Big[j(\mathcal{S}u_n(t), \widetilde{u}_n(t), u_n(t)) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \Big] \\
&\qquad \le \alpha_j\|u_n(t)-\widetilde{u}_n(t)\|_X\|v-u_n(t)\|_X \\
&\qquad\quad + c_j(\|\mathcal{S}u_n(t)\|_Y, \|\widetilde{u}_n(t)\|_X)\|u_n(t)-\widetilde{u}_n(t)\|_X
\end{aligned}$$

and, using (16), (40) and the properties of the function c_j it follows that

$$\begin{aligned}
&j(\mathcal{S}u_n(t), u_n(t), u_n(t)) - j(\mathcal{S}u_n(t), u_n(t), v) \\
&\quad + j(\mathcal{S}u_n(t), \widetilde{u}_n(t), v) - j(\mathcal{S}u_n(t), \widetilde{u}_n(t), \widetilde{u}_n(t)) \\
&\quad \le \alpha_j\delta_n\|v-u_n(t)\|_X + N'\delta_n.
\end{aligned} \tag{43}$$

Here, again, N' is a positive constant which does not depend on n and t.

Finally,

$$\begin{aligned}
&\langle f(t), v-u_n(t)\rangle - \langle f(t), v-\widetilde{u}_n(t)\rangle \\
&\quad = \langle f(t), \widetilde{u}_n(t)-u_n(t)\rangle \le \|f(t)\|_{X^*}\|\widetilde{u}_n(t)-u(t)\|_X.
\end{aligned}$$

and, therefore, (40) shows that

$$\langle f(t), v - u_n(t)\rangle - \langle f(t), v - \widetilde{u}_n(t)\rangle \le N''\delta_n, \tag{44}$$

with N'' being a positive constant which does not depend on n and t.

Then, combining inequalities (41)–(44) we find that

$$\begin{aligned}&\langle Au_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v) - j(\mathcal{S}u_n(t), u_n(t), u_n(t)) \\ &\quad +(N + N' + N'')\delta_n + (\alpha_j + L_A)\delta_n\|v - u_n(t)\|_X \ge \langle f(t), v - u_n(t)\rangle.\end{aligned} \tag{45}$$

Therefore, using (45) and notation

$$\varepsilon_n = \max\Big\{\,(N + N' + N'')\delta_n, (\alpha_j + L_A)\delta_n, \delta_n\,\Big\} \tag{46}$$

we see that

$$\begin{aligned}&\langle Au_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v) - j(\mathcal{S}u_n(t), u_n(t), u_n(t)) \\ &\quad +\varepsilon_n(1 + \|v - u_n(t)\|_X) \ge \langle f(t), v - u_n(t)\rangle.\end{aligned} \tag{47}$$

On the other hand, (39) and (46) imply that

$$d(u_n(t), K) \le \varepsilon_n \tag{48}$$

and, since $\widetilde{u}_n = \Lambda u_n$, assumption (S_3) and notation (40) imply that

$$\delta_n \to 0. \tag{49}$$

Then, using (46) and (49) we find that

$$\varepsilon_n \to 0. \tag{50}$$

We now combine relations (47), (48) and (50) to see that condition (S_2) is satisfied, which concludes the proof. □

We now have all the ingredients to provide the proof of Theorem 3.

Proof. Assume (2)–(7) and (14). We use Lemmas 3 and 3 to see that the following implications hold: $(S_2) \implies (S_3) \implies (S_1)$. We conclude from above that the statement (S_2) implies the statement (S_1).

Assume now that, in addition, (13) holds. Then, Lemmas 3 and the converse part in Lemma 3 show that the following implications hold: $(S_1) \implies (S_3) \implies (S_2)$. It follows from here that, in this case the statement (S_1) implies the statement (S_2), which concludes the proof. □

Remark 1. We end this section with the remark that, under assumptions (2)–(7), (13) and (14), Theorem 3 provides necessary and sufficient conditions which describe the convergence of any sequence $\{u_n\} \subset C([0,T];X)$ to the solution u of Problem $\mathcal{P}$ and, therefore, it represents a convergence criterion. Note that this theorem was obtained under the additional assumptions (13) and (14) which are not necessary in the statement of Theorem 2. Removing or relaxing these assumptions is an interesting problem which deserves to be investigated in the future.

4 A penalty method

Since Problem $\mathcal{P}$ is governed by a set of constraints K, for both theoretical and numerical reasons, it is useful to approximate it by using a penalty method. A classical penalty method consists in replacing a constrainted problem by an unconstrainted problem depending on a penalty parameter, which have a unique solution that converges to the solution of the original problem, as the penalty parameter converges to zero. Various results in the study of penalty methods for variational inequalities can be found in [7,8,12,22], for instance.

In this section we illustrate the use of Theorem 3 in the study of a penalty method associated to inequality (1) and, to this end, we need some preliminaries. First, we recall that, since X is assumed to be a strictly convex space, the normalized duality map $\mathcal{J}\colon X \to 2^{X^*}$, defined by

$$\mathcal{J}x = \{\, x^* \in X^* \mid \langle x^*, x\rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2 \,\} \qquad \forall\, x \in X,$$

is a single-valued operator. Details can be found in [4, Proposition 1.3.27] and [27, Proposition 32.22]. Therefore, $\mathcal{J}\colon X \to X^*$ and

$$\langle \mathcal{J}x, x\rangle = \|x\|_X^2 \qquad \forall\, x \in X. \tag{51}$$

Consider now the projection operator P_K defined by (31), denote by I_X the identity mapping of X and let $P : X \to X^*$ be the operator given by

$$P = \mathcal{J}(I_X - P_K). \tag{52}$$

Then, following [19, p.267], it results that P is a penalty operator of K, that is, P is bounded, demicontinuous, monotone and $K = \{\, x \in X \mid Px = 0_{X^*} \,\}$.

Consider now a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\lambda_n > 0 \quad \forall\, n \in \mathbb{N}, \tag{53}$$

$$\lambda_n \to 0, \quad \text{as} \ \ n \to \infty \tag{54}$$

and, for each $n \in \mathbb{N}$, consider the following problem.

Problem $\mathcal{P}_n$. *Find a function $u_n \in C([0,T];X)$ such that, for all $t \in [0,T]$, the inequality below holds:*

$$\langle Au_n(t), v - u_n(t)\rangle + \frac{1}{\lambda_n}\langle Pu_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v)\rangle \quad (55)$$
$$- j(\mathcal{S}u_n(t), u_n(t), u_n(t)) \geq \langle f(t), v - u_n(t)\rangle \quad \forall\, v \in X.$$

Note that, in contrast to Problem $\mathcal{P}$, in Problem $\mathcal{P}_n$ the constraint $u(t) \in K$ is removed and is replaced with an additional term which contains the penalty parameter λ_n. For this reason, we refer to Problem $\mathcal{P}_n$ as a penalty problem associated to Problem $\mathcal{P}$.

We have the following existence, uniqueness and convergence result.

Theorem 3. *Assume that (2)–(7), (14), (53) and (54). Then:*

a) *For each $n \in \mathbb{N}$ there exists a unique solution $u_n \in C([0,T];X)$ to Problem $\mathcal{P}_n$.*

b) *The solution u_n of Problem $\mathcal{P}_n$ converges to the solution u of Problem $\mathcal{P}$, i.e.,*

$$u_n \to u \quad \text{in} \quad C([0,T];X), \quad \text{as} \quad n \to \infty. \quad (56)$$

Proof. a) Let $n \in \mathbb{N}$. As already mentioned, the operator P defined by (52) is bounded, demicontinuous and mootone. Therefore, using a standard result (Proposition 27.6 in [27]) it follows that P is pseudomonotone. So, since $\lambda_n > 0$, the sum $A + \frac{1}{\lambda_n} P$ is a pseudomnotone operator too, for each $n \in \mathbb{N}$. The unique solvability of Problem $\mathcal{P}_n$ follows now as a direct consequence of Theorem 2.

b) Let $n \in \mathbb{N}$ and $t \in [0,T]$. Then, using the properties of the operator P, for any $v \in K$ we have

$$\langle Pu_n(t), v - u_n(t)\rangle = \langle Pu_n(t) - Pv, v - u_n(t)\rangle \leq 0$$

and, combining this inequality with (55) and (53), we deduce that the sequence $\{u_n\}$ satisfies condition (15)(b) with $\varepsilon_n = 0$. Therefore, Lemma 3 guarantees that the bounds (16) hold.

Next, we use (55) and assumption (14) to see that

$$\frac{1}{\lambda_n}\langle Pu_n(t), u_n(t) - v\rangle \leq \langle Au_n(t), v - u_n(t)\rangle$$
$$+j(\mathcal{S}u_n(t), u_n(t), v) - j(\mathcal{S}u_n(t), u_n(t), u_n(t)) + \langle f(t), u_n(t) - v\rangle$$
$$\leq \Big[\|Au_n(t)\|_{X^*} + \|f(t)\|_{X^*} + \big(c_j(\|\mathcal{S}u_n(t)\|_Y, \|u_n(t)\|_X)\big)\Big]\|u_n(t) - v\|_X$$

and, using the bounds (16) we deduce that

$$\frac{1}{\lambda_n}\langle Pu_n(t), u_n(t) - v\rangle \le E\|u_n(t) - v\|_X, \tag{57}$$

with E being a positive constant which does not depende on n, t and v.

On the the hand, using (52) and (51) we have

$$\langle Pu_n(t), u_n(t) - P_K u_n(t)\rangle = \|u_n(t) - P_K u_n(t)\|_X^2. \tag{58}$$

We now write (57) with $v = P_K u_n(t) \in K$ and use (58) to deduce that

$$\|u_n(t) - P_K u_n(t)\|_X \le E\lambda_n. \tag{59}$$

Inequality (59) combined with assumption (54) shows that condition (15)(a) is satisfied, with $\varepsilon_n = E\lambda_n$.

To conclude, the sequence $\{u_n\}$ satisfies both conditions (15)(a) and (15)(b) and, therefore, it satisfies the statement (S_2). We now use Theorem 3 in order obtain the convergence (56), which concludes the proof. □

Remark 2. A pointwise convergence result for penalty problems in the study of history-dependent inequalities of the form (1) has been obtained in [23, p. 168]. There, the proof was based on arguments of pseudomonotonicity and compactness. In contrast, the convergence result (56) is an uniform convergence result and, moreover, its proof results as a consequence of the convergence criterion in Theorem 3.

5 Well-posedness results

Well-posedness concepts of nonlinear problems represent an important topic in Analysis which has known a significant development in the last decades. Originating in the papers of Tykhonov [25] and Levitin-Polyak [14] (where the well-posedness of minimization problems was considered), well-posedness concepts have been extended to a large number of problems, including nonlinear equations, inequality problems, inclusions, fixed point problems, and optimal control problems. In particular, the well-posedness of variational inequalities was studied for the first time in [16,17]. Comprehensive references in the field are [5, 11, 15, 26, 28] and, more recently, [20].

The well-posedness concepts depend on the problem considered, vary from author to author, and even from paper to paper. Nevertheless, all these concepts are based on two main ingredients: the existence and uniqueness of the solution to the corresponding problem and the convergence to it of a special class of sequences, the so-called approximating sequences. In this section we introduce two well-posedness concepts in the study of Problem $\mathcal{P}$ and discuss their relationship. We start with the following definition.

Definition 1. A sequence $\{u_n\} \subset C([0,T];X)$ is said to be:

a) *a $\mathcal{T}_1$-approximting sequence* if $u_n(t) \in K$ for any $n \in \mathbb{N}$, $t \in [0,T]$, and there exists a sequence $0 \le \varepsilon_n \to 0$ such that

$$\begin{aligned} &\langle Au_n(t), v - u_n(t)\rangle + j(\mathcal{S}u_n(t), u_n(t), v) - j(\mathcal{S}u_n(t), u_n(t), u_n(t)) \\ &\quad + \varepsilon_n \|v - u_n(t)\|_X \ge \langle f(t), v - u_n(t)\rangle \quad \forall\, v \in K,\ n \in \mathbb{N},\ t \in [0,T]. \end{aligned} \tag{60}$$

b) a $\mathcal{T}_2$-approximating sequence if there exists a sequence $0 \le \varepsilon_n \to 0$ such that (15) holds, i.e., it satisfies condition (S_2).

It is easy to see that any $\mathcal{T}_1$-approximating sequence is a $\mathcal{T}_2$-approximating sequence. The example below shows that the converse is not true.

Example 1. Consider the history-dependent inequality (1) in the particular case when $X = Y = \mathrm{I\!R}$, $K = [0,1]$ for all $t \in [0,1]$, $Au = u$ for all $u \in \mathrm{I\!R}$, $j(y,u,v) = yv$ for all $y,\ u,\ v \in \mathrm{I\!R}$, $f(t) = t$ for all $t \in [0,1]$ and

$$\mathcal{S}u(t) = k\int_0^t u(s)\,ds \qquad \forall\, u \in C([0,T]),\ t \in [0,1],$$

whith k being an positive constant such that $k \ge 2$. Then, problem $\mathcal{P}$ consists to find a function $u \in C([0,T])$ such that

$$u(t) \in [0,1], \quad \Big(u(t) + k\int_0^t u(s)\,ds - t\Big)(v - u(t)) \ge 0 \tag{61}$$

for all $v \in [0,1]$, $t \in [0,1]$. It is easy to check that the function

$$u(t) = \frac{1}{k}\,(1 - e^{-kt}) \qquad \forall\, t \in [0,1] \tag{62}$$

is the unique solution of this inequality. Moreover, the sequence $\{u_n\} \subset C([0,T])$ defined by

$$u_n(t) = \frac{1}{k}\,(1 - e^{-kt}) + \frac{1}{kn} \qquad \forall\, n \in \mathbb{N},\ t \in [0,1]$$

is a $\mathcal{T}_1$-approximating sequence since $u_n(t) \in K$ for all $n \in \mathbb{N}$, $t \in [0,1]$ and, in addition, inequality (60) holds with $\varepsilon_n = \frac{k+1}{kn}$. In contrast, the sequence $\{u_n'\} \subset C([0,T])$ defined by

$$u_n'(t) = \frac{1}{k}\,(1 - e^{-kt}) - \frac{1}{n} \qquad \forall\, n \in \mathbb{N},\ t \in [0,1]$$

is not a $\mathcal{T}_1$-approximating sequence since $u_n(0) = -\frac{1}{n} \notin K$, for any $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $t \in [0,1]$. We note that $d(u_n'(t), K) \leq |u_n'(t) - u(t)| = \frac{1}{n}$ and, moreover, $\{u_n'\}$ satisfies inequality (15)(b) with $\varepsilon_n = \frac{k+1}{n}$. We conclude from above that $\{u_n'\}$ satisfies condition (S_2) and, therefore, it is a $\mathcal{T}_2$-approximating sequence.

We now proceed with the following definition, for $i = 1, 2$.

Definition 2. Problem $\mathcal{P}$ is said to be $\mathcal{T}_i$-*well-posed* if it has a unique solution u and every $\mathcal{T}_i$-approximating sequence converges in $C([0,T];X)$ to u.

Denote

$$\mathcal{S}_{\mathcal{P}} = \Big\{ \{u_n\} \subset C([0,T];X) \ : \ u \to u \ \text{ in } \ C([0,T];X) \Big\},$$

$$\mathcal{S}_{\mathcal{T}_i} = \Big\{ \{u_n\} \subset C([0,T];X) \ : \{u_n\} \text{ is a } \mathcal{T}_i\text{-approximating sequence} \Big\},$$

for $i = 1, 2$. Then, Defintion 5 states that, for $i = 1, 2$, Problem $\mathcal{P}$ is $\mathcal{T}_i$-well-posed if and only if $\mathcal{S}_{\mathcal{T}_i} \subset \mathcal{S}_{\mathcal{P}}$.

We are now turn to our main result in this section.

Theorem 4. *Assume (2)–(7). Then, the following statements hold.*

a) *Problem $\mathcal{P}$ is $\mathcal{T}_1$-well-posed, i.e., $\mathcal{S}_{\mathcal{T}_1} \subset \mathcal{S}_{\mathcal{P}}$.*

b) *If, moreover, (14) holds, then Problem $\mathcal{P}$ is $\mathcal{T}_2$-well-posed, i.e., $\mathcal{S}_{\mathcal{T}_2} \subset \mathcal{S}_{\mathcal{P}}$.*

c) *If, in addition, (13) and (14) hold, then $\mathcal{S}_{\mathcal{T}_2} = \mathcal{S}_{\mathcal{P}}$.*

Proof. a) The unique solvability of Problem $\mathcal{P}$ follows from Theorem 2. Let $\{u_n\} \subset C([0,T];X)$ be a $\mathcal{T}_1$-approximating sequence. Then, (60) holds with $0 \leq \varepsilon_n \to 0$. Let $n \in \mathbb{N}$ and $t \in [0,T]$. We take $v = u_n(t)$ in (1) and $v = u(t)$ in (60), then we add the resulting inequalities to obtain that

$$\begin{aligned}
&\langle Au_n(t) - Au(t), u_n(t) - u(t)\rangle \\
&\quad \leq j(\mathcal{S}u_n(t), u_n(t), u(t)) - j(\mathcal{S}u_n(t), u_n(t), u_n(t)) \\
&\qquad + j(\mathcal{S}u(t), u(t), u_n(t)) - j(\mathcal{S}u(t), u(t), u(t)) + \varepsilon_n \|u_n(t) - u(t)\|_X.
\end{aligned}$$

We now use assumptions (3)(a) and (5)(b) to deduce that

$$(m_A - \alpha_j)\|u_n(t) - u(t)\|_X \leq \beta_j \|\mathcal{S}u_n(t) - \mathcal{S}u(t)\|_X + \varepsilon_n.$$

Then, the smallness condition (6), assumption (4) on the operator $\mathcal{S}$ and the Gronwall argument leads to an inequality of the form

$$\|u_n(t) - u(t)\|_X \leq C\varepsilon_n$$

with some constant C which does not depend on n and t. This implies that $u_n \to u$ in $C([0,T];X)$. Therefore, using Definition 5 with $i = 1$ we find that Problem $\mathcal{P}$ is $\mathcal{T}_1$-well-posed.

b) Definition 5 b) shows that any $\mathcal{T}_2$-approximating sequence satisfies the statement (S_2). Therefore, using Theorem 3 it follows that, under the additional assumption (14), any $\mathcal{T}_2$-approximating sequence converges in $C([0,T];X)$ to the solution u. Hence, using Definition 5 with $i = 2$ we find that Problem $\mathcal{P}$ is $\mathcal{T}_2$-well-posed,

c) Assume now that (13) and (14) hold. Then, Theorem 3 and Definition 5 b) show that $\{u_n\} \in \mathcal{S}_{\mathcal{T}_2}$ if and ony if $\{u_n\} \in \mathcal{S}_{\mathcal{P}}$, which concludes the proof. $\square$

A validation of Theorem 5 is as follows.

Example 2. Consider the history-dependent variational inequality (61) in Example 5. Recall that the solution of this ineqaulity is the function (62) and, moreover, the sequences $\{u_n\}$ and $\{u'_n\}$ are $\mathcal{T}_1$- and $\mathcal{T}_2$-approximating sequences, respectively. It is easy to see that these sequences converge uniformly to the solution (62), which validate the statement of Theorem 5. We end this section with the following comments.

Remark 3. 1) Definition 5 shows that the inclusion $\mathcal{S}_{\mathcal{T}_1} \subset \mathcal{S}_{\mathcal{T}_2}$ always holds. Moreover, Example 5 shows that this inclusion is strict.

2) Definition 5 combined with inclusion $\mathcal{S}_{\mathcal{T}_1} \subset \mathcal{S}_{\mathcal{T}_2}$ shows that the $\mathcal{T}_2$-well-posedness of Problem $\mathcal{P}$ implies its $\mathcal{T}_1$-well-posedness. Nevertheless, Theorem 5 states that the $\mathcal{T}_1$-well-posedness of Problem $\mathcal{P}$ arises under less restrictive assumptions than those used to prove its $\mathcal{T}_2$-well-posedness.

3) The equality $\mathcal{S}_{\mathcal{T}_2} = \mathcal{S}_{\mathcal{P}}$ in Theorem 5 c) shows that, under the additional assumptions (13) and (14), among all the concepts which make inequality (1) well-posed, the $\mathcal{T}_2$-well-posedness concept generates the largest set of approximating sequences. Indeed, to prove this statement, we consider a different well-posedness concept, say the $\mathcal{T}$-well-posedness concept, defined by the set of approximating sequences $\mathcal{S}_{\mathcal{T}}$. Then, if Problem $\mathcal{P}$ is $\mathcal{T}$-well posed we have $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{P}}$ and, since $\mathcal{S}_{\mathcal{T}_2} = \mathcal{S}_{\mathcal{P}}$, we deduce that $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{T}_2}$, which ends the proof.

6 An example

In this section we present an example of history-dependent variational inequality of the form (1) for which our results in Sections 3–5 hold. The inequality represents the variational formulation of the following history-dependent boundary value problem with unilateral constraints.

Problem $\mathcal{M}$. Find $u : [0,T] \times \Omega \to \mathbb{R}$ and $\xi : [0,T] \times \Omega \to \mathbb{R}$ such that

$$\left.\begin{aligned}
&u(t) \le g,\\
&\mu\Delta u(t) + \int_0^t b(t-s)\Delta u(s)\,ds + \xi(t) + f_0(t) \ge 0,\\
&(u(t)-g)\Big(\mu\Delta u(t) + \int_0^t b(t-s)\Delta u(s)\,ds + \xi(t) + f_0(t)\Big) = 0\\
&-\xi(t) = p(u(t))
\end{aligned}\right\}\ \text{in}\ \ \Omega,$$

$$u(t) = 0 \qquad \text{on}\ \ \Gamma.$$

Here $\Omega \subset \mathbb{R}^2$ is a regular domain with boundary Γ, $[0,T]$ represents the time interval of interest with $T > 0$, g and μ are positive constants, and f_0, p are given functions which will be described below. This problem models the equilibrium of a viscoelastic membrane which occupes the domain Ω, is fixed on its boundary and is in contact along its surface with an obstacle, the so-called foundation. The unknown u is the vertical displacement of the membrane, μ is the Lamé coefficient and f_0 represents the density of applied body force. The obstacle is assumed to be made of a rigid body covered of a layer of deformable material with thickness g. The unknown ξ represents the reaction of this layer. The model above is obtained by taking into account the equilibrium equation, the normal compliance contact condition for the deformable layer and the Signorini contact condition for the rigid body. It represents a two-dimensional version of various models of contact studied in [20, 23], for instance.

We now turn to the variational formulation of Problem $\mathcal{M}$ and, to this end, we use the short hand notation X for the Sobolev space $H_0^1(\Omega)$ endowed with the inner product

$$(u,v)_X = (\nabla u, \nabla v)_{L^2(\Omega)^2} \qquad \forall\, u,\, v \in X$$

and the associated norm $\|\cdot\|_X$. Recall that the Friedrichs-Poincaré inequality guarantees that X is a Hilbert space. We denote in what follows by X^* the dual of X and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. Also, we use $\boldsymbol{x}$ to represent a typical point in $\Omega \cup \Gamma$ and, for simplicity, we sometimes skip the dependence of various functions on the spatial variable $\boldsymbol{x}$.

Next, we consider the following assumptions on the data p and f_0.

$$\left\{\begin{array}{l} p\colon \Omega \times \mathbb{R} \to \mathbb{R}_+ \text{ is such that} \\ \quad \text{(a) there exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \leq L_p |r_1 - r_2| \\ \qquad\quad \text{for all } r_1,\ r_2 \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \quad \text{(b) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))\,(r_1 - r_2) \geq 0 \\ \qquad\quad \text{for all } r_1,\ r_2 \in \mathbb{R},\ \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \quad \text{(c) } p(\cdot, r) \text{ is measurable on } \Omega \text{ for all } r \in \mathbb{R}, \\ \quad \text{(d) } p(\boldsymbol{x}, r) = 0 \ \text{ if and only if } \ r \leq 0,\ \text{ a.e. } \boldsymbol{x} \in \Omega. \end{array}\right. \tag{63}$$

$$f_0 \in C([0,T]; L^2(\Omega)). \tag{64}$$

Moreover, we recall the inequalities

$$\mu > 0, \quad g > 0. \tag{65}$$

We now define the set K, the operators $A : X \to X^*$ and $\mathcal{S} : C([0,T]; X) \to C([0,T]; X^*)$, and the functions $j : X^* \times X \times X \to \mathbb{R}$, $f : [0,T] \to X^*$ by equalities

$$K = \{\, v \in X : \ v \leq g \ \text{ a.e. in } \Omega \,\}, \tag{66}$$

$$\langle Au, v \rangle = \mu \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega p(u) v \, dx \qquad \forall\, u,\ v \in X, \tag{67}$$

$$\langle \mathcal{S}u(t), v \rangle = \int_\Omega \Big(\int_0^t b(t-s) \nabla u(s)\, ds \Big) \cdot \nabla v \, dx \tag{68}$$
$$\forall\, u \in C([0,T]; X),\ t \in [0,T],\ v \in X,$$

$$j(y, u, v) = \langle y, v \rangle \qquad \forall\, y \in X^*,\ u,\ v \in X, \tag{69}$$

$$\langle f(t), v \rangle_V = \int_\Omega f_0(t)\, v \, dx \qquad \forall\, v \in X,\ t \in [0,T]. \tag{70}$$

With these notation, by using standard arguments we deduce the following variational formulation of Problem $\mathcal{M}$, in terms of displacement.

Problem $\mathcal{M}^V$. *Find a displacement field $u\colon [0,T] \to X$ such that, for all $t \in [0,T]$, the inequality below holds:*

$$u(t) \in K, \quad \langle Au(t), v - u(t)\rangle + j(\mathcal{S}u(t), u(t), v) - j(\mathcal{S}u(t), u(t), v) \qquad (71)$$
$$\geq \langle f(t), v - u(t)\rangle \quad \forall\, v \in K.$$

It is easy to see that, under assumptions (63)–(65), the set K, the operators A, $\mathcal{S}$ and the functions j, f defined by (66)–(70) satisfy conditions (2)–(7) on the spaces $X = H_0^1(\Omega)$ and $Y = X^*$, with $m_A = \mu$, $\alpha_j = 0$ and $\beta_j = 1$. Therefore, Theorem 2 guarantees the unique solvability of problem $\mathcal{M}^V$. Moreover, conditions (13) and (14) are satisfied, with $L_A = \mu$ and $c_j(r,s) = r$ for all $r,\ s \in \mathbb{R}_+$. Hence, the convergence criterion provided by Theorem 3 can be used in this case. It allows us to identify various sequences which convergence to the solution of problem $\mathcal{M}^V$ and to provide various mechanical interpretation. For instance, using the arguments in [23] it is possible to use Theorem 3 in order to show that the solution of the problem $\mathcal{M}^V$ depends continuously on the Lamé coefficient μ and the thickness g. In addition, the penalty method in Theorem 4 works in the study of Problem $\mathcal{M}^V$ which, in addition, is both $\mathcal{T}_1$- and $\mathcal{T}_2$-well-posed.

References

[1] H. Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* 18 (1968), 115-175.

[2] H. Brézis, Problèmes unilatéraux, *J. Math. Pures Appl.* 51 (1972), 1-168.

[3] A. Capatina, *Variational Inequalities and Frictional Contact Problems*, Advances in Mechanics and Mathematics, Vol. **31**, Springer, New York, 2014.

[4] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[5] A.L. Dontchev and T. Zolezzi, *Well-posed Optimization Problems*, Lecture Notes Mathematics **1543**, Springer, Berlin, 1993.

[6] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[7] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[8] R. Glowinski, J.-L. Lions and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[9] J. Gwinner, B. Jadamba, A.A. Khan and F. Raciti, *Uncertainty Qualification in Variational Inequalities. Theory, Numerics and Applications*, CRC Press, Boca Raton, 2022.

[10] W. Han and B.D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, second edition, Springer-Verlag, 2013.

[11] X.X. Huang and X.Q. Yang, Generalized Levitin-Polyak well-posedness in constrained optimization, *SIAM J. Optim.* 17 (2006), 243-258.

[12] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

[13] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Classics in Applied Mathematics **31**, SIAM, Philadelphia, 2000.

[14] E.S. Levitin and B.T. Polyak, Convergence of minimizing sequences in conditional extremum problem, *Soviet Math. Dokl.* 7 (1966), 764-767.

[15] R. Lucchetti, *Convexity and Well-posed Problems*, CMS Books in Mathematics, Springer-Verlag, New York, 2006.

[16] R. Lucchetti and F. Patrone, A characterization of Tychonov well-posedness for minimum problems with applications to variational inequalities, *Numer. Funct. Anal. Optim.* 3 (1981), 461-476.

[17] R. Lucchetti and F. Patrone, Some properties of “well-posedness” variational inequalities governed by linear operators, *Numer. Funct. Anal. Optim.* 5 (1983), 349-361.

[18] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.

[19] D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Sijthoff and Noordhoff International Publishers, Alpen aan den Rijn, 1978.

[20] M. Sofonea, *Well-posed Nonlinear Problems. A Study of Mathematical Models of Contact*, Advances in Machanics and Mathematics **50**, Birkhäuser, Cham, 2023.

[21] M. Sofonea and A. Matei, History-dependent quasivariational inequalities arising in Contact Mechanics, *European J. Appl. Math.* 22 (2011), 471-491.

[22] M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series **398**, Cambridge University Press, 2012.

[23] M. Sofonea and S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, 2018.

[24] M. Sofonea and D.A. Tarzia, Convergence results for history-dependent variational inequalities, *Axioms* 13 (2024), 316.

[25] A.N. Tykhonov, On the stability of functional optimization problems, *USSR Comput. Math. Math. Phys.* 6 (1966), 631-634.

[26] Y.B. Xiao, N.J. Huang and M.M. Wong, Well-posedness of hemivariational inequalities and inclusion problems, *Taiwanese J. Math.* 15 (2011), 1261-1276.

[27] E. Zeidler, *Nonlinear Functional Analysis and Applications II A/B, Springer*, New York, 1990.

[28] T. Zolezzi, Extended well-posedness of optimization problems, *J. Optim. Theory Appl.* 91 (1996), 257-266.