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DO PERFECT POWERS REPEL PARTITION NUMBERS?*

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In memory of Haim Brezis who made outstanding contributions to mathematics

Abstract

In 2013 Zhi-Wei Sun conjectured that $p(n)$ is never a power of an integer when $n > 1$. We confirm this claim in many cases. We also observe that integral powers appear to repel the partition numbers. If $k > 1$ and $\Delta_k(n)$ is the distance between $p(n)$ and the nearest kth power, then for every $d \geq 0$ we conjecture that there are at most finitely many n for which $\Delta_k(n) \leq d$. More precisely, for every $\varepsilon > 0$, we conjecture that

$$M_k(d) := \max\{n \ : \ \Delta_k(n) \leq d\} = o(d^{\varepsilon}).$$

In k-power aspect with d fixed, we also conjecture that if k is sufficiently large, then

$$M_k(d) = \max\{n \ : \ p(n) - 1 \leq d\}.$$

In other words, 1 generally appears to be the closest kth power among the partition numbers.

Keywords: partition function, perfect powers.
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1 Introduction

A *partition* of a non-negative integer n is any non-increasing sequence of positive integers that sum to n, and the *partition function* $p(n)$ denotes their number. The long history of $p(n)$ (for example, see [2]) is marked with celebrated contributions by mathematicians such as Euler, Hardy, and Ramanujan. Indeed, we have Euler's famous recurrence relation, which asserts for positive n that

$$\begin{aligned} p(n) &= \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^{k+1} p\left(n - \frac{3k^2+k}{2}\right) \\ &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots . \end{aligned}$$

Hardy and Ramanujan famously proved the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}},$$

in work which gave birth to the "circle method", a tool which is now ubiquitous in analytic number theory. Their approach was later perfected by Rademacher, who obtained an exact formula as an infinite convergent series

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

Here $I_{\frac{3}{2}}(\cdot)$ is a modified Bessel function of the first kind, and $A_k(n)$ is a Kloosterman-type sum. Extending beyond the world asymptotics, exact formulas, and recurrence relations, Ramanujan famously proved that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

representing the first cases of three infinite families of congruences modulo arbitrary powers of 5, 7, and 11. These congruences have inspired an entire field of study in combinatorics and the theory of modular forms.

2 Sun's Conjecture

Despite its long history, the partition function continues to be a source of beautiful and fascinating problems. Here we refine one such problem, the following 2013 conjecture of Zhi-Wei Sun (see Conjecture 8.9 (iii) of [4] and also [5, 6]).[1]

Conjecture (Sun)**.** *If $n > 1$, then $p(n)$ is not a kth power of an integer for any $k > 1$.*

Although we are unable to prove this conjecture, we show that recent deep work of Bennett and Siksek [3] on Diophantine equations offers strong evidence supporting its truth. Their work precludes many partition numbers from being perfect powers. To make this precise, we define the following special set of positive integers

$$S := \left\{n \,:\, p(n) = x^2 + \ell^a, \text{ with } x \in \mathbb{Z},\, 2 \leq \ell < 100 \text{ prime},\, a \geq 1 \text{ and } \ell \nmid x\right\}.$$

Theorem 1. *For integers $n \in S$, the following are true.*
(1) We have that $p(n)$ is not a kth power for any $k \geq 3$.
(2) Sun's Conjecture for squares implies the conjecture for all kth powers, where $k > 1$.

Proof of Theorem 1. Bennett and Siksek consider Diophantine equations of the form

$$x^2 + q^\alpha = y^k.$$

They prove (see Theorem 1 of [3]) that if x, y, q, α and k are solutions in the positive integers with $2 \leq q < 100$ prime, $q \nmid x$, $k \geq 3$, then (q, α, y, k) is a member of an explicit finite list

$$(2, 1, 3, 3), (2, 2, 5, 3), (2, 5, 3, 4), \ldots, (89, 1, 5, 3), (97, 2, 12545, 3), (97, 1, 7, 4).$$

Applied to our setting, we replace q by ℓ, and note that if $n \in S$, then we have the identity

$$p(n) = x^2 + \ell^a.$$

Therefore, if $p(n) = y^k$ is a kth power, then apart from the finitely many possible candidates given by this explicit finite list, it must be that $k \in \{1, 2\}$. By brute force, we checked that none of the numbers y^k from this list are actual partition numbers. This is an easy finite calculation, as $p(n)$ is an increasing sequence in n. Therefore, both claims follow. □

[1] In the case of squares, T. Amdeberhan independently made the same conjecture/speculation in 2017 [1].

Example. Using Theorem 1 (1), we show that $p(3), p(4), \ldots, p(15) = 176$ are not kth powers for any $k \geq 3$. The point is that this conclusion is made without factorizing these explicit values. To this end, we list the relevant numbers of the form $x^2 + \ell^a$ that are used to define S. Ordered by size, they are

$$\{3, 4, 5, \ldots, 36\} \cup \{38, 39, 40, \ldots, 63\} \cup \{65, 66, 67, \ldots, 120\} \\ \cup \{122, 123, 124, \ldots, 135\} \cup \{137, 138\} \cup \{140, 141, \ldots, 155\} \\ \cup \{157, 158, \ldots, 164\} \cup \{167, 168, \ldots, 183\} \cup \ldots.$$

The positive integers not exceeding $p(15) = 176$ that are missed are:

$$1, 2, 37, 64, 121, 136, 139, 156, 165, 166.$$

Since the partition numbers $p(3) = 3, p(4) = 5, \ldots, p(15) = 176$ are not among these ten values, Theorem 1 (1) implies that none are kth powers for any $k \geq 3$.

Remark. Theorem 1 does not apply to all n. Indeed, among the first 20 partition numbers, it does not apply for the three values $p(2) = 2, p(16) = 231$, and $p(19) = 490$. There are no positive integer solutions to

$$x^2 + \ell^a \in \{2, 231, 490\},$$

where $\ell \leq 100$ is prime and $\ell \nmid x$.

3 Further Conjectures: Do kth powers repel partitions numbers?

To refine Sun's Conjecture, it makes sense to search for "near misses", partition numbers that are close to kth powers. A brief scan of a table of values of $p(n)$ does not reveal many near misses. In fact, the partition numbers appear to be repelled by the kth powers. Indeed, the perfect squares up to 250 are:

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ \mathbf{100},\ 121,\ 144,\ 169,\ 196,\ 225,\ \ldots$$

while the partition numbers in this range are:

$$1,\ 1,\ 2,\ 3,\ 5,\ 7,\ 11,\ 15,\ 22,\ 30,\ 42,\ 56,\ 77,\ \mathbf{101},\ 135,\ 176,\ 231, \ldots.$$

Apart from the perfect square 1, the only notable near miss is the number $p(13) = 101$.

To quantify this observation, we define $\Delta_k(n) \geq 0$ to be the distance between $p(n)$ and its nearest kth power. Namely, we let

$$\Delta_k(n) := \min\left\{|p(n) - m^k| \ : \ m \in \mathbb{Z}\right\}. \tag{1}$$

Table 1 includes these values for squares, cubes, and fourth powers for

$p(10) = 42$, $p(20) = 627$, $p(30) = 5604$, $p(40) = 37338$, and $p(50) = 204226$.

n	$\Delta_2(n)$	$\Delta_3(n)$	$\Delta_4(n)$
10	6	15	26
20	2	102	2
30	21	228	957
40	89	1401	1078
50	78	1153	9745

Table 1: Sample values of $\Delta_k(n)$

Based on numerics performed on a computer, we make the following conjecture.

Conjecture 1. *If $k > 1$ and $d \geq 0$, then there are at most finitely many n for which*

$$\Delta_k(n) \leq d.$$

Assuming this conjecture, we study the "growth" of $\Delta_k(n)$. To this end, we define

$$M_k(d) := \max\{n \ : \ \Delta_k(n) \leq d\}, \tag{2}$$

the last n for which $p(n)$ is within d of a kth power. Table 2 gives conjectured values for $d \leq 10^{70}$.

d	$M_2(d)$	$M_3(d)$	$M_4(d)$	$M_5(d)$	$M_6(d)$	$M_7(d)$	$M_8(d)$	$M_{50}(d)$	$M_{100}(d)$
0	1	1	1	1	1	1	1	1	1
10^0	35	5	7	2	2	2	2	2	2
10^5	201	133	87	82	64	71	64	45	45
10^{10}	527	295	265	247	227	258	196	135	135
10^{15}	1100	705	482	454	445	388	387	279	269
10^{20}	2058	1019	806	745	654	653	633	444	444
10^{25}	2595	1525	1203	1052	971	978	890	663	662
10^{30}	3804	2135	1636	1564	1337	1244	1280	941	941
10^{35}	5030	2815	2444	1930	1886	1747	1620	1239	1221
10^{40}	6340	3714	2849	2513	2366	2246	2047	1565	1562
10^{45}	8253	4424	3516	3178	2866	2754	2685	2170	2170
10^{50}	9646	5479	4314	3726	3537	3411	3134	2556	2368
10^{55}	11524	6808	5229	4802	4169	3933	3815	2909	2833
10^{60}	13723	8088	6117	5318	4854	4629	4506	3501	3382
10^{65}	15516	8924	6961	6403	5676	5502	5232	4129	3884
10^{70}	18237	10252	8084	7056	6628	6268	6098	4665	4502

Table 2: Conjectured values of $M_k(d)$

Based on these numerics, we make the following refinement of Conjecture 1.

Conjecture 2. *If $k > 1$, then the following are true.*
(1) We have that $M_k(0) = 1$.
(2) For every $\varepsilon > 0$, we have that $M_k(d) = o(d^\varepsilon)$.

Remark. Conjecture 2 (1) is a recapitulation of Sun's Conjecture, and claim (2) asserts that

$$\lim_{d\to+\infty} \frac{M_k(d)}{d^\varepsilon} = 0.$$

Conjecture 2 asserts that the $M_k(d)$ numbers are bounded by any positive power of d. It is natural to seek lower bounds. In Figure 1, we offer the values of $M_k(d)$ for $d \leq 10^{70}$ in the form of a graph to illustrate the slow growth of these values.

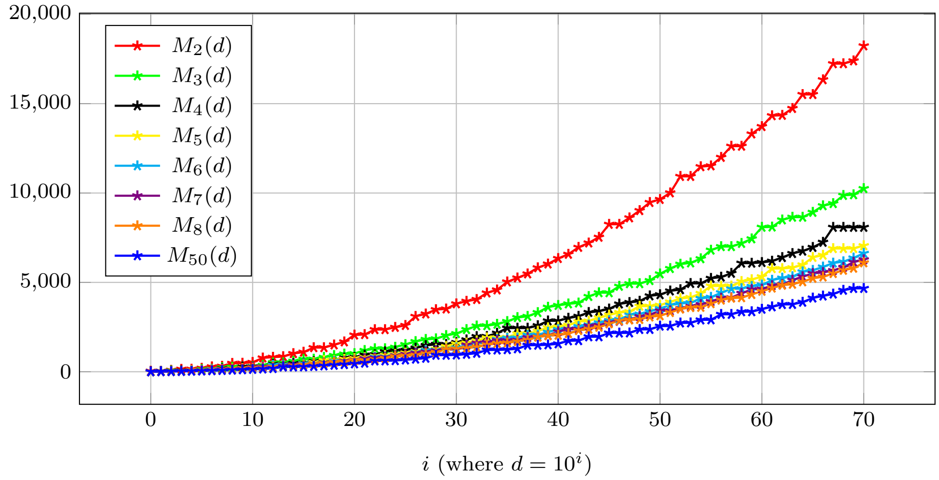

Figure 1: Growth of $M_k(d)$ with $d \leq 10^{70}$

It is tempting to seek a model that asymptotically gives the values $M_k(d)$ as $d \to +\infty$. A natural approach is to apply the standard non-linear least squares fitting method. We did so using the values at powers of 10 with $d \leq N$, and we computed approximations $f_k(N; d)$ that are polynomials in $\log(d)$. For $k = 50$ and $N \in \{10^{12}, 10^{70}\}$, we obtain two very different approximations

$$\begin{aligned} f_{50}(10^{12}; d) &\approx 2.018 \times 10^{-13} \cdot (\log d)^3 + 0.174 \cdot (\log d)^2 \\ &\qquad + 1.836 \cdot \log d + 0.748, \\ f_{50}(10^{70}; d) &\approx 1.91 \times 10^{-8} \cdot \log(d)^5 - 8.77 \times 10^{-6} \cdot \log(d)^4 \\ &\qquad + 1.25 \times 10^{-3} \cdot \log(d)^3 + 0.1118 \cdot \log(d)^2 \\ &\qquad + 2.260 \cdot \log(d) + 8.946. \end{aligned}$$

It is natural to speculate whether the $M_k(d)$ are asymptotically polynomials in $\log(d)$ as $d \to \infty$. In Figure 2 we juxtapose some of the $M_k(d)$ in Table 2 with $\log(d)$ and $\log(d)^2/5$. The trends in this figure suggest that these values are growing much faster than any scalar multiple of $\log(d)$, but do not preclude the possibility that they are asymptotically higher degree (perhaps quadratic or cubic) polynomials in $\log(d)$.

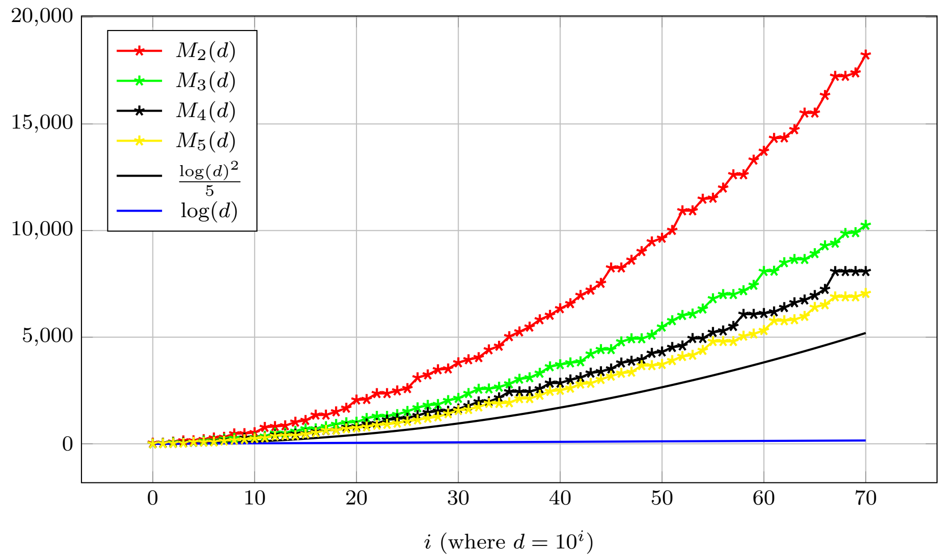

Figure 2: Growth of $M_k(d)$ with $k \in \{2, 3, 4, 5\}$

It would be very interesting to carry out even further numerics to address the following problem.

Problem. For each k, conjecture an asymptotic formula for the $M_k(d)$.

To conclude, we offer a curious conjecture about the sequences

$$\{M_1(d), M_2(d), M_3(d), \dots\}$$

for each fixed d. The first two rows of Table 2 suggest that these sequences decrease and stabilize at the values 1 and 2 respectively. We have done many further numerics, and we believe that each of these sequences eventually stabilizes. Table 3 below suggests that these limiting values are the positive integers in order repeated with curious multiplicities.

d	$M_2(d)$	$M_3(d)$	$M_4(d)$	$M_5(d)$	$M_6(d)$	$M_7(d)$	$M_8(d)$	$M_{50}(d)$	$M_{100}(d)$
0	1	1	1	1	1	1	1	1	1
1	35	5	7	2	2	2	2	2	2
2	35	22	20	9	3	3	3	3	3
3	35	22	20	9	3	3	3	3	3
4	35	22	20	9	4	4	4	4	4
5	35	22	20	9	4	4	4	4	4
6	35	22	20	9	5	5	5	5	5

Table 3: Conjectured values of $M_k(d)$

Conjecture 3. *For each fixed non-negative integer d, there is an integer L_d for which*

$$M_k(d) = L_d$$

for all sufficiently large k.

We refine this claim by offering a conjectural partition theoretic description of the limiting values and their multiplicities, which is based on the number of partitions of n without parts of size 1, denoted by $p_1(n)$. Clearly, the generating function for $p_1(n)$ is

$$\begin{aligned}\sum_{n=0}^{\infty} p_1(n)q^n &= \prod_{n=2}^{\infty} \frac{1}{1-q^n} \\ &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + \dots.\end{aligned}$$

In terms of $p_1(n)$, we define the the auxiliary partition

$$\psi(n) = \sum_{j=1}^{n} p_1(j), \tag{3}$$

the number of non-empty partitions of size $\leq n$ without parts of size 1. It is not difficult to confirm that its generating function is given by

$$\begin{aligned}\Psi(q) = \sum_{n=1}^{\infty} \psi(n)q^n &= \sum_{n=2}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= q^2 + 2q^3 + 4q^4 + 6q^5 + 10q^6 + 14q^7 + 21q^8 + 29q^9 + 41q^{10} + \dots.\end{aligned}$$

The astute reader will notice that $\psi(n) = p(n) - 1$ for all n. In loose analogy with the $M_k(d)$, we define the function

$$L(d) := \max\{n \ : \ \psi(n) \leq d\} = \max\{n \ : \ p(n) - 1 \leq d\}. \tag{4}$$

Clearly, we have that

$$L(d) = \begin{cases} 1 & \text{for } d = 0, \\ 2 & \text{for } d = 1, \\ 3 & \text{for } d = 2, 3, \\ 4 & \text{for } d = 4, 5, \\ 5 & \text{for } d = 6, 7, 8, 9, \\ \vdots & \\ n & \text{repeated } p_1(n+1) \text{ times.} \end{cases}$$

We believe that the values of $L(d)$ are the limiting values L_d which make up the alleged curious sequence of all the positive integers in order with curious multiplicities, a claim which essentially means that $p(1) = 1$ is generally the closest kth power among the partition numbers.

Conjecture 4. *For each non-negative integer d, there is a positive integer N_d such that for $k \geq N_d$ we have*

$$M_k(d) = L(d).$$

Remark. In Table 4 below, we give the conjectured minimal values that can be taken for N_d in Conjecture 4. We believe that $N_d = O(\log(d))$.

d	0	1	[2,6]	[7,21]	[22,89]
N_d	2	5	6	8	11
d	[90,156]	[157,1500]	[1501,1582]	[1583,2274]	[2275,2534]
N_d	12	14	15	16	17
d	[2535,72928]	[72929,84593]	[84594,106335]	[106336,270343]	[270344,708529]
N_d	20	21	22	23	24

Table 4: Conjectured minimal values for N_d in Conjecture 4

4 Concluding thoughts

Conjectures about the asymptotics and congruence properties of $p(n)$ have driven research in analytic number theory and the theory of modular forms. Indeed, the partition numbers gave birth to the "circle method" in analytic number theory through the work of Hardy and Ramanujan, and has inspired the development of Hecke operators and the theory of modular forms congruences through the work of Atkin and many others. It is our hope that the conjectures presented here will inspire further deep advances in the field.

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References

[1] T. Amdeberhan, How close do partitions get to perfect squares?, Math Overflow, https://mathoverflow.net/questions/263080/, Feb. 25, 2017.

[2] G.E. Andrews, *The theory of partitions*, Cambridge Univ. Press, Cambridge, 1998.

[3] M.A. Bennett and S. Siksek, Differences between perfect powers: prime power gaps, *Algebra Number Theory* 17(10) (2023), 1789-1846.

[4] Z.-W. Sun, *New conjectures in number theory and combinatorics (in Chinese)*, Harbin University of Technology Press, Harbin, 2021.

[5] Z.-W. Sun, A000041 $a(n)$ is the number of partitions of n, OEIS, https://oeis.org/A000041, Dec. 2, 2013.

[6] Z.-W. Sun, Can the partition function $p(n)$ take perfect power values?, Math Overflow, https://mathoverflow.net/questions/315828/, Nov. 21, 2018.