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NONLINEAR CRITERIA FOR STABILITY OF NONAUTONOMOUS DYNAMICS - A NEW ZABCZYK-ROLEWICZ TYPE APPROACH*

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Dedicated to the memory of Professor Haïm Brezis

Abstract

We present new Zabczyk-Rolewicz type methods of exploring the uniform exponential stability of nonautonomous dynamics. First, we give a characterization of Zabczyk type for uniform exponential stability of discrete nonautonomous systems. Next, we obtain a new criterion of Zabczyk-Rolewicz type for uniform exponential stability, using general nonlinear trajectories. In addition, we discuss consequences for uniformly bounded discrete nonautonomous systems. Furthermore, we apply our results to deduce the famous stability theorem of Rolewicz for continuous-time dynamics.

Keywords: uniform stability, uniform exponential stability, nonautonomous dynamics, evolution family, nonlinear trajectory.

MSC: 34D20, 37C20, 39A30, 93D20.

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1 Introduction

In the last decades, the criteria for detecting the existence of various asymptotic behaviors of dynamical systems have had a notable impact on the development of important methods in control theory (see [1–55]). In this framework, a remarkable class of methods has been built around results known as Datko-Pazy-type, Przyłuski-Rolewicz-type and Zabczyk-type theorems, respectively. The roots of these methods are anchored in a collection of famous studies started in the seventies by Datko [5–7], Pazy [30] and Zabczyk [50], being followed and consolidated in the eighties by Przyłuski and Rolewicz [33], Przyłuski [34] and Rolewicz [35, 36]. Although the first results on this topic were developed for autonomous or nonautonomous systems (see the foundations in [2, 3, 5–8, 15, 18, 28–30, 32–36, 50] as well as the more recent contributions from [10, 20, 21, 38, 42–44, 49]), starting in the end of the 90s, the studies were extended for variational dynamics (see [1, 11, 12, 22–24, 26, 27, 37–41, 45–47] and the references therein). Moreover, even if the pioneering studies focused on stability properties, we emphasize that in the last twenty years, novel approaches have been developed for instability (see [1, 10, 12, 26, 27]) as well as for more complex phenomena such as dichotomous or trichotomous behaviors (see [1, 10, 41, 45–47]). For a detailed description of the history of these methods and their evolution over the years we refer to Sasu, Megan and Sasu [40] and to the recent work Dragičević, Sasu, Sasu and Şirianţu [12].

We recall that in the setting of autonomous dynamics the Zabczyk-type theorems (see [50]) can be stated as:

Theorem 1. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X and let $\mathcal{Z}$ be the class of all convex and increasing functions $Z : \mathbb{R}_+ \to \mathbb{R}_+$ with $Z(0) = 0$. The following properties are equivalent:*

(i) $\mathbf{T}$ is uniformly exponentially stable (i.e. there are $K, \nu > 0$ such that $\|T(t)\| \leq Ke^{-\nu t}$, for all $t \geq 0$);

(ii) there is $Z \in \mathcal{Z}$ such that for each $x \in X$ there is $\lambda(x) > 0$ with

$$\sum_{k=0}^{\infty} Z(\lambda(x)||T(k)x||) < \infty;$$

(iii) there is $Z \in \mathcal{Z}$ such that for each $x \in X$ there is $\lambda(x) > 0$ with

$$\int_0^{\infty} Z(\lambda(x)||T(t)x||)dt < \infty.$$

We note that for $Z(t) = t^2$ the equivalence $(i) \iff (iii)$ from Theorem 1 was obtained for the first time by Datko [5] (for X a Hilbert space) and by Pazy [30] for $Z(t) = t^p$ ($p \geq 1$). An elegant approach for these Datko-Pazy type results was provided by Pritchard and Zabczyk in [32], while a slight different nice proof was given by Curtain and Pritchard in [3]. Furthermore, we also refer to the recent book Curtain and Zwart [4] for Datko's theorem and interesting subsequent applications. By dropping the hypothesis that Z is convex and taking $\lambda(x) = 1$, for all $x \in X$, Littman gave a different proof for $(i) \iff (iii)$ in [18]. A beautiful method that unified these theorems in a more general framework, was introduced by Neerven [28], extending the belonging of the trajectories $t \mapsto ||T(t)x||$ to a general Banach function space (from a certain class) compared to their belonging to an L^p-space (as stated in the Datko-Pazy theorems) or to an Orlicz space (as essentially arose in the Zabczyk-type results). For more technical details on these methods we refer to [20–22, 25, 29]. We stress that a general approach to the Zabczyk-type techniques for stability of variational systems was presented in Sasu, Megan and Sasu [40]. Zabczyk type theorems for exponential trichotomy were obtained in Sasu and Sasu [41], for variational systems as well. Furthermore, a completely new and distinct method for the Zabczyk-type results, for both stability and instability, in the framework of variational dynamics, was recently developed by Dragičević, Sasu, Sasu and Şirianţu in [12].

In the setting of the nonautonomous dynamics, we recall two major results. Thus, we first recall the Przyłuski-Rolewicz theorem (see [33]):

Theorem 2. *Let $\mathbf{\Phi} = \{\Phi(j,i)\}_{j,i\in\mathbb{N},\, j\geq i}$ be a discrete evolution family on a Banach space X. $\mathbf{\Phi}$ is uniformly exponentially stable (i.e. there are $K, \nu > 0$ such that $||\Phi(j,i)|| \leq Ke^{-\nu(j-i)}$, for all $j \geq i$) if and only if there is $p \in [1,\infty)$ such that for each $x \in X$*

$$\sup_{i\in\mathbb{N}} \sum_{j=i}^{\infty} ||\Phi(j,i)x||^p < \infty. \tag{1}$$

Following the ideas from the autonomous case, we stress that in Theorem 2, one can replace in (1) the trajectories $j \mapsto ||\Phi(j,i)x||^p$ by some more general maps of the form $j \mapsto Z(||\Phi(j,i)x||)$, where $Z \in \mathcal{Z}$ and $\lim_{t\to\infty} Z(t) = \infty$. In this context, even more extensive approaches can be given, by working with functions of two variables, as it was considered in Sasu [42]. Moreover, for diverse methods which led to the generalizations of the above result to variational dynamics we refer to [12, 38, 40, 45].

On the other hand, a very general approach is the one intiated in Rolewicz's paper [35], which can be unitarily given as:

Theorem 3. *Let $\mathcal{R}$ denote the set of all mappings $R : (0,\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ with $R(\xi,\cdot)$ nondecreasing and continuous with $R(\xi,0) = 0$ and $R(\xi,\tau) > 0$, for all $\tau > 0$ and all $\xi > 0$ and respectively $R(\cdot,\tau)$ is nondecreasing, for all $\tau \geq 0$. Let $\mathcal{U} = \{U(s,\tau)\}_{s\geq\tau\geq 0}$ be an evolution family (see Definition 2 below) on a Banach space X. The following properties are equivalent:*

(i) *$\mathcal{U}$ is uniformly exponentially stable (see Definition 3 below);*

(ii) *there is $R \in \mathcal{R}$ such that for each $x \in X$ there is $\lambda(x) > 0$ with*

$$\sup_{i\in\mathbb{N}} \sum_{j=i}^{\infty} R(\lambda(x), ||U(j,i)x||) < \infty;$$

(iii) *there is $R \in \mathcal{R}$ such that for each $x \in X$ there is $\lambda(x) > 0$ with*

$$\sup_{\tau\geq 0} \int_{\tau}^{\infty} R(\lambda(x), ||U(\xi,\tau)x||)\, d\xi < \infty.$$

We note that the equivalence $(i) \Longleftrightarrow (iii)$ from Theorem 3 was proved for the first time by Rolewicz in [35] (see also [36]). Generalized versions of the equivalence $(i) \Longleftrightarrow (ii)$ from Theorem 3 were obtained in Sasu [42–44]. A unified and general approach for the result in Theorem 3 was provided in Sasu and Sasu [38], as a consequence of a study performed for variational dynamics.

In the particular case $R(\xi,\tau) = \tau^p$, $(p \geq 1)$ the equivalence $(i) \Longleftrightarrow (iii)$ from Theorem 3 yields the original theorem of Datko for evolution families (see [6]). For a version of Datko's theorem in the case of differential equations we refer to Daleckiĭ and Kreĭn [8]. A Datko type method for nonlinear operators was introduced by Ichikawa in [15]. Furthermore, we mention that a nice presentation of the Datko type results for nonautononous dynamics was given in Chicone and Latushkin [2]. Generalizations of the Datko-type theorems for evolution families, based on the theory of Banach sequence and function spaces and on trajectory approaches, were obtained in Megan, Sasu and Sasu [20, 21]. Based on distinct arguments, a generalization of Theorem 3 was presented in Sasu [44]. The first Datko type result for stability in the case of skew-product semiflows was given in Megan, Sasu and Sasu [24] with a direct proof. Other approaches for the Datko-type and Rolewicz-type results for stability of variational dynamics were given in Megan, Sasu and Sasu [22, 23], relying on trajectory arguments and on their belongings to suitable Banach sequence or function spaces.

In recent years, the theorems of Datko-type, Zabczyk-type and Rolewicz-type were treated from many perspectives for both uniform and nonuniform behaviors as well as for both nonautononous and variational dynamics (see Backes and Dragičević [1], Dragičević [9–11], Megan, Sasu and Sasu [26, 27], Sasu [37, 39], Sasu, Megan and Sasu [40], Sasu and Sasu [38, 41, 47, 48], Sasu [45, 46]). We emphasize that the Datko-type theorems have remarkable applications in control theory - in this sense we refer to some recent advances in this area due to Jacob and Wegner [16], Jacob, Möller and Wyss [17], Marinoschi [19], Zabczyk [51]. Furthermore, the Datko-type results have proved their effectiveness in establishing various robustness properties for nonuniform strong behaviors (see Dragičević [10]). Moreover, as already mentioned, for variational dynamics, new techniques in this area were developed in Dragičević, Sasu, Sasu and Şirianţu [12] which also provided novel applications in exploring the robustness of stability and instability properties.

The aim of this paper is to present a new method of exploring the stability of discrete-time nonautonomous dynamics via Zabczyk-Rolewicz type conditions. First we obtain a new characterization for uniform exponential stability of general discrete nonautonomous systems that relies on the convergence of some associated series of trajectories and on Zabczyk-type techniques (Theorem 4). As a consequence, we deduce a criterion of Przyłuski-Rolewicz type for uniform exponential stability of uniformly bounded systems (Corollary 1). Next, we develop a Zabczyk-Rolewicz type method of exploring the uniform exponential stability. The approach is presented in two stages: first we present a technical lemma which ensures the closure of the set of all vectors with the property that certain associated series (of nonlinear trajectories corresponding to a sequence of operators) are convergent (Lemma 1). After that, we obtain a new characterization of Zabczyk-Rolewicz type for uniform exponential stability of discrete nonautonomous systems, without making any additional assumption on their coefficients (Theorem 5). As a consequence, we deduce a Zabczyk-Rolewicz type criterion for uniformly bounded systems (Corollary 2) and furthermore, as an application, we present a proof for the Rolewicz's theorem for stability of evolution families (Section 3).

2 Trajectory criteria and stability of discrete nonautonomous dynamics

In this section we present new Zabczyk type and Przyłuski-Rolewicz type characterizations for uniform exponential stability of nonautonomous dynamics. Our study is performed for general discrete nonautonomous systems and we deduce consequences for uniformly bounded systems.

Let X be a Banach space and let $\mathcal{B}(X)$ denote the space of all bounded linear operators on X. Let $\|\cdot\|$ denote the norm on X and on $\mathcal{B}(X)$. For each $x_0 \in X$ and $r > 0$ we set $D(x_0, r) := \{y \in X : \|y - x_0\| \leq r\}$.

Let $\{A(n)\}_{n\in\mathbb{N}} \subset \mathcal{B}(X)$. Consider the discrete nonautonomous system

$$(A) \qquad x(n+1) = A(n)x(n), \quad n \in \mathbb{N}$$

and the associated *evolution family*, i.e.

$$\Phi_A(j,n) = \begin{cases} A(j-1)\cdots A(n), & j > n \\ I, & j = n \end{cases} \tag{2}$$

where I is the identity operator on X. Let Δ denote the set of all pairs $(j,n) \in \mathbb{N}\times\mathbb{N}$ with $j \geq n$. We recall the classical stability notions:

Definition 1. We say that (A) is

(i) *uniformly bounded* if there is $M > 0$ such that $\|A(n)\| \leq M$, for all $n \in \mathbb{N}$;

(ii) *uniformly stable* if there is $L \geq 1$ such that $\|\Phi_A(j,n)\| \leq L$, for all $(j,n) \in \Delta$;

(iii) *uniformly exponentially stable* if there are $K \geq 1$ şi $\nu > 0$ such that $\|\Phi_A(j,n)\| \leq Ke^{-\nu(j-n)}$, for all $(j,n) \in \Delta$.

Notation Let $\mathcal{Q}$ denote the class of all continuous nondecreasing functions $q : \mathbb{R}_+ \to \mathbb{R}_+$ with $q(0) = 0$ and $q(t) > 0$, for all $t > 0$.

Our first main result is a new Zabczyk type theorem for uniform exponential stability:

Theorem 4. *The system (A) is uniformly exponentially stable if and only if there are $q \in \mathcal{Q}$, $x_0 \in X$, $r > 0$ and $\lambda > 0$ such that for every $x \in D(x_0, r)$ the following two properties hold:*

$$\sup_{n\in\mathbb{N}} \sum_{j=n}^{\infty} q(\|\Phi_A(j,n)x\|) \leq \lambda; \tag{3}$$

$$\inf_{n\in\mathbb{N}} q\left(\frac{1}{\|A(n)x\|+1}\right) > 0. \tag{4}$$

Proof. Necessity. Let $K \geq 1$ şi $\nu > 0$ be such that

$$\|\Phi_A(j,n)\| \leq Ke^{-\nu(j-n)}, \quad \forall (j,n) \in \Delta. \tag{5}$$

Taking $q(t) = t$, for all $t \geq 0$, $x_0 = 0$ and $r = 1$ and using (5) we observe that

$$\|A(n)x\| + 1 \leq Ke^{-\nu} + 1, \quad \forall n \in \mathbb{N}, \forall x \in D(0,1)$$

so

$$\inf_{n\in\mathbb{N}} q\left(\frac{1}{\|A(n)x\|+1}\right) \geq \frac{1}{K+1}, \quad \forall x \in D(0,1). \tag{6}$$

Moreover, for every $x \in D(0,1)$, we have that

$$\sum_{j=n}^{\infty} q(\|\Phi_A(j,n)x\|) \leq K\sum_{j=n}^{\infty} e^{-\nu(j-n)} = \frac{K}{1-e^{-\nu}}, \quad \forall n \in \mathbb{N}. \tag{7}$$

Taking $\lambda := K/(1-e^{-\nu})$, from (6) and (7) we have that the necessity part is completed.

Sufficiency. Let $q \in \mathcal{Q}$, $x_0 \in X$, $r > 0$ and $\lambda > 0$ be such that for every $x \in D(x_0, r)$ relations (3) and (4) hold. The demonstration will be made in three steps.

Step 1. We show that (A) is uniformly bounded.

Let $x \in D(x_0, r)$. Supposing to the contrary that $\sup_{n\in\mathbb{N}} \|A(n)x\| = \infty$ we would have that there is $(k_n)_n \subset \mathbb{N}$ such that

$$\|A(k_n)x\| \underset{n\to\infty}{\longrightarrow} \infty. \tag{8}$$

Since q is continuous, from (8) we have that

$$q\left(\frac{1}{\|A(k_n)x\|+1}\right) \underset{n\to\infty}{\longrightarrow} q(0) = 0$$

which contradicts the hypothesis (4). Thus, it follows that

$$K(x) := \sup_{n\in\mathbb{N}} \|A(n)x\| < \infty, \quad \forall x \in D(x_0, r). \tag{9}$$

Let now $x \in X \setminus \{0\}$. Then, by using (9), we obtain

$$\begin{aligned}\|A(n)\frac{rx}{\|x\|}\| &\leq \|A(n)(x_0 + \frac{rx}{\|x\|})\| + \|A(n)x_0\| \\ &\leq K\big(x_0 + \frac{rx}{\|x\|}\big) + K(x_0)\end{aligned}$$

which implies

$$\|A(n)x\| \leq \left[K\big(x_0 + \frac{rx}{\|x\|}\big) + K(x_0)\right]\frac{\|x\|}{r}, \quad \forall n \in \mathbb{N}. \tag{10}$$

Setting

$$\delta(x) := \left[K\big(x_0 + \frac{rx}{\|x\|}\big) + K(x_0)\right]\frac{\|x\|}{r}, \qquad \text{for } x \neq 0$$

and $\delta(0) = 1$, from relation (10) it yields

$$\|A(n)x\| \leq \delta(x), \quad \forall n \in \mathbb{N}, \forall x \in X. \tag{11}$$

From (11) and by the uniform boundedness principle we deduce that there is $M > 0$

$$\|A(n)\| \leq M, \quad \forall n \in \mathbb{N}. \tag{12}$$

Step 2. We show that (A) is uniformly stable.

Let $h \in \mathbb{N}^*$ be such that

$$h > \frac{\lambda}{q(1)}. \tag{13}$$

Let $n \in \mathbb{N}$ and $m \geq n + h$. If $x \in D(x_0, r)$, then by using relation (12) we have

$$\begin{aligned}\|\Phi_A(m,n)x\| &\leq \|\Phi_A(m,j)\| \, \|\Phi_A(j,n)x\| \leq \\ &\leq M^h \, \|\Phi_A(j,n)x\|, \quad \forall j \in \{m-h+1, \ldots, m\}\end{aligned}$$

which implies

$$\frac{1}{M^h}\, \|\Phi_A(m,n)x\| \leq \|\Phi_A(j,n)x\|, \quad \forall j \in \{m-h+1, \ldots, m\}. \tag{14}$$

Since q is nondecreasing, by (14) it follows that

$$q\bigg(\frac{1}{M^h}\, \|\Phi_A(m,n)x\|\bigg) \leq q\bigg(\|\Phi_A(j,n)x\|\bigg), \quad \forall j \in \{m-h+1, \ldots, m\}$$

which based on (3) yields that

$$hq\left(\frac{1}{M^h}\,\|\Phi_A(m,n)x\|\right) \leq \sum_{j=m-h+1}^{m} q\left(\|\Phi_A(j,n)x\|\right) \leq \lambda. \tag{15}$$

From relations (13) and (15), since q is nondecreasing, it follows that

$$\|\Phi_A(m,n)x\| \leq M^h, \quad \forall x \in D(x_0,r). \tag{16}$$

Let now $x \in X \setminus \{0\}$. Then, by relation (16) we obtain that

$$\|\Phi_A(m,n)\frac{rx}{\|x\|}\| \leq \|\Phi_A(m,n)(x_0+\frac{rx}{\|x\|})\| + \|\Phi_A(m,n)x_0\| \leq 2M^h$$

which implies

$$\|\Phi_A(m,n)x\| \leq \frac{2M^h}{r}\,\|x\|, \quad \forall x \in X$$

and hence

$$\|\Phi_A(m,n)\| \leq \frac{2M^h}{r}, \quad \forall m \geq n+h, \forall n \in \mathbb{N}. \tag{17}$$

Taking into account that

$$\|\Phi_A(m,n)\| \leq M^{h-1}, \quad \forall m \in \{n,\dots,n+h-1\} \tag{18}$$

denoting by $L := \max\left\{\frac{2M^h}{r}, M^{h-1}\right\}$, from (17) and (18) we deduce that

$$\|\Phi_A(m,n)\| \leq L, \quad \forall (m,n) \in \Delta. \tag{19}$$

Step 3. We show that (A) is uniformly exponentially stable.

Let $\ell \in \mathbb{N}^*$ be such that

$$\ell > \frac{\lambda}{q\left(\frac{r}{2eL}\right)}. \tag{20}$$

Let $n \in \mathbb{N}$ and $x \in D\left(\frac{x_0}{L}, \frac{r}{L}\right)$. Then, using (19) we have that

$$\|\Phi_A(n+\ell,n)x\| \leq \|\Phi_A(j,n)Lx\|, \quad \forall j \in \{n+1,\dots,n+\ell\}. \tag{21}$$

Since q is nondecreasing, by using (21), the fact that $Lx \in D(x_0, r)$ and the hypothesis (3), we deduce that

$$\ell q(\|\Phi_A(n+\ell, n)x\|) \leq \sum_{j=n+1}^{n+\ell} q(\|\Phi_A(j,n)Lx\|) \leq \lambda. \tag{22}$$

From relations (20) and (22), considering that q is nondecreasing, we get

$$\|\Phi_A(n+\ell, n)x\| \leq \frac{r}{2eL}, \quad \forall x \in D\left(\frac{x_0}{L}, \frac{r}{L}\right). \tag{23}$$

Let now $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Then, using relation (23) we have

$$\|\Phi_A(n+\ell, n)\frac{rx}{L\|x\|}\| \leq \|\Phi_A(n+\ell, n)(\frac{x_0}{L} + \frac{rx}{L\|x\|})\| + \|\Phi_A(n+\ell, n)\frac{x_0}{L}\| \leq \frac{r}{eL}.$$

This implies

$$\|\Phi_A(n+\ell, n)x\| \leq \frac{1}{e}\|x\|, \quad \forall x \in X, \forall n \in \mathbb{N}$$

or equivalently

$$\|\Phi_A(n+\ell, n)\| \leq \frac{1}{e}, \quad \forall n \in \mathbb{N}. \tag{24}$$

Finally, we set $\nu = \dfrac{1}{\ell}$ and $N = Le$. Now, for $(m,n) \in \Delta$, we note that $m = n + k\ell + j$, for some $k \in \mathbb{N}$ and $j \in \{0, \ldots, \ell - 1\}$. It follows by (19) and (24) that

$$\|\Phi_A(m,n)\| \leq L\,\|\Phi_A(n+k\ell, n)\| \leq L\frac{1}{e^k} \leq Ne^{-\nu(m-n)}$$

and this completes the proof. □

Corollary 1. *If the system (A) is uniformly bounded, then (A) is uniformly exponentially stable if and only if there are $q \in \mathcal{Q}$, $x_0 \in X$, $r > 0$ and $\lambda > 0$ such that for every $x \in D(x_0, r)$*

$$\sup_{n \in \mathbb{N}} \sum_{j=n}^{\infty} q(\|\Phi_A(j,n)x\|) \leq \lambda. \tag{25}$$

Proof. Let $M > 0$ be such that $\|A(n)\| \leq M$, for all $n \in \mathbb{N}$.

Let $x_0 \in X$ and $r > 0$. Then

$$\|A(n)x\| \leq M\,\|x\| \leq M(\|x_0\| + r), \quad \forall x \in D(x_0, r), \forall n \in \mathbb{N}. \tag{26}$$

In particular, from (26) it yields that

$$\inf_{n\in\mathbb{N}} q\left(\frac{1}{\|A(n)x\|+1}\right) \geq q\left(\frac{1}{M(\|x_0\|+r)}\right), \quad \forall x \in D(x_0, r). \tag{27}$$

Hence, from (25) and (27), by Theorem 4, we get the conclusion. □

Before proving the next main theorem, we need a technical result:

Lemma 1. *Let $q \in \mathcal{Q}$ and $\{D(n)\}_{n\in\mathbb{N}} \subset \mathcal{B}(X)$. Then, for every $n \in \mathbb{N}$ and $h \in \mathbb{N}^*$ the set*

$$U_{n,h} = \{x \in X \,:\, \sum_{j=n}^{\infty} q(\|D(j)x\|) \leq h\}$$

is closed.

Proof. Let $n \in \mathbb{N}$ and $h \in \mathbb{N}^*$. For $m \in \mathbb{N}, m \geq n$, let

$$V_{n,h}^m = \{x \in X \,:\, \sum_{j=n}^{m} q(\|D(j)x\|) \leq h\}.$$

It is straightforward that

$$U_{n,h} = \bigcap_{m\geq n} V_{n,h}^m. \tag{28}$$

Let $m \in \mathbb{N}, m \geq n$. We show that $V_{n,h}^m$ is closed.

Let $(x_k)_{k\in\mathbb{N}} \subset V_{n,h}^m$ with $\lim\limits_{k\to\infty} x_k = x \in X$. Denote $\gamma := \max\{\|D(j)\| : j \in \{n,\ldots,m\}\}$ and $T := \gamma(\|x\|+1)$.

Let $\varepsilon > 0$. By the uniform continuity of q on $[0,T]$, it yields that there is $\delta \in (0,1)$ such that for all $t,\tau \in [0,T]$ with $|t-\tau| < \delta$ we have

$$|q(t) - q(\tau)| < \frac{\varepsilon}{m-n+1}. \tag{29}$$

Let $\ell \in \mathbb{N}$ be such that

$$\|x_\ell - x\| < \frac{\delta}{\gamma+1}.$$

Then, we observe that

$$\|D(j)x_\ell\| \leq \gamma\|x_\ell\| \leq \gamma\left(\frac{\delta}{\gamma+1} + \|x\|\right) < T, \quad \forall j \in \{n,\ldots,m\}. \tag{30}$$

Furthermore, we also have that

$$\|D(j)x\| \le \gamma\|x\| < T \tag{31}$$

and

$$\begin{aligned} | \, \|D(j)x_\ell\| - \|D(j)x\| \, | &\le \|D(j)(x_\ell - x)\| \\ &\le \gamma \frac{\delta}{\gamma+1}, \quad \forall j \in \{n, \dots, m\}. \end{aligned} \tag{32}$$

From relations (29)-(32) it follows

$$\begin{aligned} q(\|D(j)x\|) &\le |q(\|D(j)x\|) - q(\|D(j)x_l\|)| + q(\|D(j)x_\ell\|) \\ &< \frac{\varepsilon}{m-n+1} + q(\|D(j)x_\ell\|), \quad \forall j \in \{n, \dots, m\}. \end{aligned} \tag{33}$$

Since $x_\ell \in V^m_{n,h}$, from (33) we deduce that

$$\sum_{j=n}^{m} q(\|D(j)x\|) < \varepsilon + h, \quad \forall \varepsilon > 0. \tag{34}$$

Letting $\varepsilon \to 0$ in (34) it yields

$$\sum_{j=n}^{m} q(\|D(j)x\|) \le h,$$

so $x \in V^m_{n,h}$. This shows that $V^m_{n,h}$ is closed, for all $m \in \mathbb{N}, m \ge n$.

Then, by (28), we obtain that $U_{n,h}$ is closed. $\square$

Notation We denote by $\mathcal{F}$ the class of all mappings $\varphi : \mathbb{N}^* \times \mathbb{R}_+ \to \mathbb{R}_+$ with the property that for every $n \in \mathbb{N}^*$, $\varphi(n, \cdot) \in \mathcal{Q}$.

The next main result is a new characterization of Zabczyk-Rolewicz type for uniform exponential stability:

Theorem 5. *The system* (A) *is uniformly exponentially stable if and only if there is* $\varphi \in \mathcal{F}$ *such that for every* $x \in X$ *there exists* $\alpha(x) \in \mathbb{N}^*$ *with*

$$\sup_{n\in\mathbb{N}} \sum_{j=n}^{\infty} \varphi(\alpha(x), \|\Phi_A(j,n)x\|) < \infty \tag{35}$$

and

$$\inf_{n\in\mathbb{N}} \varphi\left(\alpha(x), \frac{1}{\|A(n)x\|+1}\right) > 0. \tag{36}$$

Proof. Necessity. Let $K \geq 1$ and $\nu > 0$ be such that

$$\|\Phi_A(j,n)\| \leq K e^{-\nu(j-n)}, \quad \forall (j,n) \in \Delta. \tag{37}$$

We take $\varphi : \mathbb{N}^* \times \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(n,s) = s$. Then $\varphi \in \mathcal{F}$.

Let $x \in X$ and $\alpha(x) \in \mathbb{N}^*$. Then, using (37) it yields

$$\sum_{j=n}^{\infty} \varphi(\alpha(x), \|\Phi_A(j,n)x\|) \leq \frac{K}{1-e^{-\nu}}\|x\|, \quad \forall n \in \mathbb{N} \tag{38}$$

and respectively

$$\varphi\left(\alpha(x), \frac{1}{\|A(n)x\|+1}\right) = \frac{1}{\|A(n)x\|+1} \geq \frac{1}{Ke^{-\nu}\|x\|+1}, \quad \forall n \in \mathbb{N}. \tag{39}$$

Thus, by relations (38) and (39), the necessity part is completed.

Sufficiency. Let $\varphi \in \mathcal{F}$ be a function such that for every $x \in X$ there exists $\alpha(x) \in \mathbb{N}^*$ such that relations (35) and (36) hold.

Let $i, h \in \mathbb{N}^*$. We consider the set

$$Z_h^i = \{x \in X \,:\, \sup_{n \in \mathbb{N}} \sum_{j=n}^{\infty} \varphi(i, \|\Phi_A(j,n)x\|) \leq h\}.$$

Furthermore, for every $n \in \mathbb{N}$ let

$$U_{n,h}^i = \{x \in X \,:\, \sum_{j=n}^{\infty} \varphi(i, \|\Phi_A(j,n)x\|) \leq h\}.$$

We note that

$$Z_h^i = \bigcap_{n \in \mathbb{N}} U_{n,h}^i. \tag{40}$$

Let $n \in \mathbb{N}$. By Lemma 1 applied for $q = \varphi(i, \cdot)$ and for

$$D_n(j) = \begin{cases} \Phi_A(j,n), & j \geq n \\ 0, & j < n \end{cases}$$

we deduce that $U_{n,h}^i$ is closed. Hence, by (40) it follows that every Z_h^i is closed, for all $i, h \in \mathbb{N}^*$.

The demonstration in what follows will be organized in three steps.

Step 1. We show that

$$X = \bigcup_{i,h \in \mathbb{N}^*} Z_h^i. \tag{41}$$

Let $x \in X$ and let $\alpha(x) \in \mathbb{N}^*$ be such that (35) holds. It yields that there is $h \in \mathbb{N}^*$ such that

$$\sup_{n \in \mathbb{N}} \sum_{j=n}^{\infty} \varphi(\alpha(x), \|\Phi_A(j,n)x\|) \leq h.$$

This shows that $x \in Z_h^{\alpha(x)}$. In this way, we have shown that (41) holds.

Step 2. We show that (A) is uniformly bounded.

Let $x \in X$ and let $\alpha(x) \in \mathbb{N}^*$ be such that (36) holds. Supposing to the contrary that

$$\sup_{n \in \mathbb{N}} \|A(n)x\| = \infty$$

it would follow that there exists $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ such that $\|A(k_n)x\| \underset{n \to \infty}{\longrightarrow} \infty$. Then, since $\varphi(\alpha(x), \cdot) \in \mathcal{Q}$, using its continuity, we obtain that

$$\varphi\left(\alpha(x), \frac{1}{\|A(k_n)x\| + 1}\right) \underset{n \to \infty}{\longrightarrow} \varphi(\alpha(x), 0) = 0$$

which contradicts the hypothesis (36).

We deduce in this way that

$$\sup_{n \in \mathbb{N}} \|A(n)x\| < \infty, \quad \forall x \in X. \tag{42}$$

By the uniform boundedness principle, we get from (42) that (A) is uniformly bounded.

Step 3. We show that (A) is uniformly exponentially stable.

From (41) and the Baire category theorem, it yields that there are $i, h \in \mathbb{N}^*$ and $x_0 \in Z_h^i, r > 0$ such that

$$D(x_0, r) \subset Z_h^i. \tag{43}$$

Consider

$$q : \mathbb{R}_+ \to \mathbb{R}_+, \quad q(t) = \varphi(i, t).$$

Then, from (43) we get that

$$\sup_{n\in\mathbb{N}}\sum_{j=n}^{\infty} q(\|\Phi_A(j,n)x\|) \leq h, \quad \forall x \in D(x_0, r). \tag{44}$$

Since (A) is uniformly bounded, from (44) and Corollary 1 we deduce that (A) is uniformly exponentially stable. $\square$

Corollary 2. *If (A) is uniformly bounded, then (A) is uniformly exponentially stable if and only if there is $\varphi \in \mathcal{F}$ such that for every $x \in X$ there exists $\alpha(x) \in \mathbb{N}^*$ with*

$$\sup_{n\in\mathbb{N}}\sum_{j=n}^{\infty} \varphi(\alpha(x), \|\Phi_A(j,n)x\|) < \infty.$$

Proof. The conclusion yields by the argumentation from the proof of Theorem 5. $\square$

3 Applications to stability of continuous-time dynamics

In this section we will apply the central results from the previous section in order to provide a new approach to the Rolewicz type criteria for continuous-time nonautonomous dynamics. More precisely, we will deduce the theorem of Rolewicz for uniform exponential stability.

Let X be a Banach space. We denote by $\tilde{\Delta}$ the set of all pairs $(t,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $t \geq \tau$. In what follows our study is devoted to evolution families which are strongly continuous in the first variable, as introduced in:

Definition 2. $\mathcal{U} = \{U(s,\tau)\}_{s\geq\tau\geq 0} \subset \mathcal{B}(X)$ is called *an evolution family* if it satisfies:

- (i) $U(\tau,\tau) = I$, for all $\tau \geq 0$ and $U(s,\xi)U(\xi,\tau) = U(s,\tau)$, for all (s,ξ), $(\xi,\tau) \in \tilde{\Delta}$;
- (ii) there are $C, \omega > 0$ with $\|U(t,\tau)\| \leq Ce^{\omega(t-\tau)}$, for all $(t,\tau) \in \tilde{\Delta}$;
- (iii) for each $x \in X$ and $\tau \geq 0$, the function $s \mapsto U(s,\tau)x$ is continuous on $[\tau, \infty)$.

Definition 3. An evolution family $\mathcal{U} = \{U(t,\tau)\}_{t\geq\tau\geq 0}$ is *uniformly exponentially stable* if there are $K \geq 1$ and $\nu > 0$ such that

$$\|U(t,\tau)\| \leq Ke^{-\nu(t-\tau)}, \quad \forall (t,\tau) \in \tilde{\Delta}.$$

Consider now an evolution family $\mathcal{U} = \{U(t,\tau)\}_{t\geq\tau\geq 0}$ and associate to it the discrete system:

$$(A_{\mathcal{U}}) \qquad x(n+1) = U(n+1,n)x(n), \quad n \in \mathbb{N}.$$

Then, we observe that

$$\Phi_{A_{\mathcal{U}}}(j,n) = U(j,n), \quad \forall (j,n) \in \Delta. \tag{45}$$

Remark 1. From Definition 2 (ii) we deduce that $(A_{\mathcal{U}})$ is uniformly bounded. Furthermore, we note that $\mathcal{U}$ is uniformly exponentially stable if and only if $(A_{\mathcal{U}})$ is uniformly exponentially stable.

Notation We denote by $\mathcal{N}$ the set of all functions $N : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the properties:

- (a) for every $s > 0$, $N(s,\cdot) \in \mathcal{Q}$;
- (b) for every $\tau \geq 0$, $N(\cdot,\tau)$ is nondecreasing.

As a consequence of the stability results from Section 2, we deduce the theorem of Rolewicz:

Theorem 6. *$\mathcal{U}$ is uniformly exponentially stable if and only if there exists $N \in \mathcal{N}$ such that for every $x \in X$ there is $\lambda(x) > 0$ such that*

$$\sup_{\tau\geq 0} \int_\tau^\infty N(\lambda(x), \|U(s,\tau)x\|)\, ds < \infty. \tag{46}$$

Proof. Necessity. This is immediate. We present it for the sake of clarity.

Let $K, \nu > 0$ be given by Definition 3. Let $N(s,\tau) = \tau$, for all $s > 0$ and $\tau \geq 0$. It yields that

$$\sup_{\tau\geq 0} \int_\tau^\infty N(\lambda(x), \|U(s,\tau)x\|)\, ds = \sup_{\tau\geq 0} \int_\tau^\infty \|U(s,\tau)x\|\, ds \leq \frac{K\|x\|}{\nu}, \quad \forall x \in X.$$

Thus the necessity is proved.

Sufficiency. This relies on Corollary 2. Indeed, assume that $N \in \mathcal{N}$ is such that for every $x \in X$ there is $\lambda(x) > 0$ such that (46) holds. Let $C, \omega > 0$ be some constants given by Definition 2 (ii), i.e.

$$\|U(s,\tau)\| \leq Ce^{\omega(s-\tau)}, \quad \forall (s,\tau) \in \tilde{\Delta}. \tag{47}$$

Take

$$\varphi : \mathbb{N}^* \times \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi(n,\tau) = N\left(\frac{1}{n}, \frac{\tau}{Ce^{\omega}}\right).$$

From $N \in \mathcal{N}$ we get that $\varphi \in \mathcal{F}$.

Let $x \in X$ and let $\lambda(x)$ be a strictly positive constant that satisfies (46). We set

$$\kappa(x) := \sup_{\tau \geq 0} \int_\tau^\infty N(\lambda(x), \|U(s,\tau)x\|)\ ds.$$

Let $h \in \mathbb{N}^*$ be such that

$$\frac{1}{h} < \lambda(x).$$

Let $n \in \mathbb{N}$ and $j \geq n+1$. From (47) we get that

$$\|U(j,n)x\| \leq Ce^{\omega}\|U(s,n)x\|, \quad \forall s \in [j-1, j]. \tag{48}$$

Since $N \in \mathcal{N}$, from (48) it yields that

$$\begin{aligned}
\varphi(h, \|U(j,n)x\|) &= N\left(\frac{1}{h}, \frac{\|U(j,n)x\|}{Ce^{\omega}}\right) \\
&\leq N\left(\lambda(x), \frac{\|U(j,n)x\|}{Ce^{\omega}}\right) \\
&\leq N(\lambda(x), \|U(s,n)x\|), \quad \forall s \in [j-1,j].
\end{aligned}$$

By integrating on $[j-1, j]$ we obtain

$$\varphi(h, \|U(j,n)x\|) \leq \int_{j-1}^{j} N(\lambda(x), \|U(s,n)x\|) ds. \tag{49}$$

It follows from (49) that

$$\begin{aligned}
\sum_{j=n+1}^{\infty} \varphi(h, \|U(j,n)x\|) &\leq \sum_{j=n+1}^{\infty} \int_{j-1}^{j} N(\lambda(x), \|U(s,n)x\|) ds \\
&= \int_n^\infty N(\lambda(x), \|U(s,n)x\|) ds \\
&\leq \kappa(x).
\end{aligned} \tag{50}$$

Hence, from (50) we get that

$$\sum_{j=n}^{\infty} \varphi(h, \|U(j,n)x\|) \leq \varphi(h, \|x\|) + \kappa(x). \tag{51}$$

Thus, since n has been arbitrary, we conclude from (51) that

$$\sup_{n\in\mathbb{N}} \sum_{j=n}^{\infty} \varphi(h, \|U(j,n)x\|) \leq \varphi(h, \|x\|) + \kappa(x). \tag{52}$$

From (52) and (45) it yields via Corollary 2 that $(A_{\mathcal{U}})$ is uniformly exponentially stable. It follows then from Remark 1 that $\mathcal{U}$ is uniformly exponentially stable. □

Remark 2. Besides the original proof (see [35, 36]), other approaches to demonstrate Rolewicz-type stability theorems were presented in [25, 38, 44, 48, 49] for nonautonomous dynamics and respectively in [1, 12, 23, 37, 38, 40] for variational dynamics.

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