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TRACE CONJUNCTION INEQUALITIES*

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*In blessed memory of Haim Brezis
and his joy of asking and solving problems*

Abstract

Trace conjunction integrals are introduced and studied. They appear in trace conjunction inequalities which unify the Hardy inequality on a halfspace and the classical Gagliardo trace inequality. At the endpoint they satisfy a Bourgain-Brezis-Mironescu formula for smooth maps, which raises some new open problems.

Keywords: fractional Sobolev space, Hardy inequality, trace inequality.
MSC: 46E35.

1 Introduction

The *homogeneous Sobolev space*

$$\dot{W}^{1,p}(\mathbf{R}^N_+) := \left\{ u\colon \mathbf{R}^N_+ \to \mathbf{R} \;\middle|\; u \text{ is weakly differentiable and } \int_{\mathbf{R}^N_+} |\mathrm{D}u|^p < \infty \right\},$$

with $N \in \mathbf{N} \setminus \{0, 1\}$, $\mathbf{R}^N_+ = \mathbf{R}^{N-1} \times (0, \infty)$ and $p > 1$, consists a priori of equivalence classes of merely measurable functions that should not have well-defined restrictions to Lebesgue null sets. However the integrability

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condition on the derivative has long been known to allow the definition of *traces* on lower-dimensional sets such as the boundary $\partial \mathbf{R}^N_+ \simeq \mathbf{R}^{N-1}$ [36] (see also [6, lem. 9.9; 41, prop. 6.2.3]).

Since the seminal work of Gagliardo [18] (see also [1, 34, 35] for $p = 2$), it has been known that if $p > 1$, every function $u \in \dot{W}^{1,p}(\mathbf{R}^N_+)$ has a trace $v = \operatorname{tr}_{\partial \mathbf{R}^N_+} u \in \dot{W}^{1-1/p,p}(\partial \mathbf{R}^N_+)$, with a *homogeneous fractional Sobolev space* $\dot{W}^{s,p}(\mathbf{R}^\ell)$ defined for $\ell \in \mathbf{N} \setminus \{0\}$, $s \in (0, 1)$ and $p \in [1, \infty)$ as

$$\dot{W}^{s,p}(\mathbf{R}^\ell) \coloneqq \Big\{ v \colon \mathbf{R}^\ell \to \mathbf{R} \;\Big|\; \iint\limits_{\mathbf{R}^\ell \times \mathbf{R}^\ell} \frac{|v(x) - v(y)|^p}{|x - y|^{\ell + sp}} \,\mathrm{d}x \,\mathrm{d}y < \infty \Big\}. \tag{1}$$

Moreover, the trace operator $\operatorname{tr}_{\partial \mathbf{R}^N_+} \colon \dot{W}^{1,p}(\mathbf{R}^N_+) \to \dot{W}^{1-1/p,p}(\partial \mathbf{R}^N_+)$ is continuous: there is some constant $C \in (0, \infty)$ such that if $u \in \dot{W}^{1,p}(\mathbf{R}^N_+)$ and $v = \operatorname{tr}_{\partial \mathbf{R}^N_+} u$, the *Gagliardo integral* which appears in the definition (1) of the fractional Sobolev space $\dot{W}^{1-1/p,p}(\partial \mathbf{R}^N_+)$ satisfies the trace inequality

$$\iint\limits_{\partial \mathbf{R}^N_+ \times \partial \mathbf{R}^N_+} \frac{|v(x) - v(y)|^p}{|x - y|^{N + p - 2}} \,\mathrm{d}x \,\mathrm{d}y \le C \int_{\mathbf{R}^N_+} |\mathrm{D}u|^p. \tag{2}$$

The trace operator $\operatorname{tr}_{\partial \mathbf{R}^N_+}$ also has a linear continuous right inverse that provides an extension of boundary data. At the endpoint $p = 1$, the range of trace turns out to be $L^1(\partial \mathbf{R}^N_+)$ [18] (see also [26]) without a corresponding linear continuous right inverse [31] (see also [32]).

The aim of the present work is to introduce the *trace conjunction integral* that connects a function to its trace quantitatively and qualitatively. Our starting point is the inequality

$$\int_{\partial \mathbf{R}^N_+} \Big(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|^p}{|x - y|^{N + p - 1}} \,\mathrm{d}y \Big) \,\mathrm{d}x \le C \int_{\mathbf{R}^N_+} |\mathrm{D}u|^p, \tag{3}$$

for $u \in \dot{W}^{1,p}(\mathbf{R}^N_+)$ and $v = \operatorname{tr}_{\partial \mathbf{R}^N} u$ (see Theorem 1 below). The trace conjunction integral is the quantity on the left-hand side of (3). It is a kind of mixed double integral which integrates the distance between the boundary values and the interior values.

Trace conjunction integrals have already been known to appear in Fubini type arguments, where one writes

$$\iint\limits_{\mathbf{R}^N \times \mathbf{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \,\mathrm{d}x \,\mathrm{d}y = \int_{\mathbf{R}^{N-\ell}} \int_{\mathbf{R}^\ell \times \{x''\}} \int_{\mathbf{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \,\mathrm{d}y \,\mathrm{d}x' \,\mathrm{d}x'', \tag{4}$$

with $x = (x', x'')$, before applying a triangle inequality argument to rewrite the two innermost integrals of the right-hand side of (4) as a double integral on $(\mathbf{R}^\ell \times \{x''\})^2$ (see for example [39, prop. 5.9]); a similar argument proves that the trace conjunction integral on the left-hand side of (3) controls the Gagliado energy on the left-hand side of (2) (Theorem 5), so that (3) implies (2). The trace conjunction inequality (3) turns out to provide an interesting route to the trace inequality (2), with a proof of (3) that avoids the need to introduce somehow artificial interior intermediate points in the middle of the proof and treat similarly two symmetric terms.

The trace conjunction inequality (3) also contains the information that v is the trace of u, in the sense that if $v\colon \partial\mathbf{R}^N_+ \to \mathbf{R}$ is any function for which the left-hand side of (3) is finite, then v is necessarily the trace of u (Theorem 7).

The trace conjunction inequality (3) is also connected to the classical *Hardy inequality* [19; 20, th. 327] (see also [15, 22, 27])

$$\int_{\mathbf{R}^N_+} \frac{|v(x') - u(x)|^p}{x_N^p} \,\mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_{\mathbf{R}^N_+} |\mathrm{D}u|^p, \tag{5}$$

for $u \in \dot{W}^{1,p}(\mathbf{R}^N_+)$, $v = \operatorname{tr}_{\partial\mathbf{R}^N_+} u$ and with $x = (x', x_N)$. Indeed, a straightforward argument shows that the left-hand side of the Hardy inequality (5) is controlled by the trace conjunction integral on the left-hand side of (3), so that the trace conjunction inequality (3) implies the Hardy inequality (5). Compared with the Hardy inequality (5), the trace conjunction inequality (3) is invariant under suitable change of coordinates by a diffeomorphism, which makes it more appealing in more geometrical contexts.

As the classical trace (2) and Hardy (5) inequalities could be derived from the trace conjunction inequality (3), the latter can be derived from the two former in the sense that the left-hand side of (3) can be controlled by the left-hand side of (2) and (5) (Theorem 10).

Besides the first-order Sobolev spaces $\dot{W}^{1,p}(\mathbf{R}^N_+)$ considered above, other flavours of Sobolev spaces are known to have traces. We prove trace conjunction inequalities similar to (3) for some wider range of weighted Sobolev spaces (Theorem 1) and fractional Sobolev spaces (Theorem 4).

As further applications, the sharp connection between a function and its trace expressed by the trace conjunction integral could be used to prove in the critical case $p = N$ and $s = 1 - 1/p$, or even more generally $sp = N - 1$, that the trace works well in the framework of functions of vanishing mean oscillation [10; 24, prop. 2.8].

The final part of the present work starts from the *Bourgain-Brezis-*

Mironescu formula and characterisation of Sobolev spaces [3; 5; 7, th. 6.2]: if $u \in \dot{W}^{1,p}(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$, then

$$\lim_{s \underset{<}{\to} 1} (1-s) \iint\limits_{\mathbf{R}^N \times \mathbf{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x \,\mathrm{d}y = \frac{2\pi^{\frac{N-1}{2}} \Gamma\bigl(\frac{p+1}{2}\bigr)}{p\,\Gamma\bigl(\frac{N+p}{2}\bigr)} \int_{\mathbf{R}^N} |\mathrm{D}u|^p. \tag{6}$$

and conversely, if the function $u \colon \mathbf{R}^N \to \mathbf{R}$ is measurable and if

$$\liminf_{s \underset{<}{\to} 1} (1-s) \iint\limits_{\mathbf{R}^N \times \mathbf{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x \,\mathrm{d}y < \infty,$$

then $u \in \dot{W}^{1,p}(\mathbf{R}^N)$ and

$$\frac{2\pi^{\frac{N-1}{2}} \Gamma(\frac{p+1}{2})}{p\,\Gamma(\frac{N+p}{2})} \int_{\mathbf{R}^N} |\mathrm{D}u|^p \le \liminf_{s \underset{<}{\to} 1} (1-s) \iint\limits_{\mathbf{R}^N \times \mathbf{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x \,\mathrm{d}y < \infty.$$

For trace conjunction integrals, we prove that if $u \in C^1_c(\bar{\mathbf{R}}^N_+)$, then (Theorem 11)

$$\lim_{s \underset{<}{\to} 1} (1-s) \int_{\partial \mathbf{R}^N_+} \biggl(\int_{\mathbf{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}y \biggr) \,\mathrm{d}x = \frac{\pi^{\frac{N-1}{2}} \Gamma\bigl(\frac{p+1}{2}\bigr)}{p\,\Gamma\bigl(\frac{N+p}{2}\bigr)} \int_{\partial \mathbf{R}^N_+} |\mathrm{D}u|^p. \tag{7}$$

Compared to the classical Bourgain-Brezis-Mironescu formula (6), the right-hand side of its trace conjunction counterpart (7) features an integral that is only performed on the *boundary* but an integrand involving both the *tangential* and *normal* components of the derivative. The identity 7 is currently only proved for *smooth* functions.

Open Problem 1. Does (7) still hold when u belongs to some suitable Sobolev space?

Thanks to the relationship between the conjunction integral and the Gagliardo energy, we also prove in Theorem 13 that if $p > 1$ and if

$$\liminf_{s \underset{<}{\to} 1} (1-s) \int_{\partial \mathbf{R}^N_+} \biggl(\int_{\mathbf{R}^N_+} \frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}y \biggr) \,\mathrm{d}x < \infty, \tag{8}$$

then one has $v \in \dot{W}^{1,p}(\partial \mathbf{R}^N_+)$; following [16, 33, 40], when $p = 1$ and (8) holds, the function v lies in the space of functions of bounded variation $BV(\partial \mathbf{R}^N_+)$ (Theorem 14).

We do not expect these results to be complete, as they do not provide any of the information on the normal derivative that the identity (7) suggests. This raises the question about more precise results.

Open Problem 2. Prove that the condition (61) implies the existence of a normal derivative of u in some weak sense.

At the other endpoint of the range $s \in (0, 1)$ it would also make sense to study the limit

$$\lim_{\substack{s \to 0 \\ >}} s \int_{\partial \mathbf{R}^N_+} \left(\int_{\mathbf{R}^N_+} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, \mathrm{d}y \right) \mathrm{d}x,$$

in the spirit of the *Maz′ya-Shaposhnikova formula* for fractional Sobolev spaces [25].

Further interesting problems involve studying the counterparts of the conjunction integral inspired by other the characterisation of Sobolev spaces. The *Nguyen formula* [29] (see also [4, 9, 30])

$$\lim_{\delta \to 0} \iint\limits_{\substack{x, y \in \mathbf{R}^N \\ |u(x) - u(y)| \ge \delta}} \frac{\delta^p}{|x - y|^{N+p}} \, \mathrm{d}x \, \mathrm{d}y = \frac{2\pi^{\frac{N-1}{2}} \Gamma(\frac{p+1}{2})}{p\, \Gamma(\frac{N+p}{2})} \int_{\mathbf{R}^N} |\mathrm{D}u|^p, \tag{9}$$

hints at investigating the limit

$$\lim_{\delta \to 0} \iint\limits_{\substack{x \in \partial \mathbf{R}^N_+, y \in \mathbf{R}^N_+ \\ |v(x) - u(y)| \ge \delta}} \frac{\delta^p}{|x - y|^{N+p}} \, \mathrm{d}x \, \mathrm{d}y. \tag{10}$$

Similarly, the *Brezis-Van Schaftingen-Yung formula* [13] (see also [14])

$$\begin{aligned} \lim_{\lambda \to \infty} \lambda^p \left| \left\{ (x, y) \in \mathbf{R}^N \times \mathbf{R}^N \;\middle|\; \frac{|u(x) - u(y)|^p}{|x - y|^{N+p}} \ge \lambda^p \right\} \right| \\ = \frac{2\pi^{\frac{N-1}{2}} \Gamma(\frac{p+1}{2})}{N\, \Gamma(\frac{N+p}{2})} \int_{\mathbf{R}^N} |\mathrm{D}u|^p. \end{aligned} \tag{11}$$

suggests studying the limit

$$\lim_{\lambda \to \infty} \lambda^p \left| \left\{ (x, y) \in \partial \mathbf{R}^N_+ \times \mathbf{R}^N_+ \;\middle|\; \frac{|v(x) - u(y)|^p}{|x - y|^{N+p}} \ge \lambda^p \right\} \right|. \tag{12}$$

Even more generally, it would make sense to study conjunction integral counterparts for the full family of *Brezis-Seeger-Van Schaftingen-Yung formulae* that includes (9) and (11) [11, 12] (see also [17]).

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2 First-order trace conjunction inequality

Wotivated by Uspenskiĭ's weighted version of the trace inequality (2) [38] (see also [28]): if the function $u\colon \mathbf{R}^N_+ \to \mathbf{R}$ is weakly differentiable and if

$$\int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(z)|^p}{z_N^{1-(1-s)p}} \,\mathrm{d}z < \infty,$$

then u has a trace $v = \operatorname{tr}_{\partial \mathbf{R}^N_+} u$ and

$$\iint\limits_{\partial \mathbf{R}^N_+ \times \partial \mathbf{R}^N_+} \frac{|v(x) - v(y)|^p}{|x - y|^{N-1+sp}} \,\mathrm{d}y \,\mathrm{d}x \le C \int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(z)|^p}{z_N^{1-(1-s)p}} \,\mathrm{d}z, \tag{13}$$

we will prove the following trace conjunction inequality (3).

Theorem 1. *If $N \in \mathbf{N} \setminus \{0, 1\}$, if $s \in (0, 1)$ and if $p \in (1, \infty)$, then every weakly differentiable function $u\colon \mathbf{R}^N_+ \to \mathbf{R}$ satisfying*

$$\int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(z)|^p}{z_N^{1-(1-s)p}} \,\mathrm{d}z < \infty$$

has a trace $v = \operatorname{tr}_{\partial \mathbf{R}^N_+} u$ satisfying

$$\int_{\partial \mathbf{R}^N_+} \Bigl(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}y \Bigr) \,\mathrm{d}x \le \frac{\pi^{\frac{N-1}{2}} \Gamma(\frac{sp+1}{2})}{\Gamma(\frac{N+sp}{2})} \Bigl(\frac{N}{s}\Bigr)^p \int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(z)|^p}{z_N^{1-(1-s)p}} \,\mathrm{d}z. \tag{14}$$

The main ingredient in the proof Theorem 1 will be the following weighted Hardy inequality.

Proposition 2. *If $N \in \mathbf{N} \setminus \{0, 1\}$, if $s \in (0, 1)$, if $p \in (1, \infty)$ and if $u \in C^1(\bar{\mathbf{R}}^N_+)$, then*

$$\int_{\mathbf{R}^N_+} \frac{|u(x) - u(0)|^p}{|x|^{N+sp}} \,\mathrm{d}x \le \Bigl(\frac{N}{s}\Bigr)^p \int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(x)|^p x_N^p}{|x|^{N+sp}} \,\mathrm{d}x. \tag{15}$$

In the classical Hardy inequality, the left-hand side of the estimate (15) is controlled as

$$\int_{\mathbf{R}^N_+} \frac{|u(x) - u(0)|^p}{|x|^{N+sp}} \,\mathrm{d}x \le \frac{1}{s^p} \int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(x)|^p}{|x|^{N-(1-s)p}} \,\mathrm{d}x; \tag{16}$$

the integrand on the right-hand side in (15) is smaller than in (16); its stronger decay in the tangential direction will be crucial in the proof of Theorem 1.

Proof of Proposition 2. Defining first the vector field $\xi\colon \mathbf{R}^N_+ \to \mathbf{R}^N$ at every point $x = (x', x_N) \in \mathbf{R}^N_+ = \mathbf{R}^{N-1} \times (0, \infty)$ by

$$\xi(x) \coloneqq \frac{x_N e_N}{|x|^{N+sp}} - \Bigl(1 + \frac{N}{sp}\Bigr) \frac{x_N^2 x}{|x|^{N+sp+2}}, \tag{17}$$

where $e_1, \dots, e_N$ is the canonical basis of $\mathbf{R}^N$, we compute for every $x \in \mathbf{R}^N_+$

$$\begin{aligned} \operatorname{div} \xi(x) &= \frac{1}{|x|^{N+sp}} - (N + sp) \frac{x_N^2}{|x|^{N+sp}} \\ &\qquad - \Bigl(1 + \frac{N}{sp}\Bigr)(2 + N - (N + sp + 2)) \frac{x_N^2}{|x|^{N+sp+2}} \\ &= \frac{1}{|x|^{N+sp}}. \end{aligned} \tag{18}$$

We also have from (17) for every $x \in \mathbf{R}^N_+$

$$|\xi(x)|^2 = \frac{x_N^2}{|x|^{2(N+sp)}} + \biggl(\Bigl(\frac{N}{sp}\Bigr)^2 - 1\biggr) \frac{x_N^4}{|x|^{2(N+sp+1)}} \le \Bigl(\frac{N}{sp}\Bigr)^2 \frac{x_N^2}{|x|^{2(N+sp)}},$$

so that

$$|\xi(x)| \le \frac{N}{sp} \frac{x_N}{|x|^{N+sp}}, \tag{19}$$

and

$$\xi(x) \cdot x = -\frac{N x_N^2}{sp|x|^{N+sp}}. \tag{20}$$

We then get for each $R > r > 0$, by the divergence theorem

$$\begin{aligned} \int_{(\mathbb{B}^N_R(0) \setminus \mathbb{B}^N_r(0)) \cap \mathbf{R}^N_+} &|u(x) - u(0)|^p \operatorname{div} \xi(x) \,\mathrm{d}x \\ &= -p \int_{(\mathbb{B}^N_R(0) \setminus \mathbb{B}^N_r(0)) \cap \mathbf{R}^N_+} |u(x) - u(0)|^{p-1} \mathrm{D}u(x)[\xi(x)] \,\mathrm{d}x \\ &\qquad + \int_{\partial \mathbb{B}^N_R(0) \cap \mathbf{R}^N_+} |u(x) - u(0)|^p \frac{\xi(x) \cdot x}{|x|} \,\mathrm{d}x \\ &\qquad - \int_{\partial \mathbb{B}^N_r(0) \cap \mathbf{R}^N_+} |u(x) - u(0)|^p \frac{\xi(x) \cdot x}{|x|} \,\mathrm{d}x. \end{aligned} \tag{21}$$

The first term in the right-hand side of (21) is controlled by Young's inequality and by (19) as

$$
\begin{aligned}
-p\int_{(\mathbb{B}_R^N(0)\setminus\mathbb{B}_r^N(0))\cap\mathbf{R}_+^N} &|u(x)-u(0)|^{p-1}\,\mathrm{D}u(x)[\xi(x)]\,\mathrm{d}x\\
&\le \Bigl(1-\frac{1}{p}\Bigr)\int_{(\mathbb{B}_R^N(0)\setminus\mathbb{B}_r^N(0))\cap\mathbf{R}_+^N}\frac{|u(x)-u(0)|^p}{|x|^{N+sp}}\,\mathrm{d}x \\
&\quad+\frac{1}{p}\Bigl(\frac{N}{s}\Bigr)^p\int_{\mathbf{R}_+^N}\frac{|\mathrm{D}u(x)|^p x_N^p}{|x|^{N+sp}}\,\mathrm{d}x.
\end{aligned}
\tag{22}
$$

The second term in the right-hand side of (21) is nonnegative in view of (20):

$$
\int_{\partial\mathbb{B}_R^N(0)\cap\mathbf{R}_+^N}|u(x)-u(0)|^p\,\frac{\xi(x)\cdot x}{|x|}\,\mathrm{d}x\le 0. \tag{23}
$$

The last term in the the right-hand side of (21) can be controlled thanks to (20) again as

$$
\begin{aligned}
-\int_{\partial\mathbb{B}_r^N(0)\cap\mathbf{R}_+^N}&|u(x)-u(0)|^p\,\frac{\xi(x)\cdot x}{|x|}\,\mathrm{d}x\\
&=\frac{N}{s}\int_{\partial\mathbb{B}_r^N(0)\cap\mathbf{R}_+^N}\frac{|u(x)-u(0)|^p\,x_N^2}{|x|^{N+sp+1}}\,\mathrm{d}x\\
&\le\frac{N}{sp}\|\mathrm{D}u\|_{L^\infty(\mathbf{R}_+^N)}^p\int_{\partial\mathbb{B}_N^N\cap\mathbf{R}_+^N}\frac{1}{|x|^{N-(1-s)p-1}}\,\mathrm{d}x\\
&\le\frac{N|\partial\mathbb{B}_1^N|}{2sp}\|\mathrm{D}u\|_{L^\infty(\mathbf{R}_+^N)}^p r^{(1-s)p}.
\end{aligned}
\tag{24}
$$

It follows thus from (21), (22), (23) and (24) that

$$
\begin{aligned}
\int_{(\mathbb{B}_R^N(0)\setminus\mathbb{B}_r^N(0))\cap\mathbf{R}_+^N}&\frac{|u(x)-u(0)|^p}{|x|^{N+sp}}\,\mathrm{d}x\\
&\le\Bigl(\frac{N}{s}\Bigr)^p\int_{\mathbf{R}_+^N}\frac{|\mathrm{D}u(x)|^p\,x_N^p}{|x|^{N+sp}}\,\mathrm{d}x+\frac{N|\partial\mathbb{B}_1^N|}{2s}\|\mathrm{D}u\|_{L^\infty(\mathbf{R}_+^N)}^p r^{(1-s)p}.
\end{aligned}
\tag{25}
$$

Letting $R\to\infty$ and $r\to 0$ in (25) we reach the conclusion (15) since $s<1$. □

The constant in the estimate (14) of Theorem 1 will also rely on the following computation.

Lemma 3. *If $N \in \mathbf{N} \setminus \{0,1\}$, $s \in (0,1)$, $p \in [1,\infty)$ and $x \in \mathbf{R}^N_+$, then*

$$\int_{\mathbf{R}^{N-1}} \frac{1}{|x-y|^{N+sp}} \,\mathrm{d}y = \frac{\pi^{\frac{N-1}{2}} \Gamma(\frac{sp+1}{2})}{\Gamma(\frac{N+sp}{2})\, x_N^{sp+1}}.$$

Proof. Integrating in spherical coordinates and applying the change of variable $r = x_N \sqrt{t^{-1} - 1}$, we get

$$\begin{aligned}\int_{\mathbf{R}^{N-1}} \frac{1}{|x-y|^{N+sp}} \,\mathrm{d}y &= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^\infty \frac{r^{N-2}}{(x_N^2 + r^2)^{\frac{N+sp}{2}}} \,\mathrm{d}r \\ &= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})\, x_N^{sp+1}} \int_0^1 (1-t)^{\frac{N-1}{2}-1} t^{\frac{sp+1}{2}-1} \,\mathrm{d}t \\ &= \frac{\pi^{\frac{N-1}{2}} \mathrm{B}(\frac{N-1}{2}, \frac{sp+1}{2})}{\Gamma(\frac{N-1}{2})\, x_N^{sp+1}} = \frac{\pi^{\frac{N-1}{2}} \Gamma(\frac{sp+1}{2})}{\Gamma(\frac{N+sp}{2})\, x_N^{sp+1}},\end{aligned}$$

in view of the volume formula for the unit sphere $\partial\mathbb{B}_1^{N-1}$ and the properties of the Beta function. □

We can now prove Theorem 1.

Proof of Theorem 1. By density we can assume that $u \in C^1(\bar{\mathbf{R}}^N_+)$. By Proposition 2 and Lemma 3, we have then

$$\begin{aligned}\int_{\partial\mathbf{R}^N_+} \Bigl(\int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(x)|^p}{|x-y|^{N+sp}} \,\mathrm{d}y \Bigr) \,\mathrm{d}x &\le \Bigl(\frac{N}{s}\Bigr)^p \int_{\partial\mathbf{R}^N_+} \Bigl(\int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(y)|^p\, y_N^p}{|x-y|^{N+sp}} \,\mathrm{d}y \Bigr) \,\mathrm{d}x \\ &= \Bigl(\frac{N}{s}\Bigr)^p \int_{\mathbf{R}^N_+} \Bigl(\int_{\partial\mathbf{R}^N_+} \frac{|\mathrm{D}u(y)|^p\, y_N^p}{|x-y|^{N+sp}} \,\mathrm{d}x \Bigr) \,\mathrm{d}y \\ &= \frac{\pi^{\frac{N-1}{2}} \Gamma(\frac{sp+1}{2})}{\Gamma(\frac{N+sp}{2})} \Bigl(\frac{N}{s}\Bigr)^p \int_{\mathbf{R}^N_+} \frac{|\mathrm{D}u(y)|^p}{y_N^{1-(1-s)p}} \,\mathrm{d}y,\end{aligned}$$

which proves the announced inequality (14). □

3 Fractional trace conjunction inequality

The fractional Sobolev spaces not only appear as traces of first-order Sobolev spaces, but also have traces on their own. Indeed [2; 37, th. 12] (see also

[23, th. 9.14]), if $sp > 1$, then every function $u \in \dot{W}^{s,p}(\mathbf{R}^N_+)$ has trace $v = \operatorname{tr}_{\partial \mathbf{R}^N_+} u \in \dot{W}^{s-1/p,p}(\partial \mathbf{R}^N_+)$ satisfying

$$\iint\limits_{\partial \mathbf{R}^N_+ \times \partial \mathbf{R}^N_+} \frac{|v(x) - v(y)|^p}{|x - y|^{N-2+sp}} \,\mathrm{d}x \,\mathrm{d}y \le C \iint\limits_{\mathbf{R}^N_+ \times \mathbf{R}^N_+} \frac{|u(y) - u(z)|^p}{|y - z|^{N+sp}} \,\mathrm{d}y \,\mathrm{d}z. \tag{26}$$

We prove a corresponding trace conjunction inequality.

Theorem 4. *For every $N \in \mathbf{N} \setminus \{0, 1\}$, $s \in (0, 1)$ and $p \in [1, \infty)$ satisfying $sp > 1$, there exists a constant $C \in (0, \infty)$ such every $u \in \dot{W}^{s,p}(\mathbf{R}^N_+)$ has a trace $v = \operatorname{tr}_{\partial \mathbf{R}^N_+} u$ satisfying*

$$\int_{\partial \mathbf{R}^N_+} \left(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|^p}{|x - y|^{N-1+sp}} \,\mathrm{d}y \right) \mathrm{d}x \le C \iint\limits_{\mathbf{R}^N_+ \times \mathbf{R}^N_+} \frac{|u(y) - u(z)|^p}{|y - z|^{N+sp}} \,\mathrm{d}y \,\mathrm{d}z. \tag{27}$$

Even though the proof of Theorem 4 will not be based on a Hardy inequality, our proof will rely on a the strategy devised by Brezis, Mironescu and Ponce for fractional Hardy inequalities [8; 27, Lemma 2].

Proof of Theorem 4. By density, we can assume that $u \in C^1_c(\bar{\mathbf{R}}^N_+)$ so that in particular we have

$$\int_{\partial \mathbf{R}^N_+} \left(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|^p}{|x - y|^{N-1+sp}} \,\mathrm{d}y \right) \mathrm{d}x < \infty, \tag{28}$$

with $v = u|_{\partial \mathbf{R}^N_+}$. Defining for $\lambda > 0$ the set

$$A_\lambda \coloneqq \bigl\{ (x, y, z) \in \partial \mathbf{R}^N_+ \times \mathbf{R}^N_+ \times \mathbf{R}^N_+ \bigm| |z - x| \le \lambda |x - y| \bigr\}, \tag{29}$$

we have by convexity and the triangle inequality

$$\begin{aligned} \iiint\limits_{A_\lambda} \frac{|u(x) - u(y)|^p}{|x - y|^{2N-1+sp}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \le 2^{p-1} &\iiint\limits_{A_\lambda} \frac{|u(x) - u(z)|^p}{|x - y|^{2N-1+sp}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\ &+ 2^{p-1} \iiint\limits_{A_\lambda} \frac{|u(y) - u(z)|^p}{|x - y|^{2N-1+sp}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z. \end{aligned} \tag{30}$$

For the left-hand side of (30), we note that for each $x \in \partial \mathbf{R}^N_+$ and $y \in \mathbf{R}^N_+$,

$$\int\limits_{\substack{z \in \mathbf{R}^N_+ \\ |z - x| \le \lambda |x - y|}} \frac{1}{|x - y|^{2N-1+sp}} \,\mathrm{d}z = \frac{\lambda^N |\mathbb{B}^N_1|}{2 |x - y|^{N-1+sp}}, \tag{31}$$

whereas for the first term in the right-hand side of (30) we have

$$
\begin{aligned}
\int\limits_{\substack{y\in\mathbf{R}^N_+\\|z-x|\le\lambda|x-y|}} \frac{1}{|x-y|^{2N-1+sp}}\,\mathrm{d}y &= \frac{|\partial\mathbb{B}^N_1|}{2}\int_{|z-x|/\lambda}^{\infty}\frac{1}{r^{N+sp}}\,\mathrm{d}r\\
&= \frac{N|\mathbb{B}^N_1|\lambda^{N-1+sp}}{2(N-1+sp)|z-x|^{N-1+sp}}.
\end{aligned}
\tag{32}
$$

In order to treat the second term in the right-hand side of (30), we note that if $|z-x|\le\lambda|x-y|$, then by the triangle inequality

$$
(1-\lambda)|x-y|\le|x-y|-|z-x|\le|y-z|\le|x-y|+|z-x|\le(1+\lambda)|x-y|, \tag{33}
$$

and hence, if $\lambda<1$

$$
\begin{aligned}
\int\limits_{\substack{x\in\partial\mathbf{R}^N_+\\|z-x|\le\lambda|x-y|}} \frac{1}{|x-y|^{2N-1+sp}}\,\mathrm{d}x &\le \int\limits_{\substack{x\in\partial\mathbf{R}^N_+\\(1-\lambda)|x-y|\le|y-z|}} \frac{(1+\lambda)^{2N-1+sp}}{|y-z|^{2N-1+sp}}\,\mathrm{d}x\\
&\le |\mathbb{B}^{N-1}_1|\frac{(1+\lambda)^{2N-1+sp}}{(1-\lambda)^{N-1}|y-z|^{N+sp}}.
\end{aligned}
\tag{34}
$$

Inserting (31), (32) and (34) into (30) we get, in view of (28),

$$
\begin{aligned}
\Bigl(\frac{1}{2^{p-1}}-\frac{N\lambda^{sp-1}}{N-1+sp}\Bigr)&\int_{\partial\mathbf{R}^N_+}\Bigl(\int_{\mathbf{R}^N_+}\frac{|u(x)-u(y)|^p}{|x-y|^{N-1+sp}}\,\mathrm{d}y\Bigr)\,\mathrm{d}x\\
&\le \frac{2|\mathbb{B}^{N-1}_1|}{|\mathbb{B}^N_1|}\frac{(1+\lambda)^{2N-1+sp}}{\lambda^N(1-\lambda)^{N-1}}\iint\limits_{\mathbf{R}^N_+\times\mathbf{R}^N_+}\frac{|u(y)-u(z)|^p}{|y-z|^{N+sp}}\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}
\tag{35}
$$

Since $sp>1$ we can fix $\lambda\in(0,1)$ so that $N2^{p-1}\lambda^{sp-1}<N-1+sp$ and get the estimate (27) as a consequence of (35). □

4 Controling the Gagliardo integral by the conjunction integral

The conjunction integral controls the Gagliardo integral so that the estimates Theorem 1 and Theorem 4 imply their classical counterparts (13) and (26) respectively.

Theorem 5. *If $N \in \mathbf{N}\setminus\{0,1\}$, if $s \in (0,1)$, if $p \in [1,\infty)$ and if the functions $u\colon \mathbf{R}^N_+ \to \mathbf{R}$ and $v\colon \partial\mathbf{R}^N_+ \to \mathbf{R}$ are measurable, then*

$$\iint\limits_{\partial\mathbf{R}^N_+\times\partial\mathbf{R}^N_+} \frac{|v(x)-v(y)|^p}{|x-y|^{N-1+sp}}\,\mathrm{d}x\,\mathrm{d}y \le \frac{2\cdot 3^{2N-1+sp}\Gamma(\frac{N+2}{2})}{4^{sp}\pi^{\frac{1}{2}}\Gamma(\frac{N+1}{2})}\int_{\partial\mathbf{R}^N_+}\biggl(\int_{\mathbf{R}^N_+}\frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\biggr)\,\mathrm{d}x. \tag{36}$$

Theorem 5 will follow from the more general result.

Lemma 6. *If $N \in \mathbf{N}\setminus\{0,1\}$, if $s \in (0,1)$, if $p \in [1,\infty)$ and if the functions $u\colon \mathbf{R}^N_+ \to \mathbf{R}$ and $v,w\colon \partial\mathbf{R}^N_+ \to \mathbf{R}$ are measurable, then*

$$\begin{aligned}\iint\limits_{\partial\mathbf{R}^N_+\times\partial\mathbf{R}^N_+} &\frac{|v(x)-w(x)|^p}{|x-z|^{N-1+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &\le \frac{2\cdot 3^{2N-1+sp}\Gamma(\frac{N+2}{2})}{4^{sp}\pi^{\frac{1}{2}}\Gamma(\frac{N+1}{2})}\biggl(\int_{\partial\mathbf{R}^N_+}\biggl(\int_{\mathbf{R}^N_+}\frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\biggr)\,\mathrm{d}x\\ &\qquad + \int_{\partial\mathbf{R}^N_+}\biggl(\int_{\mathbf{R}^N_+}\frac{|w(z)-u(y)|^p}{|z-y|^{N+sp}}\,\mathrm{d}y\biggr)\,\mathrm{d}z\biggr).\end{aligned} \tag{37}$$

Proof. We define the set

$$A \coloneqq \Bigl\{(x,y,z)\in\partial\mathbf{R}^N_+\times\mathbf{R}^N_+\times\partial\mathbf{R}^N_+ \Bigm| |y-\tfrac{x+z}{2}| \le \tfrac{|z-x|}{4}\Bigr\}$$

and we write, thanks to the triangle inequality,

$$\begin{aligned}\iiint\limits_A \frac{|v(x)-w(z)|^p}{|x-z|^{2N-1+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \le 2^{p-1}&\iiint\limits_A \frac{|v(x)-u(y)|^p}{|x-z|^{2N-1+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &+2^{p-1}\iiint\limits_A \frac{|u(y)-w(z)|^p}{|x-z|^{2N-1+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.\end{aligned} \tag{38}$$

We first compute

$$\int\limits_{\substack{y\in\mathbf{R}^N_+\\ |y-\frac{x+z}{2}|\le\frac{|x-z|}{4}}} \frac{1}{|x-z|^{2N-1+sp}}\,\mathrm{d}y \ge \frac{|\mathbb{B}^N_1|}{2\cdot 4^N|x-z|^{N-1+sp}}. \tag{39}$$

Next, we have for each $(x, y, z) \in A$,

$$\tfrac{1}{4}|x - z| \le |\tfrac{x-z}{2}| - |y - \tfrac{x+z}{2}| \le |x - y| \le |\tfrac{x-z}{2}| + |y - \tfrac{x+z}{2}| \le \tfrac{3}{4}|x - z|,$$

and therefore for every $x \in \partial \mathbf{R}^N_+$ and $y \in \mathbf{R}^N_+$

$$\begin{aligned}\int\limits_{\substack{z \in \partial \mathbf{R}^N_+\\ |y - \frac{x+y}{2}| \le \frac{|x-z|}{4}}} \frac{1}{|x - z|^{2N-1+sp}} \,\mathrm{d}z &\le \int\limits_{\substack{z \in \partial \mathbf{R}^N_+\\ |x-z| \le 4|x-y|}} \frac{3^{2N-1+sp}}{4^{2N-1+sp}|x - y|^{2N-1+sp}} \,\mathrm{d}z\\ &\le \frac{3^{2N-1+sp}|\mathbb{B}^{N-1}_1|}{4^{N+sp}|x - y|^{N+sp}}.\end{aligned} \tag{40}$$

Similarly, we have for every $y \in \mathbf{R}^N_+$ and every $z \in \partial \mathbf{R}^N_+$,

$$\int\limits_{\substack{x \in \partial \mathbf{R}^N_+\\ |y - \frac{x+z}{2}| \le \frac{|x-z|}{4}}} \frac{1}{|x - z|^{2N-1+sp}} \,\mathrm{d}x \le \frac{3^{2N-1+sp}|\mathbb{B}^{N-1}_1|}{4^{N+sp}|y - z|^{N+sp}}. \tag{41}$$

Inserting (39), (40) and (41) into (38), we get

$$\begin{aligned}&\iint\limits_{\partial \mathbf{R}^N_+ \times \partial \mathbf{R}^N_+} \frac{|v(x) - w(x)|^p}{|x - z|^{N-1+sp}} \,\mathrm{d}x \,\mathrm{d}y\\ &\quad\le \frac{2 \cdot 3^{2N-1+sp}|\mathbb{B}^{N-1}_1|}{|\mathbb{B}^N_1| 4^{sp}} \biggl(\int_{\partial \mathbf{R}^N_+} \Bigl(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}y\Bigr) \,\mathrm{d}x\\ &\qquad\qquad + \int_{\partial \mathbf{R}^N_+} \Bigl(\int_{\mathbf{R}^N_+} \frac{|w(z) - u(y)|^p}{|y - z|^{N+sp}} \,\mathrm{d}y\Bigr) \,\mathrm{d}z\biggr),\end{aligned}$$

we get the conclusion (37). □

5 Characterising the trace

The next result shows that the finitiness of the trace conjunction integral characterises the trace.

Theorem 7. *If $N \in \mathbf{N} \setminus \{0, 1\}$, if $s \in (0, 1)$, if $p \in [1, \infty)$, if the functions $u \colon \mathbf{R}^N_+ \to \mathbf{R}$ and $v, w \colon \partial \mathbf{R}^N_+ \to \mathbf{R}$ are measurable and if*

$$\int_{\partial \mathbf{R}^N_+} \Bigl(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}y\Bigr) \,\mathrm{d}x < \infty$$

and

$$\int_{\partial \mathbf{R}^N_+} \biggl(\int_{\mathbf{R}^N_+} \frac{\lvert w(x) - u(y)\rvert^p}{\lvert x - y\rvert^{N + sp}} \,\mathrm{d} y\biggr) \,\mathrm{d} x < \infty,$$

then $v = w$ *almost everywhere in* $\partial \mathbf{R}^N_+$.

The proof of Theorem 7 will rely on Lemma 6 and the next lemma.

Lemma 8. *If* $N \in \mathbf{N} \setminus \{0, 1\}$, *if* $s \in (0, 1)$, *if* $p \in [1, \infty)$, *if the functions* $v, w \colon \partial \mathbf{R}^N_+ \to \mathbf{R}$ *are measurable and if*

$$\iint\limits_{\partial \mathbf{R}^N_+ \times \partial \mathbf{R}^N_+} \frac{\lvert v(x) - w(y)\rvert^p}{\lvert x - y\rvert^{N - 1 + sp}} \,\mathrm{d} y \,\mathrm{d} x < \infty, \tag{42}$$

then $v = w$ *almost everywhere.*

Proof. By the assumption (42) and Fubini's theorem, $v, w \in L^p_{\mathrm{loc}}(\partial \mathbf{R}^N_+)$. We define for $\delta > 0$ the functions $v_\delta, w_\delta \colon \partial \mathbf{R}^N_+ \to \mathbf{R}$ for each $z \in \partial \mathbf{R}^N_+$ by

$$v_\delta (z) \coloneqq \fint_{\mathbb{B}^{N-1}_\delta (z)} v \qquad \text{and} \qquad w_\delta (z) \coloneqq \fint_{\mathbb{B}^{N-1}_\delta (z)} w.$$

We have for every $z \in \mathbf{R}^N$ and $\delta > 0$, by Jensen's inequality

$$\begin{aligned} \lvert v_\delta (z) - w_\delta (z)\rvert^p &\le \biggl\lvert \fint_{\mathbb{B}^{N-1}_\delta (z)} \fint_{\mathbb{B}^{N-1}_\delta (z)} \lvert v(x) - w(y)\rvert \,\mathrm{d} x \,\mathrm{d} y\biggr\rvert^p\\ &\le \fint_{\mathbb{B}^{N-1}_\delta (z)} \fint_{\mathbb{B}^{N-1}_\delta (z)} \lvert v(x) - w(y)\rvert^p \,\mathrm{d} x \,\mathrm{d} y\\ &\le (2\delta)^{N - 1 + sp} \fint_{\mathbb{B}^{N-1}_\delta (z)} \fint_{\mathbb{B}^{N-1}_\delta (z)} \frac{\lvert v(x) - w(y)\rvert^p}{\lvert x - y\rvert^{N - 1 + sp}} \,\mathrm{d} x \,\mathrm{d} y. \end{aligned} \tag{43}$$

and hence, integrating (43),

$$\begin{aligned} &\int_{\partial \mathbf{R}^N_+} \lvert v_\delta - w_\delta\rvert^p\\ &\quad \le \frac{2^{N - 1 + sp}}{\delta^{N - 1 - sp} \lvert \mathbb{B}^{N - 1}_1\rvert^2} \int_{\partial \mathbf{R}^N_+} \iint\limits_{\mathbb{B}^N_\delta (z) \times \mathbb{B}^N_\delta (z)} \lvert v(x) - w(y)\rvert^p \,\mathrm{d} y \,\mathrm{d} x \,\mathrm{d} z\\ &\quad \le \frac{2^{N - 1 + sp}}{\delta^{N - 1 - sp} \lvert \mathbb{B}^{N - 1}_1\rvert^2} \iint\limits_{\partial \mathbf{R}^N_+ \times \partial \mathbf{R}^N_+} \int_{\mathbb{B}^{N-1}_\delta (\frac{x + y}{2})} \frac{\lvert v(x) - w(y)\rvert^p}{\lvert x - y\rvert^{N - 1 + sp}} \,\mathrm{d} z \,\mathrm{d} y \,\mathrm{d} x\\ &\quad \le \frac{2^{N - 1 + sp} \delta^{sp}}{\lvert \mathbb{B}^{N - 1}_1\rvert} \iint\limits_{\partial \mathbf{R}^N_+ \times \partial \mathbf{R}^N_+} \frac{\lvert v(x) - w(y)\rvert^p}{\lvert x - y\rvert^{N - 1 + sp}} \,\mathrm{d} y \,\mathrm{d} x. \end{aligned} \tag{44}$$

Therefore, we deduce from (44) and (42) that

$$\int_{\mathbf{R}^N} |v - w|^p \le \lim_{\delta \to 0} \int_{\mathbf{R}^N} |v_\delta(z) - w_\delta(z)|^p = 0, \tag{45}$$

which implies that $v = w$ almost everywhere on $\partial \mathbf{R}^N_+$. □

Proof of Theorem 7. This follows from Lemma 6 and Lemma 8. □

6 Controlling the Hardy integral by the conjunction integral

We now show how the conjunction inequality implies a Hardy inequality.

Theorem 9. *If $N \in \mathbf{N} \setminus \{0, 1\}$, if $s \in (0, 1)$, if $p \in [1, \infty)$, and if the functions $u \colon \mathbf{R}^N_+ \to \mathbf{R}$ and $v \colon \partial \mathbf{R}^N_+ \to \mathbf{R}$ are measurable, then*

$$\int_{\mathbf{R}^N_+} \frac{|v(x') - u(x)|^p}{x_N^{sp+1}} \,\mathrm{d}x \le C \int_{\partial \mathbf{R}^N_+} \left(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x \right) \mathrm{d}y, \tag{46}$$

for some constant C depending only on N, s and p.

Proof. Defining the set

$$A \coloneqq \bigl\{ (x, y) = (x, y', y_N) \in \partial \mathbf{R}^N_+ \times \mathbf{R}^N_+ \bigm| |x - y'| \le 3y_N/4 \bigr\},$$

we have by convexity

$$\begin{aligned} \iint\limits_A \frac{|v(y') - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x \,\mathrm{d}y \le 2^{p-1} \iint\limits_A \frac{|v(y') - v(x)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x \,\mathrm{d}y \\ + 2^{p-1} \iint\limits_A \frac{|v(x) - u(y)|^p}{|x - y|^{N+sp}} \,\mathrm{d}x \,\mathrm{d}y. \end{aligned} \tag{47}$$

For the integral in the left-hand side of (47), we note that if $|x - y'| \le y_N/2$, then

$$|x - y|^2 = |x - y'|^2 + y_N^2 \le \frac{25}{16} y_N^2$$

and thus

$$\int\limits_{\substack{x \in \partial \mathbf{R}^N_+ \\ |x - y'| \le 3y_N/4}} \frac{1}{|x - y|^{N+sp}} \,\mathrm{d}x \ge \int\limits_{\substack{x \in \partial \mathbf{R}^N_+ \\ |x - y'| \le 3y_N/4}} \frac{1}{(5y_N/4)^{N+sp}} \,\mathrm{d}x = \frac{3^{N-1} 4^{sp+1} |\mathbb{B}^{N-1}_1|}{5^{N+sp} y_N^{sp+1}}, \tag{48}$$

whereas for the first term in the right-hand side of (47), we have

$$\begin{aligned}\int_{4|x-y'|/3}^{\infty} \frac{1}{|x-y|^{N+sp}} \,\mathrm{d}x &\le \int_{4|x-y'|/3}^{\infty} \frac{1}{y_N^{N+sp}} \,\mathrm{d}y_N \\ &= \frac{3^{N-1+sp}}{(N-1+sp)4^{N-1+sp}|x-y'|^{N-1+sp}}.\end{aligned} \tag{49}$$

Inserting (48) and (49) into (47), we get

$$\begin{aligned}\frac{3^{N-1}4^{sp+1}|\mathbb{B}_1^{N-1}|}{5^{N+sp}} &\int_{\mathbf{R}_+^N} \frac{|v(y')-u(y)|^p}{y_N^{sp+1}} \,\mathrm{d}x\,\mathrm{d}y \\ &\le 2^{p-1} \iint\limits_{\partial\mathbf{R}_+^N\times\partial\mathbf{R}_+^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N-1+sp}} \,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \frac{2^{p-1}3^{N-1+sp}}{(N-1+sp)4^{N-1+sp}} \int_{\partial\mathbf{R}_+^N} \Big(\int_{\mathbf{R}_+^N} \frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\Big) \,\mathrm{d}y.\end{aligned} \tag{50}$$

Thanks to Theorem 5, (46) follows then from (50). □

7 From Hardy and Gagliardo to conjunction

As a converse to Theorem 9 and Theorem 7, the conjunction integral is controlled by the Gagliardo and Hardy integrals.

Theorem 10. *If $N \in \mathbf{N} \setminus \{0,1\}$, if $s \in (0,1)$, if $p \in [1,\infty)$ and if the functions $u\colon \mathbf{R}_+^N \to \mathbf{R}$ and $v\colon \partial\mathbf{R}_+^N \to \mathbf{R}$ are measurable, then*

$$\begin{aligned}\int_{\partial\mathbf{R}_+^N} &\Big(\int_{\mathbf{R}_+^N} \frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\Big) \,\mathrm{d}y \\ &\le C\Big(\iint\limits_{\partial\mathbf{R}_+^N\times\partial\mathbf{R}_+^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N-1+sp}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbf{R}_+^N} \frac{|v(x')-u(x)|^p}{x_N^{sp+1}} \,\mathrm{d}x\Big).\end{aligned} \tag{51}$$

Proof. By convexity and the triangle inequality, we write

$$\begin{aligned}\int_{\partial\mathbf{R}_+^N} &\Big(\int_{\mathbf{R}_+^N} \frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\Big) \,\mathrm{d}y \\ &\le 2^{p-1} \int_{\partial\mathbf{R}_+^N} \Big(\int_{\mathbf{R}_+^N} \frac{|v(x)-v(y')|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\Big) \,\mathrm{d}y \\ &\qquad + 2^{p-1} \int_{\partial\mathbf{R}_+^N} \Big(\int_{\mathbf{R}_+^N} \frac{|v(y')-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}x\Big) \,\mathrm{d}y.\end{aligned} \tag{52}$$

For the first term in the right-hand side of (52), we compute

$$\int_0^\infty \frac{1}{(|x-y'|^2+y_N^2)^{\frac{N+sp}{2}}}\,\mathrm{d}y_N = \frac{1}{|x-y'|^{N-1+sp}}\int_0^\infty \frac{1}{(1+t^2)^{\frac{N+sp}{2}}}\,\mathrm{d}t, \quad (53)$$

where the integral on the right-hand side is finite if $N+sp>1$, whereas for the second term, we have

$$\begin{aligned}\int_{\partial\mathbf{R}^N_+} \frac{1}{|x-y|^{N+sp}}\,\mathrm{d}x &= \int_{\partial\mathbf{R}^N_+} \frac{1}{(|x-y'|^2+y_N^2)^{\frac{N+sp}{2}}}\,\mathrm{d}x\\ &= \frac{1}{y_N^{sp+1}}\int_{\mathbf{R}^{N-1}} \frac{1}{(|z|^2+1)^{\frac{N+sp}{2}}}\,\mathrm{d}z,\end{aligned} \quad (54)$$

where the integral on the right-hand side is finite if $sp+1>0$. Inserting (53) and (54) into (52), we get the conclusion (51). □

8 Bourgain-Brezis-Mironescu formulae

For smooth functions, the conjunction integral satisfies a counterpart of the Bourgain-Brezis-Mironescu formula (7).

Theorem 11. *If $N \in \mathbf{N}\setminus\{0,1\}$, if $p\in[1,\infty)$ and if $u \in C^1_c(\bar{\mathbf{R}}^N_+)$, then*

$$\lim_{\substack{s\to 0\\ >}}(1-s)\int_{\partial\mathbf{R}^N_+}\Bigl(\int_{\mathbf{R}^N_+}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\Bigr)\,\mathrm{d}x = \frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{p+1}{2})}{p\Gamma(\frac{N+p}{2})}\int_{\partial\mathbf{R}^N_+}|\mathrm{D}u|^p. \quad (55)$$

The constant in (55) comes from the following computation.

Lemma 12. *If $N \in \mathbf{N}\setminus\{0,1\}$, if $p\in[1,\infty)$, then*

$$\int_{\partial\mathbb{B}^N_1}|w_1|^p\,\mathrm{d}w = \frac{2\pi^{\frac{N-1}{2}}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{N+p}{2})}.$$

Proof. We have

$$\begin{aligned}\int_{\partial\mathbb{B}^N_1}|w_1|^p\,\mathrm{d}w &= |\partial\mathbb{B}^{N-1}_1|\int_0^\pi(\sin\theta)^{N-2}|\cos\theta|^p\,\mathrm{d}\theta\\ &= |\partial\mathbb{B}^{N-1}_1|\,2\int_0^{\pi/2}(\sin\theta)^{N-2}(\cos\theta)^p\,\mathrm{d}\theta\\ &= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\frac{\Gamma(\frac{N-1}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{N+p}{2})} = \frac{2\pi^{\frac{N-1}{2}}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{N+p}{2})},\end{aligned} \quad (56)$$

which proves (56). □

Proof of Theorem 11. We consider the set

$$K \coloneqq \{x \in \partial \mathbf{R}^N_+ \mid \mathbb{B}^N_1(x) \cap \operatorname{supp} u \neq \emptyset\}.$$

Since $\operatorname{supp} u \subseteq \bar{\mathbf{R}}^N_+$ is compact, the K itself is also compact. We have

$$\begin{aligned}
\int_{\mathbb{B}^N_1(x)\cap \mathbf{R}^N_+} & \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}y \\
&= \int_{\mathbb{B}^N_1 \cap \mathbf{R}^N_+} \frac{|u(x)-u(x+h)|^p}{|h|^{N+sp}} \,\mathrm{d}h \\
&= \int_0^1 \int_{\partial\mathbb{B}^N_1 \cap \mathbf{R}^N_+} \frac{|u(x)-u(x+rw)|^p}{r^{1+sp}} \,\mathrm{d}w \,\mathrm{d}r \\
&= \frac{1}{1-s} \int_0^1 \int_{\partial\mathbb{B}^N_1 \cap \mathbf{R}^N_+} \frac{|u(x)-u(x+t^{\frac{1}{1-s}}w)|^p}{t^{1+\frac{p}{1-s}-p}} \,\mathrm{d}w \,\mathrm{d}t,
\end{aligned}$$

under the change of variable $r = t^{1/(1-s)}$ and thus

$$\begin{aligned}
\lim_{\substack{s \to 1 \\ <}} (1-s) & \int_{\mathbb{B}^N_1(x)\cap \mathbf{R}^N_+} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \,\mathrm{d}y \\
&= \lim_{\substack{s \to 1 \\ <}} \int_0^1 \int_{\partial\mathbb{B}^N_1 \cap \mathbf{R}^N_+} \frac{|u(x)-u(x+t^{\frac{1}{1-s}}w)|^p}{t^{1+\frac{p}{1-s}-p}} \,\mathrm{d}w \,\mathrm{d}t \\
&= \frac{1}{(1-s)p} \int_{\partial\mathbb{B}^N_1 \cap \mathbf{R}^N_+} |\mathrm{D}u(x)[w]|^p \,\mathrm{d}w \\
&= \frac{1}{(1-s)p} \frac{\pi^{\frac{N-1}{2}} \Gamma(\frac{p+1}{2})}{\Gamma(\frac{N+p}{2})} \int_{\partial \mathbf{R}^N_+} |\mathrm{D}u|^p,
\end{aligned} \tag{57}$$

since, in view of Lemma 12,

$$\begin{aligned}
\int_{\partial\mathbb{B}^N_1 \cap \mathbf{R}^N_+} |\mathrm{D}u(x)[w]|^p \,\mathrm{d}w &= |\mathrm{D}u(x)|^p \int_{\partial\mathbb{B}^N_1 \cap \mathbf{R}^N_+} |w_1|^p \,\mathrm{d}w \\
&= \frac{|\mathrm{D}u(x)|^p}{2} \int_{\partial\mathbb{B}^N_1} |w_1|^p \,\mathrm{d}w \\
&= \frac{\pi^{\frac{N-1}{2}} \Gamma(\frac{p+1}{2})}{\Gamma(\frac{N+p}{2})} |\mathrm{D}u(x)|^p.
\end{aligned}$$

On the other hand,

$$\int_{\mathbb{B}_1^N(x)\cap\mathbf{R}_+^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y \le \int_{\mathbb{B}_1^N(x)\cap\mathbf{R}_+^N} \frac{\|\mathrm{D}u\|_{L^\infty(\mathbf{R}_+^N)}^p}{|x-y|^{N+(1-s)p}}\,\mathrm{d}y \\ \le C_1\|\mathrm{D}u\|_{L^\infty(\mathbf{R}_+^N)}^p. \tag{58}$$

In view of (57) and (58), we have by Lebesgue's dominated convergence

$$\lim_{\substack{s\to 1\\ <}} \int_{\partial\mathbf{R}_+^N} \Bigl(\int_{\mathbb{B}_1^N(x)\cap\mathbf{R}_+^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\Bigr)\,\mathrm{d}x = \frac{\pi^{\frac{N-1}{2}}\Gamma(\frac{p+1}{2})}{p\,\Gamma(\frac{N+p}{2})} \int_{\partial\mathbf{R}_+^N} |\mathrm{D}u|^p. \tag{59}$$

Finally, we also have by Lebesgue's dominated convergence theorem,

$$\lim_{\substack{s\to 1\\ <}} (1-s) \int_{\partial\mathbf{R}_+^N} \Bigl(\int_{\mathbf{R}_+^N\setminus\mathbb{B}_1^N(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\Bigr)\,\mathrm{d}x = 0, \tag{60}$$

and the conclusion (55) follows then from (59) and (60). □

A suitable asymptotic control on the conjunction integral gives some integrability of the gradient.

Theorem 13. *If $N \in \mathbf{N}\setminus\{0,1\}$, if $p\in(1,\infty)$ and if $u\colon \mathbf{R}_+^N \to \mathbf{R}$ and $v\colon \mathbf{R}_+^N \to \mathbf{R}$ are measurable and satisfy*

$$\liminf_{\substack{s\to 1\\ <}}\ (1-s) \int_{\partial\mathbf{R}_+^N} \Bigl(\int_{\mathbf{R}_+^N} \frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\Bigr)\,\mathrm{d}x < \infty, \tag{61}$$

then $v \in \dot W^{1,p}(\partial\mathbf{R}_+^N)$ and

$$\int_{\partial\mathbf{R}_+^N} |\mathrm{D}v|^p \le C \liminf_{\substack{s\to 1\\ <}}\ (1-s) \int_{\partial\mathbf{R}_+^N} \Bigl(\int_{\mathbf{R}_+^N} \frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\Bigr)\,\mathrm{d}x,$$

for some constant C depending only on N and p.

Proof. By Theorem 5, we have

$$\liminf_{\substack{s\to 1\\ <}}\ (1-s) \iint\limits_{\partial\mathbf{R}_+^N\times\partial\mathbf{R}_+^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N-1+sp}}\,\mathrm{d}x\,\mathrm{d}y \\ \le \frac{2\cdot 3^{2N-1+p}\Gamma(\frac{N+2}{2})}{4^p\pi^{\frac12}\Gamma(\frac{N+1}{2})} \liminf_{\substack{s\to 1\\ <}} (1-s) \int_{\partial\mathbf{R}_+^N} \Bigl(\int_{\mathbf{R}_+^N} \frac{|v(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}y\Bigr)\,\mathrm{d}x \\ < \infty;$$

we conclude then by the classical Bourgain-Brezis-Mironescu result [3]. □

Theorem 14. *If $N \in \mathbf{N} \setminus \{0, 1\}$ and if $u\colon \mathbf{R}^N_+ \to \mathbf{R}$ and $v\colon \mathbf{R}^N_+ \to \mathbf{R}$ are measurable and satisfy*

$$\liminf_{\substack{s \to 1 \\ <}} (1 - s) \int_{\partial \mathbf{R}^N_+} \left(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|}{|x - y|^{N+s}} \, \mathrm{d}y \right) \mathrm{d}x < \infty,$$

then $v \in BV(\partial \mathbf{R}^N_+)$ and

$$\int_{\partial \mathbf{R}^N_+} |\mathrm{D}v| \le C \liminf_{\substack{s \to 1 \\ <}} (1 - s) \int_{\partial \mathbf{R}^N_+} \left(\int_{\mathbf{R}^N_+} \frac{|v(x) - u(y)|}{|x - y|^{N+s}} \, \mathrm{d}y \right) \mathrm{d}x,$$

for some constant C depending only on N.

Proof. This follows from Theorem 5 and the corresponding results for the Gagliardo integral [16, 40]. □

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