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POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS INVOLVING UNBOUNDED VARIABLE EXPONENTS AND EXPONENTIAL GROWTH*

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The article is dedicated to the memory of Professor Haïm Brezis

Abstract

The paper provides the existence of a positive weak solution and a priori estimates for a class of parametric Dirichlet problems with unbounded variable exponents and exhibiting exponential growth. The approach relies on a special sub-supersolution method that we develop in our general setting. The bound of the admissible values of the parameter is explicitly determined. Applications to the regularity properties and asymptotic behavior with respect to the parameter are also given. Examples demonstrate the applicability of the stated results.

Keywords: Dirichlet problem, positive solution, sub-supersolution, exponential growth, supercritical growth, unbounded variable exponent.
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1 Introduction and main result

In this paper we study the Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda u^{q(x)-1} + f(x,u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

on a bounded domain Ω in $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary $\partial\Omega$ that is driven by the (negative) p-Laplacian operator $-\Delta_p$ for $p \in (1, +\infty)$ and depends on a real parameter $\lambda > 0$. Recall that $-\Delta_p$ is defined as

$$\langle -\Delta_p u, v \rangle \quad = \quad \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx, \quad \forall u, v \in W_0^{1,p}(\Omega).$$

The nonlinearity $f(x,u)$ in the right-hand side of the equation in (1) is expressed through a Carathéodory function $f : \Omega \times [0, \infty) \to \mathbb{R}$ (i.e., $f(x,s)$ is measurable in x and continuous in s). We assume the following set of hypotheses:

$$(F) \begin{cases} 0 \leq f(x,s) \leq a_1 s^{l(x)-1} e^{\alpha(x) s^{r(x)-1}}, \\ 1 < q_- \leq q(x) \leq q_+ < p, \\ p < l_- \leq l(x), \\ 1 < r(x), \\ 0 < \alpha(x) \leq a_2 K_0^{r(x)-1} \end{cases}$$

for all $(x,s) \in \Omega \times [0, \infty)$, where $a_1, a_2 > 0$, and $K_0 > 1$ are constants, q, l, r, α are continuous functions from Ω to $\mathbb{R}$, and for any function $h : \Omega \to \mathbb{R}$ we set

$$h_- := \inf_{x \in \Omega} h(x), \quad h_+ := \sup_{x \in \Omega} h(x).$$

The variable exponents $l(x)$, $r(x)$, and $\alpha(x)$ are permitted to be unbounded from above, that is to have $l_+ = \infty$, $r_+ = \infty$, and $\alpha_+ = \infty$.

Our main result reads as follows.

Theorem 1. *Assume that hypotheses* (F) *are satisfied. Then there exists a real number* $\Lambda > 0$ *such that for each* $\lambda \in (0, \Lambda]$, *problem* (1) *has a positive bounded weak solution.*

The proof of Theorem 1 is given in Section 5. Theorem 1 is applied in Section 6 to investigate the regularity of solutions and their asymptotic properties with respect to the parameter λ. Theorem 1 provides the solvability of problem (1) for large classes of nonlinear equations with general terms involving exponential growth. The wide range of applicability of Theorem 1 is demonstrated by three examples in Section 7.

The statement in (1) patterns the problems with concave-convex nonlinearities going back to Ambrosetti-Bresis-Cerami [2]. Example 1 in Section 7 illustrates the p-concave-convex structure of the considered nonlinearity in the context of unbounded variable exponents. Problems with concave-convex nonlinearities and gradient dependence are studied in [4].

A closely related work is that of Araujo-Montenegro [6] dealing with the Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{l-1} e^{\alpha u^r} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda > 0$ and $\alpha > 0$ are parameters, while $l > p$, $r > 0$, and $1 < q < p$ are constants. Theorem 1 extends the main result in [6]. Existence of positive radial solutions to semilinear elliptic equations of this type on the open unit ball can be found in [5] and on the entire space $\mathbb{R}^N$ in [9] (see also [7]).

A strong motivation for our work comes from the Trudinger-Moser inequality addressing the critical exponential growth in dimension two. In line with this, Alves-Shen [1] proves the existence of a nontrivial solution for the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = h(u) e^{\alpha_0 u^\tau} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

on a bounded domain Ω in $\mathbb{R}^2$, with constants $\alpha_0 > 0$ and $\tau \geq 2$, and a function $h(s)$ satisfying some requirements. We also mention that in Faria-Montenegro [8] the existence of radial solutions is proven for the problem

$$\begin{cases} -\Delta u = \lambda u^{q(x)-1} + f(x,u) & \text{in} \quad B \\ u > 0 & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B \end{cases}$$

on the open unit ball B in $\mathbb{R}^2$, when $\lambda \in (0, \lambda^*)$, with some $\lambda^* > 0$ and the reaction term $f(x,u)$ supercritical in the sense of Trudinger-Moser and satisfying some symmetry conditions.

Our approach relies on a special sub-supersolution method that we develop in the general setting of hypotheses (F). For the background of the sub-supersolution technique we refer to [3]. Here we provide explicit formulas for the subsolution and supersolution using essentially the parameter

λ. The bound $\Lambda > 0$ postulated in Theorem 1 is explicitly determined as $\Lambda := \min\{\lambda_*^1, \lambda_*^2\}$ with λ_*^1 and λ_*^2 given below in (9) and (12), respectively.

We outline the ideas in the proof of Theorem 1. The essential point is that corresponding to any admissible value of the parameter λ we are able to explicitly construct an ordered pair consisting of a subsolution and a supersolution to problem (1). Then along the ordered sub-supersolution interval we truncate problem (1). Through the fundamental theorem of pseudomonotone operators (see, e.g., [3]) we can resolve the truncated problem. Finally, making use of comparison arguments we show that the found solution of the truncated problem is actually a weak solution to the original problem (1). The applications to the regularity and asymptotic properties are based on the fact that the solution obtained in Theorem 1 belongs to the ordered interval formed with the subsolution and supersolution which offer a priori estimates.

The rest of the paper is organized as follows. Section 2 provides necessary background. Section 3 sets forth the construction of sub-supersolution for problem (1). Section 4 studies the auxiliary truncated problem. Section 5 contains the proof of Theorem 1. Section 6 discusses the application of Theorem 1 to the regularity of solutions and the asymptotic behavior with respect to the parameter. Section 7 presents examples illustrating the applicability of our results.

2 Preliminaries

Given $p \in (1, +\infty)$, denote $p' = p/(p-1)$ (the conjugate exponent). In the sequel, $W_0^{1,p}(\Omega)$ stands for the usual Sobolev space endowed with the norm $\|u\| = \|\nabla u\|_p$. The dual of $W_0^{1,p}(\Omega)$ is $W^{-1,p'}(\Omega)$.

Definition 1. *A function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of problem (1) if $(\lambda u^{q(\cdot)-1} + f(\cdot, u))\varphi \in L^1(\Omega)$ and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx - \int_\Omega (\lambda u^{q(x)-1} + f(x,u))\varphi dx = 0$$

for all $\varphi \in W_0^{1,p}(\Omega)$. A function $u \in W_0^{1,p}(\Omega)$ is called a subsolution (resp., supersolution) of problem (1) if $(\lambda u^{q(\cdot)-1} + f(\cdot, u))\varphi \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx - \int_\Omega (\lambda u^{q(x)-1} + f(x,u))\varphi dx \le 0 \text{ (resp., } \ge 0)$$

for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geq 0$. By a sub-supersolution of problem (1) we mean a pair of a subsolution u and a supersolution v such that $u \leq v$ almost everywhere on Ω.

The following two problems will play a major part in the next developments:

$$\begin{cases} -\Delta_p \phi = 1 & \text{in} \quad \Omega \\ \phi = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2}$$

and

$$\begin{cases} -\Delta_p u_1 = \lambda_1 |u_1|^{p-2} u_1 & \text{in} \quad \Omega \\ u_1 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3}$$

The regularity theory ensures that the unique solution to problem (2) satisfies $\phi \in C^1(\overline{\Omega})$, thus

$$\gamma := \|\phi\|_\infty < \infty. \tag{4}$$

In problem (3), λ_1 stands for the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$. The solution u_1 denotes the first eigenfunction of $-\Delta_p$ that is chosen to be positive and normalized as $\|u_1\|_\infty = 1$.

In the proof of Theorem 1 we will need the surjectivity result for pseudomonotone operators (see [3, Theorem 2.99]).

Lemma 1. *Let $A : X \to X^*$ be a pseudomonotone, bounded (in the sense it maps bounded sets into bounded sets), and coercive operator defined on a reflexive Banach space X. Then A is surjective. In particular, there exists $u \in X$ such that $Au = 0$.*

An efficient tool in our approach is the truncation operator $T = T_{\underline{u},\overline{u}} : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ associated with an ordered pair of functions $\underline{u}, \overline{u} \in W_0^{1,p}(\Omega)$ with $\underline{u} \leq \overline{u}$ almost everywhere on Ω, which is defined by

$$(Tu)(x) = \begin{cases} \underline{u}(x) & \text{if} \quad u(x) \leq \underline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\ \overline{u}(x) & \text{if} \quad \overline{u}(x) \leq u(x) \end{cases}$$

for all $u \in W_0^{1,p}(\Omega)$. It readily follows that

$$Tu = \max\{u, \underline{u}\} + \min\{u, \overline{u}\} - u, \quad \forall u \in W_0^{1,p}(\Omega)$$

showing that the map T is well defined, bounded and continuous.

The next results set the basis for the construction of sub-supersolution.

Proposition 1. *Assume that conditions (F) hold. Given any $K > 0$, for each $\lambda \in (0, \lambda_*^1(K)]$ where*

$$\lambda_*^1(K) = \left(\frac{1}{K}\right)^{\frac{p-q_-}{l_--p}} \left(\frac{p-q_-}{\gamma^{p-1}(l_- - q_-)}\right)^{\frac{l_--q_-}{l_--p}} \frac{l_- - p}{p - q_-}, \tag{5}$$

the positive number

$$M(\lambda, K) = \lambda^{1/(l_- - q_-)} \left[\frac{p-q_-}{(l_- - p)K}\right]^{1/(l_- - q_-)} \tag{6}$$

satisfies

$$\lambda M(\lambda, K)^{q_- - 1} + M(\lambda, K)^{l_- - 1} K \le \left(\frac{M(\lambda, K)}{\gamma}\right)^{p-1}, \tag{7}$$

with γ in (4).

Proof. Fix $K > 0$. For any $\lambda > 0$, inequality (7) can be written as

$$\lambda\gamma^{p-1} M(\lambda, K)^{q_- - p} + \gamma^{p-1} M(\lambda, K)^{l_- - p} K \le 1. \tag{8}$$

In turn, inequality (8) is equivalent to $\psi_\lambda(M(\lambda, K)) \le 1$, where $\psi_\lambda : (0, +\infty) \to (0, +\infty)$ is the function defined by

$$\psi_\lambda(t) = \lambda\gamma^{p-1} t^{q_- - p} + K\gamma^{p-1} t^{l_- - p}, \quad \forall t > 0.$$

Since $1 < q_- < p < l_-$ as assumed in (F), it holds

$$\lim_{t \to 0^+} \psi_\lambda(t) = \lim_{t \to \infty} \psi_\lambda(t) = +\infty.$$

Therefore the function ψ_λ admits a global minimizer. Solving the equation $\psi'_\lambda(t) = 0$ we find that the global minimizer of ψ_λ is $M(\lambda, K)$ given in (6).

Using (6), through direct computation we infer that $\psi_\lambda(M(\lambda, K)) \le 1$ if and only if $0 < \lambda \le \lambda_*^1(K)$ for $\lambda_*^1(K)$ in (5). The desired conclusion is achieved, thus completing the proof. □

Remark 1. *The proof of Proposition 1 reveals that the threshold $\lambda_*^1(K)$ for λ indicated in (5) is optimal. This is true because (7) implies $\lambda \in (0, \lambda_*^1(K)]$.*

We are going to apply Proposition 1 to problem (1) with the data K_0, a_1, and a_2 in hypotheses (F). Namely, for $K = a_1 e^{a_2}$, formula (5) yields

$$\lambda_*^1 = \lambda_*^1(a_1 e^{a_2}) = \left(\frac{1}{a_1 e^{a_2}}\right)^{\frac{p-q_-}{l_--p}} \left(\frac{p-q_-}{\gamma^{p-1}(l_- - q_-)}\right)^{\frac{l_--q_-}{l_--p}} \frac{l_- - p}{p - q_-}. \tag{9}$$

For any $\lambda \in (0, \lambda_*^1]$, from (6) and (7) it turns out that

$$M_\lambda := M(\lambda, a_1 e^{a_2}) = \lambda^{1/(l_- - q_-)} \left[\frac{p - q_-}{(l_- - p) a_1 e^{a_2}} \right]^{1/(l_- - q_-)} \tag{10}$$

and

$$\lambda M_\lambda^{q_- - 1} + M_\lambda^{l_- - 1} a_1 e^{a_2} \leq \left(\frac{M_\lambda}{\gamma} \right)^{p-1}. \tag{11}$$

Now we set

$$\lambda_*^2 := \frac{(l_- - p) a_1 e^{a_2}}{(p - q_-) K_0^{l_- - q_-}}. \tag{12}$$

Corollary 1. *Let $\Lambda := \min\{\lambda_*^1, \lambda_*^2\}$, with λ_*^1 and λ_*^2 expressed in (9) and (12), respectively. If $\lambda \in (0, \Lambda]$, for M_λ in (10) we have*

$$0 < M_\lambda \leq \frac{1}{K_0}, \tag{13}$$

with K_0 given in hypotheses (F).

Proof. Since $\lambda \leq \lambda_*^2$, by (10) and (12), we find that

$$\begin{aligned} M_\lambda &\leq (\lambda_*^2)^{\frac{1}{l_- - q_-}} \left[\frac{p - q_-}{(l_- - p) a_1 e^{a_2}} \right]^{\frac{1}{l_- - q_-}} \\ &= \left[\frac{(l_- - p) a_1 e^{a_2}}{(p - q_-) K_0^{l_- - q_-}} \right]^{\frac{1}{l_- - q_-}} \left[\frac{p - q_-}{(l_- - p) a_1 e^{a_2}} \right]^{\frac{1}{l_- - q_-}} \\ &= \frac{1}{K_0}, \end{aligned}$$

which proves (13). This completes the proof. □

3 Sub-supersolution for problem (1)

Throughout this section, we fix $\lambda \in (0, \Lambda]$, with Λ determined in Corollary 1. Using M_λ as defined in (10), we set

$$\overline{u}_\lambda := \left(\frac{M_\lambda}{\gamma} \right) \phi.$$

With a fixed real number ϵ_λ such that

$$0 < \epsilon_\lambda < \min \left\{ 1, \left(\frac{\lambda}{\lambda_1} \right)^{1/(p - q^+)}, \frac{M_\lambda}{\lambda_1^{\frac{1}{p-1}} \gamma} \right\}, \tag{14}$$

we introduce $\underline{u}_\lambda := \epsilon_\lambda u_1$.

The next statement represents the main result of this section.

Theorem 2. *Under hypotheses (F), $\underline{u}_\lambda$ is a subsolution, $\overline{u}_\lambda$ is a supersolution, and the ordered pair $(\underline{u}_\lambda, \overline{u}_\lambda)$ forms a sub-supersolution for problem (1) in the sense of Definition 1.*

Proof. On the basis of Corollary 1 and in conjunction with (4), (11) and hypotheses (F), where $K_0 > 1$, we have

$$\begin{array}{rcl}\lambda\overline{u}_\lambda(x)^{q(x)-1} + f(x, \overline{u}_\lambda(x)) & \leq & \lambda\overline{u}_\lambda(x)^{q(x)-1} + a_1\overline{u}_\lambda(x)^{l(x)-1}e^{\alpha(x)\overline{u}_\lambda(x)^{r(x)-1}} \\ & \leq & \lambda M_\lambda^{q(x)-1} + a_1 M_\lambda^{l(x)-1}e^{\alpha(x)M_\lambda^{r(x)-1}} \\ & \leq & \lambda M_\lambda^{q(x)-1} + a_1 M_\lambda^{l(x)-1}e^{\alpha(x)\left(\frac{1}{K_0}\right)^{r(x)-1}} \\ & \leq & \lambda M_\lambda^{q_- -1} + a_1 M_\lambda^{l_- -1}e^{a_2} \\ & \leq & \left(\frac{M_\lambda}{\gamma}\right)^{p-1} \\ & = & -\Delta_p\overline{u}_\lambda(x).\end{array}$$

This shows that $\overline{u}_\lambda$ is a supersolution of problem (1).

We note that hypotheses (F), $\varepsilon_\lambda < 1$ in (14) and $\|u_1\|_\infty = 1$ imply

$$\begin{array}{rcl}\lambda\underline{u}_\lambda(x)^{q(x)-1} + f(x, \underline{u}_\lambda(x)) & \geq & \lambda\underline{u}_\lambda(x)^{q(x)-1} \\ & \geq & \lambda_1\epsilon_\lambda^{p-q_+}\underline{u}_\lambda(x)^{q(x)-1} \\ & \geq & \lambda_1\epsilon_\lambda^{p-q(x)}\underline{u}_\lambda(x)^{q(x)-1} \\ & \geq & \lambda_1\underline{u}_\lambda(x)^{p-q(x)}\underline{u}_\lambda(x)^{q(x)-1} \\ & = & \lambda_1\underline{u}_\lambda(x)^{p-1} \\ & = & -\Delta_p\underline{u}_\lambda(x).\end{array}$$

Therefore $\underline{u}_\lambda$ is a subsolution of problem (1).

Using (3), the choice of ε_λ in (14), $u_1 > 0$ and $\|u_1\|_\infty = 1$, we get

$$-\Delta_p\underline{u}_\lambda(x) = \epsilon_\lambda^{p-1}\lambda_1 u_1^{p-1} \leq \epsilon_\lambda^{p-1}\lambda_1 \leq \left(\frac{M_\lambda}{\gamma}\right)^{p-1} = -\Delta_p\overline{u}_\lambda.$$

We conclude that the ordered pair $(\underline{u}_\lambda, \overline{u}_\lambda)$ is a sub-supersolution of problem (1), which completes the proof. □

4 Auxiliary truncated problem

In the present section we associate to problem (1) an auxiliary problem obtained by truncation along the ordered pair of sub-supersolution $\underline{u}_\lambda \leq \bar{u}_\lambda$

built in Theorem 2. The theory of pseudomonotone operators will enable us to guarantee the solvability of this new problem.

Fix $\lambda \in (0, \Lambda]$, with Λ determined in Corollary 1, and consider the sub-supersolution $(\underline{u}_\lambda, \overline{u}_\lambda)$ to problem (1) given by Theorem 2. Complying with the inequality $\underline{u}_\lambda \leq \bar{u}_\lambda$, it makes sense to look at the truncation operator $T = T_{\underline{u}_\lambda, \overline{u}_\lambda} : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ described in Section 2. We formulate the auxiliary truncated problem

$$\begin{cases} -\Delta_p u = \lambda (Tu)^{q(x)-1} + f(x, Tu) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

We are going to prove the existence of weak solutions to problem (15).

Theorem 3. *Assume that conditions* (F) *are fulfilled. For any* $\lambda \in (0, \Lambda]$, *there exists a weak solution* $u_\lambda \in W_0^{1,p}(\Omega)$ *to problem* (15), *that is,*

$$\int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \cdot \nabla \varphi dx - \int_\Omega (\lambda (Tu_\lambda)^{q(x)-1} + f(x, Tu_\lambda)) \varphi dx = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Proof. The definitions of T and the functions $\overline{u}_\lambda, \underline{u}_\lambda \in W_0^{1,p}(\Omega)$ entail that the notion of weak solution to problem (15) is well defined.

Let the map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be given by

$$Au = -\Delta_p u - \lambda (Tu)^{q(x)-1} - f(x, Tu), \quad \forall u \in W_0^{1,p}(\Omega). \tag{16}$$

We check that the map A is well defined. Indeed, the definitions of T and $\bar{u}_\lambda$, as well as by Corollary 1 and $K_0 > 1$, we have

$$(Tu)(x) \leq \bar{u}_\lambda(x) \leq M_\lambda \leq \frac{1}{K_0} < 1 \tag{17}$$

for almost all $x \in \Omega$, hence $(Tu)^{q(x)-1} \in L^\infty(\Omega)$. In addition, due to hypotheses (F) and the preceding estimate, we infer that

$$\begin{aligned} 0 \leq f(x, (Tu)(x)) &\leq a_1 \big((Tu)(x)\big)^{l(x)-1} e^{\alpha(x)\big((Tu)(x)\big)^{r(x)-1}} \\ &\leq a_1 e^{\alpha(x)\left(\frac{1}{K_0}\right)^{r(x)-1}} \\ &\leq a_1 e^{a_2}, \end{aligned} \tag{18}$$

so $(Tu)^{q(x)-1} + f(x, Tu) \in L^\infty(\Omega) \subset W^{-1,p'}(\Omega)$.

From (16) we observe that $u \in W_0^{1,p}(\Omega)$ is a weak solution to problem (15) if and only if u solves the equation

$$Au = 0. \tag{19}$$

Our goal is to show that the hypotheses of Lemma 1 are fulfilled for the operator A in (16).

The operator A is bounded. Indeed, by (16), (17), (18) and Hölder's inequality, we find that

$$\begin{aligned}|\langle Au,v\rangle| &= \left|\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx - \int_\Omega (\lambda (Tu)^{q(x)-1} + f(x,Tu))v dx\right| \\ &\le \|u\|^{p-1}\|v\| + C_1\|v\|_1\end{aligned}$$

for all $u, v \in W_0^{1,p}(\Omega)$, with a constant $C_1 > 0$. This amounts to saying that

$$\|Au\|_{W^{-1,p'}(\Omega)} \le \|u\|^{p-1} + C,, \quad \forall u \in W_0^{1,p}(\Omega),$$

for a constant $C > 0$, thereby the operator A is bounded.

We claim that A is a pseudomonotone operator. We need to prove for each sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ satisfying $u_n \rightharpoonup u$ and

$$\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \le 0 \tag{20}$$

that

$$\liminf_{n\to+\infty}\langle Au_n, u_n - v\rangle \ge \langle Au, u - v\rangle, \quad \forall v \in W_0^{1,p}(\Omega).$$

By the compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, the weak convergence $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ implies the strong convergence $u_n \to u$ in $L^p(\Omega)$. Thanks to the uniform boundedness of the sequence $\{Tu_n\}$, it turns out

$$\lim_{n\to+\infty}\int_\Omega (\lambda (Tu_n)^{q(x)-1} + f(x,Tu_n))(u_n - u)dx = 0.$$

Combining with (20) renders

$$\begin{aligned}&\limsup_{n\to+\infty}\langle -\Delta_p u_n, u_n - u\rangle \\ &= \limsup_{n\to+\infty}\left(\langle Au_n, u_n - u\rangle + \int_\Omega (\lambda (Tu_n)^{q(x)-1} + f(x,Tu_n))(u_n - u)dx\right) \\ &= \limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \le 0.\end{aligned}$$

This enables us to invoke the (S_+)-property of $-\Delta_p$ (see [3, Lemma 2.111]) to get the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. The continuity of the operator A entails

$$\lim_{n\to+\infty}\langle Au_n, u_n - v\rangle = \langle Au, u - v\rangle, \quad \forall v \in W_0^{1,p}(\Omega),$$

whence the pseudomonotonicity of A.

We now establish that A is a coercive operator, which means that

$$\lim_{\|u\|\to+\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty. \tag{21}$$

By (16) and (18), we arrive at

$$\begin{aligned} \langle Au, u\rangle &= \int_\Omega |\nabla u|^p dx - \int_\Omega (\lambda(Tu)^{q(x)-1} + f(x, Tu))u dx \\ &\geq \|u\|^p - C\|u\|, \quad \forall\, u \in W_0^{1,p}(\Omega), \end{aligned}$$

with a constant $C > 0$. Taking into account that $p > 1$, the above estimate proves (21).

We have shown that all the requirements to apply Lemma 1 for the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ in (16) are fulfilled. Consequently, there exists $u_\lambda \in W_0^{1,p}(\Omega)$ solving equation (19), that is, u_λ is a weak solution of problem (15). The proof is thus complete. □

5 Proof of Theorem 1

Here we are concerned with the location of the solution $u_\lambda \in W_0^{1,p}(\Omega)$ to auxiliary problem (15) given in Theorem 3.

Theorem 4. *Assume that conditions* (F) *are fulfilled and let* $\lambda \in (0, \Lambda]$. *The weak solution* $u_\lambda \in W_0^{1,p}(\Omega)$ *to problem* (15) *obtained in Theorem 3 satisfies*

$$\underline{u}_\lambda \leq u_\lambda \leq \overline{u}_\lambda \ \text{a.e. in}\ \Omega, \tag{22}$$

where $\underline{u}_\lambda$ *and* $\overline{u}_\lambda$ *constitute the sub-supersolution constructed in Theorem 2.*

Proof. We will only prove

$$\underline{u}_\lambda \leq u_\lambda \text{ a.e. in } \Omega \tag{23}$$

because the proof of the inequality $u_\lambda \leq \overline{u}_\lambda$ proceeds analogously.

The proof of (23) is carried out through comparison arguments. Since $u_\lambda, \underline{u}_\lambda \in W_0^{1,p}(\Omega)$ we have that

$$(\underline{u}_\lambda - u_\lambda)^+ := \max\{\underline{u}_\lambda - u_\lambda, 0\} \in W_0^{1,p}(\Omega),$$

thus $(\underline{u}_\lambda - u_\lambda)^+$ can be employed as a test function in (15).

From Theorem 3 we infer that

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla(\underline{u}_\lambda - u_\lambda)^+ dx$$
$$-\int_\Omega (\lambda (Tu_\lambda)^{q(x)-1} + f(x, Tu_\lambda))(\underline{u}_\lambda - u_\lambda)^+ dx = 0.$$

According to Theorem 2 we know that $\underline{u}_\lambda$ is a subsolution to problem (1), which provides

$$\int_\Omega |\nabla \underline{u}_\lambda|^{p-2}\nabla \underline{u}_\lambda \nabla(\underline{u}_\lambda - u_\lambda)^+ dx - \int_\Omega (\lambda \underline{u}_\lambda^{q(x)-1} + f(x, \underline{u}_\lambda))(\underline{u}_\lambda - u_\lambda)^+ dx \le 0.$$

By subtraction we derive

$$\begin{aligned}&\int_\Omega (|\nabla \underline{u}_\lambda|^{p-2}\nabla \underline{u}_\lambda - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\nabla(\underline{u}_\lambda - u_\lambda)^+ dx\\ &-\int_\Omega (\lambda \underline{u}_\lambda^{q(x)-1} + f(x, \underline{u}_\lambda) - (\lambda (Tu_\lambda)^{q(x)-1} + f(x, Tu_\lambda)))(\underline{u}_\lambda - u_\lambda)^+ dx\\ &\le 0. \end{aligned} \tag{24}$$

The definition of the operator $T = T_{\underline{u}_\lambda, \overline{u}_\lambda}$ ensures that

$$\begin{aligned}&\int_\Omega (\lambda \underline{u}_\lambda^{q(x)-1} + f(x, \underline{u}_\lambda) - (\lambda (Tu_\lambda)^{q(x)-1} + f(x, Tu_\lambda)))(\underline{u}_\lambda - u_\lambda)^+ dx\\ &= \int_{\{\underline{u}_\lambda > u\}} (\lambda \underline{u}_\lambda^{q(x)-1} + f(x, \underline{u}_\lambda) - (\lambda (Tu_\lambda)^{q(x)-1} + f(x, Tu_\lambda)))(\underline{u}_\lambda - u_\lambda) dx\\ &= \int_{\{\underline{u}_\lambda > u\}} (\lambda \underline{u}_\lambda^{q(x)-1} + f(x, \underline{u}_\lambda) - (\lambda \underline{u}_\lambda^{q(x)-1} + f(x, \underline{u}_\lambda)))(\underline{u}_\lambda - u_\lambda) dx = 0.\end{aligned}$$

Returning to (24) produces

$$\begin{aligned}&\int_{\{\underline{u}_\lambda > u\}} (|\nabla \underline{u}_\lambda|^{p-2}\nabla \underline{u}_\lambda - |\nabla u|^{p-2}\nabla u_\lambda)\nabla(\underline{u}_\lambda - u_\lambda) dx\\ &= \int_\Omega (|\nabla \underline{u}_\lambda|^{p-2}\nabla \underline{u}_\lambda - |\nabla u_\lambda|^{p-2}\nabla u_\lambda)\nabla(\underline{u}_\lambda - u_\lambda)^+ dx \le 0.\end{aligned}$$

If $p \ge 2$, by a well known inequality (see, e.g., [11]), there exists a positive constant c_p such that

$$\begin{aligned}&c_p\|(\underline{u}_\lambda - u_\lambda)^+\|^p = c_p \int_{\{\underline{u}_\lambda > u_\lambda\}} |\nabla(\underline{u}_\lambda - u_\lambda)|^p dx\\ &\le \int_{\{\underline{u}_\lambda > u_\lambda\}} (|\nabla \underline{u}_\lambda|^{p-2}\nabla \underline{u}_\lambda - |\nabla u|^{p-2}\nabla u_\lambda)\nabla(\underline{u}_\lambda - u_\lambda) dx.\end{aligned}$$

It follows that $(\underline{u}_\lambda - u_\lambda)^+ = 0$ almost everywhere on Ω, so (23) holds true.

If $1 < p < 2$, there exists a positive constant c_p such that

$$
\begin{aligned}
& c_p \int_{\{\underline{u}_\lambda > u_\lambda\}} \frac{|\nabla(\underline{u}_\lambda - u_\lambda)|^2}{(|\nabla \underline{u}_\lambda| + |\nabla u_\lambda|)^{2-p}} dx \\
& \leq \int_{\{\underline{u}_\lambda > u_\lambda\}} (|\nabla \underline{u}_\lambda|^{p-2} \nabla \underline{u}_\lambda - |\nabla u_\lambda|^{p-2} \nabla u_\lambda) \nabla(\underline{u}_\lambda - u_\lambda) dx
\end{aligned}
$$

(see, e.g., [11]). Therefore the Lebesgue measure $|\{\underline{u}_\lambda > u_\lambda\}|$ of the measurable set $\{\underline{u}_\lambda > u_\lambda\}$ vanishes, thus the validity of (23) is established in the case $1 < p < 2$, too. The proof is complete. $\square$

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Theorem 4 guarantees that the location of the weak solution $u_\lambda \in W_0^{1,p}(\Omega)$ to problem (15) within inequality $\underline{u}_\lambda \leq u_\lambda \leq \overline{u}_\lambda$ a.e. on Ω in (22) holds true. By the definition of the truncation operator $T = T_{\underline{u}_\lambda, \overline{u}_\lambda} : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ we have that $u = u_\lambda$ satisfies the equality $Tu_\lambda = u_\lambda$. Consequently, from (15) it is apparent that u_λ is actually a weak solution to problem (1). Taking into account that the functions $\underline{u}_\lambda$ and $\overline{u}_\lambda$ are positive and uniformly bounded, from (22) it follows that the weak solution u_λ to problem (1) is positive and bounded, which completes the proof.

6 Application to regularity of solutions and asymptotic behavior

In this section, the bounded domain Ω is supposed to be of class $C^{1,\alpha}$ for some $0 < \alpha < 1$. The enclosure $u_\lambda \in [\underline{u}_\lambda, \overline{u}_\lambda] := \{u \in W_0^{1,p}(\Omega) : \underline{u}_\lambda \leq u \leq \overline{u}_\lambda \text{ a.e. in } \Omega\}$ established in Theorem 4 provides a priori estimates that permits to investigate the regularity of the solution u_λ to problem (1) given by Theorem 1.

Theorem 5. *Assume that hypotheses (F) are satisfied. Then for every $\lambda \in (0, \Lambda]$, the weak solution u_λ given in Theorem 1 belongs to $C^{1,\beta}(\overline{\Omega})$, with some $0 < \beta = \beta(\alpha, p, N) < 1$ independent of λ. Moreover, there is a constant $C(\alpha, p, N, \Omega) > 0$ for which it holds the estimate*

$$
\|u_\lambda\|_{C^{1,\beta}(\overline{\Omega})} \leq C(\alpha, p, N, \Omega) \tag{25}
$$

independent of λ.

Proof. Let $\lambda \in (0, \Lambda]$ be fixed. By (22) and Corollary 1, we get

$$0 \leq u_\lambda \leq \overline{u}_\lambda = \left(\frac{M_\lambda}{\gamma}\right)\phi \leq M_\lambda \leq \frac{1}{K_0} < 1. \tag{26}$$

We emphasize that Lieberman's boundary regularity theorem for Dirichlet problems in [10] cannot be directly applied to problem (1) mainly because of the general growth condition in hypotheses (F). In order to overcome this difficulty we introduce the function $B_\lambda : \overline{\Omega} \to \mathbb{R}$ as

$$B_\lambda(x) = \begin{cases} \lambda u_\lambda(x)^{q(x)-1} + f(x, u_\lambda(x)) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial\Omega. \end{cases}$$

Using (26) and assumptions (F), we see that $B_\lambda \in L^\infty(\Omega)$. Accordingly, we formulate the Dirichlet problem

$$\begin{cases} -\Delta_p u = B_\lambda(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

The unique solution of (27) is $u = u_\lambda$. Lieberman's boundary regularity result (see [10, Theorem 1]) applies to equation (27) with $A(x, z, \tau) = |\tau|^{p-2}\tau$, $B(x, z, \tau) = B_\lambda(x)$, and according to (26) taking $M_0 = 1$. We obtain that $u_\lambda \in C^{1,\beta}(\overline{\Omega})$ with some $\beta = \beta(\alpha, p, N) \in (0, 1)$ independent of $\lambda \in (0, \Lambda]$. Moreover, we get estimate (25) that holds uniformly with respect to $\lambda \in (0, \Lambda]$. This completes the proof. $\square$

Next, on the basis of Theorem 5, we point out the asymptotic behavior of the solutions u_λ when the parameter λ approaches 0.

Corollary 2. *The solutions u_λ to problem (1) for $\lambda \in (0, \Lambda]$ as given in Theorem 1 exhibit the following asymptotic property*

$$\|u_\lambda\|_{C^1(\overline{\Omega})} \to 0 \quad \textit{as} \quad \lambda \to 0.$$

Proof. Estimate (25) ensures that the set $\{u_\lambda : \lambda \in (0, \Lambda]\}$ is bounded in $C^{1,\beta}(\overline{\Omega})$. The compact inclusion of $C^{1,\beta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ entails along any relabeled subsequence that $u_\lambda \to v$ in $C^1(\overline{\Omega})$ as $\lambda \to 0$, with a function $v \in C^1(\overline{\Omega})$. By (10) we have that $M_\lambda \to 0$ as $\lambda \to 0$. Then (26) provides $u_\lambda \to 0$ in $C(\overline{\Omega})$ as $\lambda \to 0$. A simple comparison confirms that $v = 0$. This concludes the proof. $\square$

7 Examples

In order to simplify the presentation we fix a continuous function $q(x)$ on Ω satisfying $1 < q_- \le q(x) \le q_+ < p$ (refer to assumptions (F)),

Example 1. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda u^{q(x)-1} + a_1 u^{l(x)-1} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{28}$$

with a constant $a_1 > 0$ and a continuous function $l(x)$ on Ω that satisfies $p < l_- \le l(x)$. Note that assumptions (F) are verified with any $\alpha(x) > 0$ and $r(x) > 1$, so Theorem 1 applies to problem (28). The statement in (28) covers the problem involving concave-convex nonlinearities treated by Ambrosetti-Brezis-Cerami [2]. Here variable exponents and supercritical growth are allowed.

Example 2. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda u^{q(x)-1} + a_1 u^p e^{au} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{29}$$

with constants $a_1 > 0$ and $a > 0$. Assumptions (F) are fulfilled by taking $l(x) \equiv p+1$, $\alpha(x) \equiv a$, $r(x) \equiv 2$, $a_2 = a$, and any $K_0 > 1$. Theorem 1 applies to problem (29) providing a positive bounded weak solution. Through problem (29) we see that Theorem 1 enables us to handle the existence of nontrivial solutions to problems with nonlinearities of exponential growth, a topic that is not covered by variational methods.

Example 3. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda u(x)^{q(x)-1} + a_1 u(x)^{p+\mathrm{dist}(x,\partial\Omega)} e^{\frac{1}{\mathrm{dist}(x,\partial\Omega)} u(x)^{\ln\left(\frac{D(\Omega)}{\mathrm{dist}(x,\partial\Omega)}\right)}} & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{30}$$

with a constant $a_1 > 0$ and a domain Ω of finite diameter $D(\Omega)$. The notation $\mathrm{dist}(x, \partial\Omega)$ stands for the distance from $x \in \Omega$ to the boundary $\partial\Omega$. Assumptions (F) are verified with $l(x) = p + \mathrm{dist}(x, \partial\Omega) + 1$, $\alpha(x) = \frac{1}{\mathrm{dist}(x,\partial\Omega)}$, $r(x) = \ln\left(\frac{D(\Omega)}{\mathrm{dist}(x,\partial\Omega)}\right) + 1$, $a_2 = D(\Omega)^{-1}$, and $K_0 = e$. Hence the function $\alpha(x)$ in hypotheses (F) needs not be bounded. Theorem 1 applies to problem (30). This example shows that Theorem 1 can resolve boundary value problems where the geometry of the domain Ω is incorporated.

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