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ALMOST GLOBAL IMPLICIT PARAMETERIZATIONS OF SURFACES AND HYPERSURFACES*

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This paper is devoted to the memory of the renowned French mathematician H. Brezis, who passed away on July 7-th, 2024. He had some special connections with Romania and his work is a source of inspiration. He was a honorary member of the Romanian Academy.

Abstract

The level set method has outstanding applications in free boundary problems or in optimal design (geometric optimization). In the computations, a crucial question is to obtain parameterizations of the involved implicitly defined curves, surfaces. The general solution has a local character. This note is devoted to the construction of almost global parameterizations, that we introduce here. First, a general geometric description is indicated, of the region generated by our parameterization technique, on the implicitly defined surfaces and hypersurfaces. Then, we formulate necessary and/or sufficient conditions for the global character of the defined parameterizations, valid in many cases of interest. The essential points in our methodology are Hamiltonian systems and the Poincaré - Bendixson theory and we also underline the constructive characteristics of our approach. To obtain global or almost global parametrizations has strong consequences to global optimization algorithms in nonlinear programming and to extending Hamiltonian techniques in shape/topology optimization, beyond dimension two (including in the computational applications).

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1 Introduction

A general and constructive local parameterization approach, based on iterated Hamiltonian systems, was developed in [25], for implicitly defined manifolds in arbitrary dimension and codimension. If $F_j \in C^1(D)$, $D \subset \mathbb{R}^d$ bounded domain, $j = \overline{1,l}$, $l \leq d-1$, are such that $F_j(x^0) = 0, j = \overline{1,l}$, for some given $x^0 \in D$ and the independence condition:

$$\frac{D(F_1, F_2, \ldots, F_l)}{D(x_1, x_2, \ldots, x_l)}(x^0) \neq 0 \tag{1}$$

is satisfied, then the (nonlinear) algebraic system:

$$F_j(x_1, x_2, \ldots, x_d) = 0, \quad j = \overline{1,l}, \tag{2}$$

defines in a neighborhood $V \subset D$ of x^0 (where condition (1) remains valid) a $(d-l)$-dimensional manifold in $\mathbb{R}^d$. A local parameterization, around x^0, of the manifold defined in (2), is obtained via the iterated Hamiltonian system ($d-l$ subsystems of dimension d):

$$\frac{\partial y_1(t_1)}{\partial t_1} = v_1(y_1(t_1)), \quad t_1 \in I_1 \subset \mathbb{R}, \tag{3}$$

$$y_1(0) = x^0;$$

$$\frac{\partial y_2(t_1,t_2)}{\partial t_2} = v_2(y_2(t_1,t_2)), \quad t_2 \in I_2(t_1) \subset \mathbb{R}, \tag{4}$$

$$y_2(t_1, 0) = y_1(t_1);$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\partial y_{d-l}(t_1,t_2,\ldots,t_{d-l})}{\partial t_{d-l}} = v_{d-l}(y_{d-l}(t_1,t_2,\ldots,t_{d-l})), \tag{5}$$

$$t_{d-l} \in I_{d-l}(t_1,\ldots,t_{d-l-1}) \subset \mathbb{R},$$

$$y_{d-l}(t_1,\ldots,t_{d-l-1},0) = y_{d-l-1}(t_1,t_2,\ldots,t_{d-l-1}).$$

In (3) - (5), we have considered the undetermined linear algebraic system with unknowns $v(x) \in \mathbb{R}^d,\ x \in V$:

$$v(x) \cdot \nabla F_j(x) = 0, \quad j = \overline{1, l}, \tag{6}$$

and $v_j(x) \in \mathbb{R}^d,\ j = \overline{1, d-l}$ denote linearly independent solutions of (6), continuous with respect to $x \in V$. The choice of such sets of independent solutions is infinite. The notations $I_1, I_2(t_1), \ldots, I_{d-l}(t_1, \ldots, t_{d-l-1})$ represent open intervals in $\mathbb{R}$ containing the origin and given by the Peano existence theorem applied to (3) - (5). They are, in fact, ordinary differential systems with parameters introduced via their initial condition, although PDE's notations are used (since just one derivative appears in each subsystem).

In the paper [25], we used special sets of such independent solutions of (6), that helped to obtain a complete constructive local parameterization theory, containing the implicit functions theorem as a particular case. Moreover, the generalization to the singular case (when (1) is not valid) is also included. In the sequel, another (simpler and "bidimensional") set of independent solutions of (6) will play the essential role to derive an almost global theoretical extension of [25], in the case of surfaces and hypersurfaces.

Other properties, related to the above approach, are: the existence intervals $I_1, I_2(t_1), \ldots, I_{d-l}(t_1, \ldots, t_{d-l-1})$ may be selected independently of the parameters, the uniqueness is also valid (in each subsystem), new derivation formulas are proved, see [25] and [26] (where applications in nonlinear programming, both theoretical and computational, are indicated as well).

In dimension two ($d = 2$), the system (3) - (5) reduces to the simplest Hamiltonian system. Under appropriate assumptions and using the Poincaré - Bendixson theory of limit cycles, [6], [21], we have established in [24], [10] that its solution is periodic and the period is differentiable with respect to the so-called *functional variations* in the system (2), see [16], [15]. The derivative can be computed via a certain adjoint system. This plays a key role in the extension of the level set method (introduced in [19], in the frame of evolutionary free boundary problems) to shape/topology optimization problems formulated in an elliptic setting, see [11], [12]. For a general presentation of such geometric optimization problems, see the monographs [14], [18] and their references.

Notice that the periodicity property for Hamiltonian systems in $\mathbb{R}^2$ ensures the global parameterization of the boundaries of the involved two dimensional domains defined as sublevel sets $F(x_1, x_2) < 0$, for instance. Related periodicity results in dimension two are already known in differential geometry, via a different approach [22], Ch.10, Ch.11. The literature on global parameterizations, especially from the algorithmic point of view, is very rich. See [20], [9] and their references. Our constructive extension to higher dimension of global parameterizations seems to be new and the necessary and/or sufficient conditions, that we discuss, put into evidence the difficult points of the subject. The examples that we have in mind are bounded connected hypersurfaces that are connected components of domain boundaries. In [23], [27], the case of closed curves in arbitrary dimension is investigated via different arguments.

In the preliminaries from Section 2, we consider just the case of planar curves ($d = 2,\ l = 1$), that plays an important role in the sequel. The last section reports our main results on regional and almost global parameterizations of implicitly defined surfaces and hypersurfaces. Our methodology here is relevant in dimension two or higher (for the manifolds), since we investigate the general system (3) - (5) with at least two subsystems. We also mention the possible applications to nonlinear programming (global optimization algorithms, see [26] for an example) or to the extension of the Hamiltonian approach to shape/topology optimal design problems beyond dimension two or to problems involving manifolds of codimension higher than one and we quote the paper [11] for the basics in this respect.

2 Preliminaries in dimension two

In $\mathbb{R}^2$, we have $d = 2,\ \ l = 1$ and just one equation in (2), that implicitely defines a curve in the plane x_1x_2. The hypothesis (1) becomes $\nabla F(x^0) \neq 0$, where $x^0 = (x_1^0, x_2^0)$ is a given point in the bounded domain $D \subset \mathbb{R}^2$ and $F \in C^1(D)$. This situation is thoroughly studied in [22], Ch.10, Ch.11. In [24], [10], we prove similar periodicity results, based on the Poincaré - Bendixson theory and, in the next Section, we partially extend them to dimension three and higher. Notice that the vector of partial derivatives $v(x_1, x_2) = (-\partial_2 F(x_1, x_2), \partial_1 F(x_1, x_2)) \neq 0$ is a solution of (6) in a neighborhood $x^0 \in V \subset D$, due to the definition of V and (1). The system (3) - (5) becomes the simplest Hamiltonian system in dimension two:

$$y_1'(t) = -\partial_2 F(y_1(t), y_2(t)), t \in I, \tag{7}$$

$$y_2'(t) = \partial_1 F(y_1(t), y_2(t)), t \in I, \tag{8}$$

$$y_1(0) = x_1^0,\ \ y_2(0) = x_2^0. \tag{9}$$

We know that (7) - (9) has a unique local solution in some maximal open interval $t \in I$ with 0 in its interior. And it gives a local parameterization around x^0 of the plane curve $F(x_1, x_2) = 0$, as in [25]. We introduce

$$G = \{(x_1, x_2) \in D; F(x_1, x_2) = 0\} \tag{10}$$

which is a compact subset of D, not necessarily connected. On G, we assume now

$$\nabla F(x_1, x_2) \neq 0, \tag{11}$$

which also ensures the linear independence condition required in (6). Moreover, G cannot contain isolated points, since they would be extremum points for F and (11) would be contradicted. Due to the well known conservation property of Hamiltonian systems, we get $0 = F(x_1^0, x_2^0) = F(y_1(t), y_2(t)),\ t \in I$, i.e. $(y_1(t), y_2(t)) \in G,\ t \in I$ and (11) gives

$$\nabla F(y_1(t), y_2(t)) \neq 0, \ \ t \in I. \tag{12}$$

Relation (12), together with $F \in C^2(\overline{D})$, is exactly the main Poincaré - Bendixson assumption for (7) - (9): there is no equilibrium point on the trajectory. We also assume a constant sign condition on ∂D:

$$|F(x_1, x_2)| > 0. \tag{13}$$

Then, again by the conservation property of the Hamiltonian, the solution $(y_1(t), y_2(t))$ has to remain in D and is bounded (another assumption of Poincaré - Bendixson type). The general structure theorem for ODE's, [1], [21] gives global existence, i.e. $I = (-\infty, \infty)$, under (12), (13). Moreover, one can argue as in [24], [11] and show that the limit cycles are not possible for Hamiltonian systems like (7) - (9). Then, the solution of (7) - (9) has to be periodic and gives a global parameterization of the closed curve in $G \subset D$, passing through x^0. In fact, we have:

Proposition 1. *The compact set $G \subset D$ is a finite union of closed curves, parameterized by (7) - (9), if some initial condition is fixed on each connected component of G.*

The complete proof can be found in [24]. The hypothesis (11) is, in fact, necessary and sufficient for the conclusion of Prop.1, if (13) is valid. Such analytic representations of domains bordered by closed curves from G (and not necessarily simply connected) are a key step in our approach in shape/topology optimization, in dimension two, and in the equivalence of such geometric optimization problems with certain optimal control problems with mixed constraints, [11]. Another essential property is the differentiability of the period corresponding to (7) - (9) with respect to functional variations of the Hamiltonian F and of the geometry given by its sublevel domains, [11]. The period of the solution $(y_1(t), y_2(t))$ is, in principle, different on each connected component of G.

3 Dimension three and higher

We discuss first the case $d = 3,\ l = 1$, which is related to surfaces in $D \subset \mathbb{R}^3$, bounded domain. The equation (2) becomes

$$F(x_1, x_2, x_3) = 0, \ \ F \in C^2(\overline{D}), \ \ F(x^0) = 0, \tag{14}$$

and is associated to a surface $S \subset D$, defined around a non critical point $x^0 \in D, \ \ \nabla F(x^0) \neq 0$. The case $d = 3,\ l = 2$ concerns general curves in D and is studied in [23], [27], where conditions to obtain closed curves are indicated (in higher dimension too).

The system (3) - (5) has now just two subsystems of dimension three, that may be chosen of the form (such a choice is not unique):

$$\begin{aligned}
x_1' &= -\partial_2 F(x_1, x_2, x_3),\ t \in I_1,\\
x_2' &= \partial_1 F(x_1, x_2, x_3),\ t \in I_1, \qquad (15)\\
x_3' &= 0,\ t \in I_1,\\
x^0 &= (x_1(0), x_2(0), x_3(0)). \qquad (16)
\end{aligned}$$

Below, the point denotes derivative with respect to $s \in I_2$:

$$\begin{aligned}
\dot\varphi &= -\partial_3 F(\varphi, \psi, \xi),\ s \in I_2,\\
\dot\psi &= 0,\ s \in I_2, \qquad (17)\\
\dot\xi &= \partial_1 F(\varphi, \psi, \xi),\ s \in I_2,\\
\varphi(0) &= x_1(t),\ \psi(0) = x_2(t),\ \xi(0) = x_3(t), \qquad (18)
\end{aligned}$$

The system (15)-(18) gives a local parameterization around x^0 of the implicitly defined manifold $F(x_1, x_2, x_3) = 0$. The Hamiltonian structure of each of its two subsystems is obvious, in the planes $x_3 = x_3^0$, respectively $\psi = x_2(t)$. We assume that the vectors $(-\partial_2 F(x_1, x_2, x_3),\ \partial_1 F(x_1, x_2, x_3))$, $(-\partial_3 F(x_1, x_2, x_3),\ \partial_1 F(x_1, x_2, x_3))$ are linearly independent in a neighborhood $x_0 \in V \subset D$. They give solutions to (6), as required in [25]. From the previous section, this condition yields the main Poincaré - Bendixson assumption: each subsystem of (15)-(18) has no equilibrium points on its trajectory. We also impose

$$|F(x_1, x_2, x_3)| > 0,\ (x_1, x_2, x_3) \in \partial D, \tag{19}$$

that ensures the second Poincaré - Bendixson hypothesis (the trajectories are bounded), due to the conservation property of Hamiltonian systems:

$$F(x_1(t), x_2(t), x_3(t)) = F(\varphi(t,s), \psi(t,s), \xi(t,s)) = F(x^0) = 0. \tag{20}$$

By [25], the system (15)-(18) gives a local parameterization around x^0 of $S \subset D$. The Poincaré - Bendixson hypotheses ensure (as in the previous section) that the trajectories associated to each subsystem of (15)-(18) are planar and periodic. However, this periodicity does not yield that the obtained parameterization of S is of global type, as the examples related to the torus, from [17], show. We also underline, that in the monograph of Thorpe [22], in dimension higher than two, just local theory is investigated.

The following result is a strengthening of the local parameterization property, giving an accurate geometric description of the achieved parameterization region, in dimension three:

Proposition 2. *Assume that $F \in C^2(\overline{D})$ satisfies the above Poincaré - Bendixson conditions. Then, the iterated Hamiltonian system (15)-(18) gives a parameterization of the region of S obtained as the union of all the nonvoid intersections S_c, of the planes $x_2 = c,\ c \in \mathbb{R}$, with S, that touch the trajectory generated by (15)-(16).*

Proof. This is a direct consequence of the way we have constructed the parameterization and the above discussion. Notice that although S is a connected component of the solution of (14) (containing x^0), its planar "slices" S_c may be not connected in the planes $x_2 = c,\ c \in \mathbb{R}$. But the number of the connected components is finite in the plane $x_2 = c$, due to Proposition 1 and the Poincaré - Bendixson assumptions for (17)-(18). In the components that touch the closed trajectory generated by (15)-(16), the intersection points provide initial conditions for the subsystem (17)-(18) and the respective components are parameterized by (15)-(18). The other components of S_c, from the plane $x_2 = c$, remain outside the parameterization range. □

We see that the parameterized region is a rich one and Proposition 2 improves local parameterization results, but the global character is not ensured.

Remark 1. *Under the Poincaré-Bendixson assumptions, a necessary and sufficient condition towards global parameterization is that the trajectory generated by (15)-(16) touches all the connected components of S_c. Here, we also use that $S = \cup S_c,\ c \in \mathbb{R}$.*

Remark 2. *If some plane $x_2 = c$ is tangent with S in certain points (which indeed happens), then they are extremum or saddle points for $F(x_1, c, x_3)$ and the Poincaré - Bendixson assumption for (17)-(18) is not true in such tangency points since $\nabla_{x_1 x_3} F(x_1, c, x_3) = 0$. Obviously, these tangency points depend on the choice of the axes and the parametrization property may be not valid in such points. See Example 1 and Remark 3 below, showing the difficulties arising from this situation.*

We notice that the Poincaré-Bendixson main hypothesis is, in fact, unrealistic for iterated Hamiltonian systems in $\mathbb{R}^3$. Fortunately, the mentioned exceptional situations have limited consequences in computations.

Example 1. *As in [17], in the torus example, we take $F(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2 + R^2 - r^2)^2 - 4R^2(x_1^2 + x_2^2) = 0,\ R = 2, r = 1,\ x^0$ is on the exterior maximal circle, parameterized by (15)-(16). There are two points of tangency of the exterior maximal circle of the torus, with vertical planes parallel to $x_1 x_3$. They are extremum points of $F(x_1, c, x_3)$. With these two starting points, the trajectory of (17)-(18) remains in its initial condition, which coincides with the corresponding "slice" through it. The parameterization in exterior tangency points is indeed given by (15)-(16), starting in x^0. There exist as well two tangency points to the interior minimal circle of the torus and they are saddle points for $F(x_1, c, x_3)$. The corresponding two planar "slices" S_c resemble each to a lemniscate. See Fig. 1. The intersection of the maximal exterior circle of the torus with each lemniscate are exactly two of the initial conditions of the subsystem (17)-(18). The associated two trajectories via (17)-(18) are global and the interior tangency point is an equilibrium point for both (and never attained). That is, the parameterization is almost global in this example: just the two interior tangency points are not parameterized, in fact.*

Remark 3. *Notice as well that, in the above example, the parts of the torus around the two extreme points of $F(x_1, c, x_3)$ are doubly parametrized. Namely, starting from initial conditions on the maximal exterior circle, before the extreme point and after it, we get via (17)-(18) the same trajectories two times. At the discrete level, this difficulty can be solved in various manners. For instance, by choosing appropriately the discretization points (that*

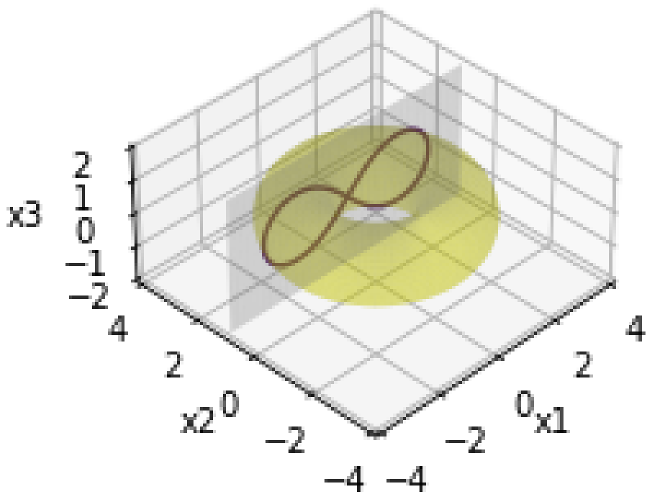

Figure 1: Intersection of the torus with an interior "tangent" plane, giving a lemniscate.

give initial conditions for the second step) on the trajectory obtained via (15)-(16).

The choice of x^0 may play a fundamental role in obtaining almost global parameterizations via (15)-(18). *Farthest* points A (on a compact surface S), are defined to be at maximal distance from some other point $A' \in S$ with respect to all the other pairs of points in S (see [7], [8] and their references, for a discussion of this and related notions). Obviously, A' is also a farthest point of S. For instance, all the points on the exterior maximal circle of the torus are farthest points, while the ellipsoid has a unique pair of farthest points.

We indicate, as well, some geometric conditions, in order to obtain an almost global parameterization: The initial condition x^0 has to be a farthest point A of S and the line $\overline{AA'}$ should be chosen the axis Ox_2 (according to the above notations, in this section). The choice of the plane x_1x_2 should define a curve in S, (by cutting S), that connects A with A' (if possible). Then, the planes $x_2 = c$, through the points of this curve AA', intersect S in any of its points. The solution of (15)-(16) should touch all the connected components of S_c, see Remark 1.

We investigate now the case of hypersurfaces $\tilde{S}$ in arbitrary dimension $\mathbb{R}^d$, that is $d \in \mathbb{N}, l = 1$:

$$\tilde{F}(x_1, x_2, \ldots, x_d) = 0, \quad \tilde{F} \in C^2(\overline{D}), \quad F(\tilde{x}^0) = 0, \tag{21}$$

for some given $\tilde{x}^0 \in D \subset \mathbb{R}^d$, bounded domain, and under the hypothesis

$$|\tilde{F}(x_1, x_2, \ldots, x_d)| > 0, \quad (x_1, x_2, \ldots, x_d) \in \partial D. \tag{22}$$

We choose now the solution of the linear algebraic system (6), with $l = j = 1$, of the following form (not unique of this type). Its advantages are the simplicity of writing and the reduction to dimension two:

$$\begin{aligned} v_1 &= (-\partial_2 \tilde{F}, \partial_1 \tilde{F},\ 0, \ldots, 0), \\ v_2 &= (-\partial_3 \tilde{F}, 0, \partial_1 \tilde{F},\ 0, \ldots, 0), \\ &\ldots \\ v_{d-1} &= (-\partial_d \tilde{F}, 0, \ldots, 0, \partial_1 \tilde{F}). \end{aligned} \tag{23}$$

We assume that the vectors in (23) are linearly independent and form a basis in the tangent space, in any point of $\tilde{S}$. In particular, it yields $\nabla \tilde{F} \neq 0$ in any point of $\tilde{S}$, which is the fundamental condition in the implicit definition of hypersurfaces, [22]. The d-dimensional subsystems of (3) - (5) are in fact the simplest Hamiltonian systems in dimension two since all the other components of the vectors $v_j,\ j = \overline{1, d-1}$ are null. Moreover, the independence condition yields the main Poincaré - Bendixson assumption: the subsystems of (3) - (5) have no equilibrium points on their trajectories. Together with (22), we obtain again that all these trajectories are planar and periodic (closed). An example in arbitrary dimension, of a simple sufficient condition (not realistic) for the vectors (23) to be independent, is $\partial_1 \tilde{F}(x_1, x_2, \ldots, x_d) \neq 0, \quad (x_1, x_2, \ldots, x_d) \in \tilde{S}$, and it has to be combined with (22). We obtain, similarly with Proposition 2:

Proposition 3. *Assume (22) and that the vectors (23) are linearly independent on $\tilde{S}$. Then, the system (3) - (5), associated to them, has periodic solutions in each variable $t_1, t_2, \ldots, t_{d-1}$ and gives a parameterization of the region of $\tilde{S}$ defined iteratively (in $d-1$ iterations): in step 1 by (3)* *and in steps* $2, \ldots, d-1$ *as the union of the nonvoid intersections* $\tilde{S}_{j,\tilde{c}_{j+2}}$ *of* $\tilde{S}$ *with the planes* $x_1 x_{j+2}$ *defined by fixing* $(x_2, \ldots, \hat{x}_{j+2}, \ldots, x_{d-1}) = (c_2, \ldots, \hat{c}_{j+2}, \ldots, c_{d-1}) = \tilde{c}_{j+2}$, $j = \overline{1, d-3}$, $d \geq 4$ *(where the notations* $\hat{x}_{j+2}$, $\hat{c}_{j+2}$ *mean that these variables are missing) and that touch a trajectory from the previous step.*

The proof is again based on the construction of the parametrization and consists of a finite number of iterations. In step 1, we get just a planar curve on $\tilde{S}$ and in each subsequent step, the region obtained in the previous step is completed by considering the intersections of $\tilde{S}$ with the planes parallel to $x_1 x_{j+2}$, passing through the points obtained in the previous step. This is achieved by solving the corresponding Hamiltonian system, with initial conditions given in the region constructed previously. All the trajectories computed in (3) - (5) are planar and periodic and represent the intersection of one of the planes introduced in Proposition 3 with $\tilde{S}$.

The results of this section describe in analytic, respectively geometric manners the region of $\tilde{S}$ that is parameterized via (3) - (5). Although $\tilde{S}$ from (21) is connected, the "slices" $\tilde{S}_{j,\tilde{c}_{j+2}}$ may not be connected. Comments similar to Remark 1, Remark 2, Remark 3 and conditions to ensure almost global parameterizations can be formulated in this case too.

4 Conclusion

Our approach is based on a reduction to dimension two, by using simple advantageous bases in the tangent plane to the manifolds. In dimension two, strong global parameterization properties are valid. But, even in dimension three, our approach may not offer global results in complex examples like certain trefoil knots (due to the condition that the choice of the plane $x_1 x_2$ should define a curve that connects A with A', by cutting S). Moreover, concerning manifolds obtained for $2 \leq l \leq d-2$ (arbitrary codimension), the existence and how to construct such "bidimensional" simple bases satisfying (6), is not clear. However, we stress that the case of surfaces and hypersurfaces, discussed in this paper, is an important one, for applications to shape/topology optimization problems in any dimension.

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about 25 years ago, at the Euler Colloquium in Potsdam (where he was the main speaker), I also asked him about his Romanian origin, that he confirmed to me: his father was born and lived in Romania and he emigrated in his young years to France. H. Brezis supported many young Romanian mathematicians as Ph.D. students. He also helped the creation and the international collaboration via the French-Romanian Conference in applied mathematics (that is organized starting with 1992, every two years, alternatively in France and in Romania and enjoys a high rate of international participation).

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