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LONG TIME SOLUTIONS OF MHD MODIFIED STOKES PROBLEMS FOR MAXWELL FLUIDS WITH VISCOSITY DEPENDING EXPONENTIALLY ON PRESSURE*

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Abstract

Exact analytical expressions are derived for the long time components of the dimensionless velocities and shear stresses corresponding to the modified Stokes problems for incompressible upper-convected Maxwell fluids whose viscosity exponentially depends on pressure. The influence of magnetic field and of the gravitational acceleration is taken into account and some known results from the literature are recovered as limiting cases. Obtained solutions can be used as tests for numerical methods that are developed for more complex flow problems and to find the required time to reach the steady state. Graphical representations showed that the fluids with pressure dependent viscosity flow more quickly in comparison with ordinary fluids.

Keywords: modified Stokes problems, Maxwell fluids, viscosity depending exponential on pressure, magnetic field.
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1 Introduction

Usually, the viscosity of fluids is assumed to be constant. However, it was experimentally proved that, at high pressures it substantially increases [6, 21, 26]. Such a situation appears, for instance, at pharmaceutical tablet manufacturing, food processing, fluid film lubrication, fuel oil pumping, micro fluidics and elastohydrodynamic lubrication [5, 6, 19, 27]. Early enough Andrade [2] and Bridgman [4] investigated the variation of viscosity with pressure for different fluids. In addition, some experimental studies of Griest et al. [12], Johnson and Cameron [15], Johnson and Greenwood [16], Bair and Winer [3] attest the dependence of viscosity of pressure. On the other hand, the influence of pressure on the fluid density is small enough and the respective fluids can be treated as incompressible.

Steady solutions for motions in rectangular domains of fluids whose viscosity depend of pressure have been established by Hron et al. [14], Rajagopal [22, 23], Prusa [20], Akyildiz and Siginer [1] and Housiadas and Georgiou [13]. Some of them were extended to unsteady motions of same fluids by Rajagopal et al. [24], Fetecau and Vieru [7], Fetecau and Bridges [8] and Fetecau et al. [9]. The authors of these papers have investigated motions of fluids with linear, exponential or power law dependence of viscosity on the pressure. The most of them contain exact expressions for the steady components of the start-up velocities. These expressions can be used to determine the necessary time to reach the steady state. This time, which is very important for the experimental researchers, can be graphically determined showing that the start-up velocities (numerical solutions) converge to their long time components for increasing values of the time t. However, in all above mentioned papers, the influence of magnetic field on the fluid behavior has not been taken into consideration.

Effects of the magnetic field on the fluids' motion are meaningful and have a lot of industrial applications. The interaction between an electrical conducting fluid in motion and a magnetic field implies effects with applications in physics, chemistry, engineering, biological fluids, plasma studies, polymer manufacturing, MHD generators and many others. The influence of magnetic field on the Couette flow of viscous fluids was early studied by Tao [28] and Katagiri [18]. Exact solutions for MHD motions of some non-Newtonian fluids between parallel plates were derived by Zahid et al. [29] and Gosh et al. [11]. In this note we determine first long time solutions of modified Stokes problems for incompressible upper-convected Maxell fluids with exponential dependence of viscosity on pressure when magnetic effects and the gravitational acceleration are taken into consideration. These solutions can be graphically used to determine the time to reach the steady state.

2 Constitutive and governing equations

Let us consider an electrically conducting incompressible upper-convected Maxwell fluid with exponential dependence of viscosity on pressure between two unbounded horizontal parallel flat plates. Its constitutive equations are [17]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \mathbf{S} + \lambda\,\frac{\delta \mathbf{S}}{\delta t} = \mu \mathrm{e}^{\alpha(p-p_0)}(\mathbf{L} + \mathbf{L}^T). \tag{1}$$

Here $\mathbf{T}$ is the stress tensor, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{I}$ is the unit tensor, $\mathbf{L} = \mathrm{grad}w$ where w is the velocity vector, p is the hydrostatic pressure, λ is the relaxation time, μ is the fluid viscosity at the reference pressure p_0 and $\alpha > 0$ is the dimensional pressure viscosity coefficient. The upper-convected derivative $\delta/\delta t$ is defined by the relation

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T, \tag{2}$$

where d/dt denotes the material time derivative. When $\alpha = 0$, the governing equations (1) correspond to ordinary incompressible upper-convected Maxwell fluids. If $\lambda = 0$, the constitutive equations of incompressible Newtonian fluids are recovered.

At the moment $t = 0$ the whole system is at rest. After this moment the inferior plate begins to oscillate in its plane with the velocity $W\cos(\omega t)$ or $W\sin(\omega t)$ or to slide in the same plane with the constant velocity W. Here, ω is the oscillations frequency. Owing to the shear the fluid begins to move and since both plates are unbounded, we are seeking for a velocity vector w and a pressure p of the form [24]

$$w = w(x,t) = w(x,t)\mathbf{e}_z,\ p = p(x), \tag{3}$$

in a convenient Cartesian coordinate system x, y and z. Here $\mathbf{e}_z$ is the unit vector along the z-axis. We also assume that $\mathbf{S}$ as well as the velocity vector w is a function of x and t only. The continuity equation is satisfied. Using the fact that the fluid has been in rest up to the initial moment $t = 0$, it is not difficult to show that the non-trivial shear stress $\eta(x,t) = S_{xz}(x,t)$ has to satisfy the partial differential equation

$$\lambda\,\frac{\partial \eta(x,t)}{\partial t} + \eta(x,t) = \mu \mathrm{e}^{\alpha(p-p_0)}\,\frac{\partial w(x,t)}{\partial x};\ 0 < x < d,\ t > 0, \tag{4}$$

where d is the distance between plates.

In the following we assume that the fluid is finitely conducting and a magnetic field of constant strength B acts orthogonal to plates. Furthermore, supposing that the magnetic permeability of the fluid is constant, the induced magnetic field can be neglected and there is no electric charge distribution in fluid, the balance of linear momentum reduces to [30]

$$\rho\,\frac{\partial w(x,t)}{\partial t}=\frac{\partial \eta(x,t)}{\partial x}-\sigma B^2 w(x,t),\quad \frac{dp(x)}{dx}=-\rho g;\quad 0<x<d,\ t>0,\tag{5}$$

where ρ is the fluid density, σ is its electrical conductivity and g is the gravitational acceleration. Integrating the second equation from the equalities (5), it results that

$$p=p(x)=\rho g(d-x)+p_0\quad\text{where } p_0=p(d).\tag{6}$$

Substituting p from the last relation in (4) one finds that

$$\lambda\,\frac{\partial \eta(x,t)}{\partial t}+\eta(x,t)=\mu \mathrm{e}^{\alpha\rho g(d-x)}\,\frac{\partial w(x,t)}{\partial x};\quad 0<x<d,\ t>0.\tag{7}$$

The unknown functions $w(x,t)$ and $\eta(x,t)$ have to satisfy the following initial and boundary conditions

$$w(x,0)=0,\ \eta(x,0)=0;\quad 0\le x\le d,\tag{8}$$

respectively,

$$w(0,t)=W\cos(\omega t),\ W\sin(\omega t)\text{ or } W,\ w(d,t)=0;\ t>0.\tag{9}$$

Using the next non-dimensional variables, functions and parameters

$$x^*=\frac{1}{d}\,x,\ t^*=\frac{\nu}{d^2}\,t,\ w^*=\frac{1}{W}\,w,\ \eta^*=\frac{d}{\mu W}\,\eta,\ \alpha^*=\alpha\rho g d,\ \omega^*=\frac{d^2}{\nu}\,\omega,\tag{10}$$

and neglecting the star notation, one finds the dimensionless forms

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial \eta(x,t)}{\partial x}-Mw(x,t);\ 0<x<1,\ t>0,\tag{11}$$

$$\mathrm{We}\,\frac{\partial \eta(x,t)}{\partial t}+\eta(x,t)=\mathrm{e}^{\alpha(1-x)}\,\frac{\partial w(x,t)}{\partial x};\ 0<x<1,\ t>0,\tag{12}$$

of the governing equations. In these last relations $\nu=\mu/\rho$ is the kinematic viscosity of the fluid while the magnetic parameter M and the Weissenberg number We (which is the ratio of the relaxation time λ and a characteristic time scale d^2/ν) are defined by the relations

$$M = \frac{\sigma B^2}{\rho}\,\frac{d^2}{\nu} = \frac{d^2}{\mu}\,\sigma B^2, \quad \mathrm{We} = \frac{\nu\lambda}{d^2}. \tag{13}$$

The dimensionless initial and boundary conditions are

$$w(x,0) = 0,\ \ \eta(x,0) = 0;\ \ 0 \le x \le 1, \tag{14}$$

$$w(0,t) = \cos(\omega t),\ \ \sin(\omega t) \quad \text{or} \quad 1, \quad w(1,t) = 0; \quad t > 0. \tag{15}$$

3 Long time solutions

The staring solutions corresponding to Stokes problems for such fluids have to satisfy the governing equations (11) and (12) with the corresponding initial and boundary conditions. These solutions describe the fluid motion some time after its initiation. After this time, the fluid behavior is described by the long time (steady state or permanent) solutions which are independent of the initial conditions but satisfy the boundary conditions and the governing equations. This is the time to reach the steady state. In practice, this time is very important for the experimental researchers who want to know the transition moment of the motion to the steady state. In order to determine this time, it is necessary and sufficient to know the long time solutions. This is the reason that, in the following we shall determine these solutions. By eliminating the shear stress $\eta(x,t)$ between Eqs. (11) and (12) one obtains the governing equation

$$\begin{aligned} \mathrm{We}\,\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial w(x,t)}{\partial t} &= \mathrm{e}^{\alpha(1-x)}\left[\frac{\partial^2 w(x,t)}{\partial x^2} - \alpha\,\frac{\partial w(x,t)}{\partial x}\right] \\ &-M\left[\mathrm{We}\,\frac{\partial w(x,t)}{\partial t} + w(x,t)\right]; \quad 0 < x < 1,\ t > 0, \end{aligned} \tag{16}$$

for the fluid velocity $w(x,t)$.

3.1 Long time solutions for the second problem of Stokes

In the next, for distinction, we denote by $w_c(x,t), w_s(x,t)$ the dimensionless start-up velocity fields corresponding to motions induced by cosine or sine oscillations of the inferior plate and by $\eta_c(x,t), \eta_s(x,t)$ the associated shear stresses. It is well known that these motions become steady in time and the corresponding solutions can be written as sum of their long time (steady state or permanent) and transient components. For instance,

$$w_c(x,t) = w_{cp}(x,t) + w_{ct}(x,t),\ w_s(x,t) = w_{sp}(x,t) + w_{st}(x,t); \quad 0 < x < 1,\ t > 0. \tag{17}$$

As we previously mentioned, in order to determine the necessary time to touch the steady state, the long time solutions $w_{cp}(x,t)$ and $w_{sp}(x,t)$ have to be known. From mathematical point of view, this is the time after which the diagrams of start-up velocities $w_c(x,t)$ and $w_s(x,t)$ overlap over those of their long time components $w_{cp}(x,t)$ and $w_{sp}(x,t)$, respectively. To determine both velocity fields in the same time we use the complex velocity

$$w_{com}(x,t) = w_{cp}(x,t) + iw_{sp}(x,t);\ 0 < x < 1, t > 0, \tag{18}$$

which has to satisfy the next boundary value problem

$$\begin{aligned} &\mathrm{We}\, \frac{\partial^2 w_{com}(x,t)}{\partial t^2} + \frac{\partial w_{com}(x,t)}{\partial t} \\ &\quad = \mathrm{e}^{\alpha(1-x)} \left[\frac{\partial^2 w_{com}(x,t)}{\partial x^2} - \alpha\, \frac{\partial w_{com}(x,t)}{\partial x} \right] \\ &\quad -M \left[\mathrm{We}\, \frac{\partial w_{com}(x,t)}{\partial t} + w_{com}(x,t) \right];\quad 0 < x < 1,\ t > 0, \end{aligned} \tag{19}$$

$$w_{com}(0,t) = \mathrm{e}^{i\omega t},\quad w_{com}(1,t) = 0;\quad t > 0. \tag{20}$$

In the equality (18), i is the imaginary unit.

Making the change of the independent variable $x = 1 + \ln \sqrt[\alpha]{r}$, one finds the following boundary value problem

$$\begin{aligned} \mathrm{We}\, \frac{\partial^2 w_{com}(r,t)}{\partial t^2} + \frac{\partial w_{com}(r,t)}{\partial t} &= \alpha^2 r\, \frac{\partial^2 w_{com}(r,t)}{\partial r^2} \\ -M \left[\mathrm{We}\, \frac{\partial w_{com}(r,t)}{\partial t} + w_{com}(r,t) \right] &= 0, \end{aligned} \tag{21}$$

$$w_{com}(1/\mathrm{e}^{\alpha},t) = \mathrm{e}^{i\omega t},\quad w_{com}(1,t) = 0;\quad t > 0. \tag{22}$$

The homogeneity of the partial differential equation (21) and the form of boundary conditions (22) suggest us to look for a solution of the form

$$w_{com}(r,t) = U(r)\mathrm{e}^{i\omega t};\quad 1/\mathrm{e}^{\alpha} < r < 1,\ t > 0. \tag{23}$$

Replacing $w_{com}(r,t)$ from Eq. (23) in (21) and bearing in mind the boundary conditions (22), it results that the complex function $U(\cdot)$ has to satisfy the boundary value problem

$$rU''(r) + \beta U(r) = 0; \quad U(1/\mathrm{e}^{\alpha}) = 1, \quad U(1) = 0, \tag{24}$$

where $\beta = -\frac{(i\omega+M)(i\omega\mathrm{We}+1)}{\alpha^2}$.

Now, using the following observation (see [30] the exercise 37 from the page 251) the general solution of the differential equation $xy'' + \lambda y = 0$ on the interval $(0,\infty)$ is

$$y = \sqrt{x}\,[c_1 J_1(2\sqrt{\lambda x}) + c_2 Y_1(2\sqrt{\lambda x})], \tag{25}$$

we can show that $U(r)$ is given by the relation

$$U(r) = \sqrt{r\mathrm{e}^{\alpha}} \times \frac{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta r}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta r})}{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta \mathrm{e}^{-\alpha}})}; \ \frac{1}{\mathrm{e}^{\alpha}} < r < 1, \tag{26}$$

where $J_1(\cdot)$ and $Y_1(\cdot)$ are Bessel functions of the first and second kind and first order.

Finally, bearing in mind the relations (18), (23) and (26), we conclude that the long time velocities $w_{cp}(x,t)$ and $w_{sp}(x,t)$ are given by the next relations

$$w_{cp} = \sqrt{\mathrm{e}^{\alpha x}}\,\mathrm{Re}\left\{\frac{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta \mathrm{e}^{\alpha(x-1)}}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta \mathrm{e}^{\alpha(x-1)}})}{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta \mathrm{e}^{-\alpha}})}\,\mathrm{e}^{i\omega t}\right\}, \tag{27}$$

$$w_{sp} = \sqrt{\mathrm{e}^{\alpha x}}\,\mathrm{Im}\left\{\frac{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta \mathrm{e}^{\alpha(y-1)}}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta \mathrm{e}^{\alpha(y-1)}})}{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta \mathrm{e}^{-\alpha}})}\,\mathrm{e}^{i\omega t}\right\}, \tag{28}$$

where Re and Im means the real and the imaginary part of that which follows.

Proceedings in the same way with the dimensionless shear stresses $\eta_c(x,t)$, $\eta_s(x,t)$ and using the relation $J_1'(z) = J_0(z) - J_1(z)/z$, we can prove that

$$\eta_{cp}(x,t) = \sqrt{\mathrm{e}^{\alpha}} \times \mathrm{Re}\left\{\frac{Y_1(2\sqrt{\beta})J_0(2\sqrt{\beta \mathrm{e}^{x-1}}) - J_1(2\sqrt{\beta})Y_0(2\sqrt{\beta \mathrm{e}^{\alpha(x-1)}})}{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta \mathrm{e}^{-\alpha}})}\,\frac{i\sqrt{i\omega+M}}{\sqrt{i\omega\mathrm{We}+1}}\,\mathrm{e}^{i\omega t}\right\}, \tag{29}$$

$$\eta_{sp}(y,t) = \sqrt{\mathrm{e}^{\alpha}} \times \mathrm{Im}\left\{\frac{Y_1(2\sqrt{\beta})J_0(2\sqrt{\beta \mathrm{e}^{x-1}}) - J_1(2\sqrt{\beta})Y_0(2\sqrt{\beta \mathrm{e}^{\alpha(x-1)}})}{Y_1(2\sqrt{\beta})J_1(2\sqrt{\beta \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\beta})Y_1(2\sqrt{\beta \mathrm{e}^{-\alpha}})}\,\frac{i\sqrt{i\omega+M}}{\sqrt{i\omega\mathrm{We}+1}}\,\mathrm{e}^{i\omega t}\right\}, \tag{30}$$

In the absence of magnetic field, when $M = 0$, the relations (27)–(30) are in accordance with those obtained by Fetecau et al. [9] using a different normalization.

Taking We $= 0$ in the above relations, the dimensionless long time solutions

$$w_{Ncp}(x,t) = \sqrt{\mathrm{e}^{\alpha x}} \times \mathrm{Re}\left\{ \frac{Y_1(2\sqrt{\gamma})J_1(2\sqrt{\gamma \mathrm{e}^{\alpha(x-1)}}) - J_1(2\sqrt{\gamma})Y_1(2\sqrt{\gamma \mathrm{e}^{\alpha(x-1)}})}{Y_1(2\sqrt{\gamma})J_1(2\sqrt{\gamma \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\gamma})Y_1(2\sqrt{\gamma \mathrm{e}^{-\alpha}})} \mathrm{e}^{i\omega t} \right\}, \tag{31}$$

$$w_{Nsp}(x,t) = \sqrt{\mathrm{e}^{\alpha x}} \times \mathrm{Im}\left\{ \frac{Y_1(2\sqrt{\gamma})J_1(2\sqrt{\gamma \mathrm{e}^{\alpha(x-1)}}) - J_1(2\sqrt{\gamma})Y_1(2\sqrt{\gamma \mathrm{e}^{\alpha(x-1)}})}{Y_1(2\sqrt{\gamma})J_1(2\sqrt{\gamma rm e^{-\alpha}}) - J_1(2\sqrt{\gamma})Y_1(2\sqrt{\gamma \mathrm{e}^{-\alpha}})} \mathrm{e}^{i\omega t} \right\}, \tag{32}$$

$$\eta_{Ncp}(x,t) = \sqrt{\mathrm{e}^{\alpha}} \times \mathrm{Re}\left\{ \frac{Y_1(2\sqrt{\gamma})J_0(2\sqrt{\gamma \mathrm{e}^{\alpha(x-1)}}) - J_1(2\sqrt{\gamma})Y_0(2\sqrt{\gamma \mathrm{e}^{\alpha(x-1)}})}{Y_1(2\sqrt{\gamma})J_1(2\sqrt{\gamma \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\gamma})Y_1(2\sqrt{\gamma \mathrm{e}^{-\alpha}})} \, i\sqrt{i\omega + M}\, \mathrm{e}^{i\omega t} \right\}, \tag{33}$$

$$\eta_{Nsp}(x,t) = \sqrt{\mathrm{e}^{\alpha}} \times \mathrm{Im}\left\{ \frac{Y_1(2\sqrt{\gamma})J_0(2\sqrt{\gamma \mathrm{e}^{\alpha(x-1)}}) - J_1(2\sqrt{\gamma})Y_0(2\sqrt{\gamma rm e^{\alpha(x-1)}})}{Y_1(2\sqrt{\gamma})J_1(2\sqrt{\gamma \mathrm{e}^{-\alpha}}) - J_1(2\sqrt{\gamma})Y_1(2\sqrt{\gamma \mathrm{e}^{-\alpha}})} \, i\sqrt{i\omega + M}\, \mathrm{e}^{i\omega t} \right\}, \tag{34}$$

corresponding to the MHD modified second problem of Stokes for incompressible Newtonian fluids with exponential dependence of viscosity on pressure are obtained. In the last relations the complex constant $\gamma = -(i\omega + M)/\alpha^2$.

3.2 Study case $\omega = 0$ (modified Stokes first problem)

The dimensionless start-up velocity and shear stress corresponding to the MHD motion of the upper-convected incompressible Maxwell fluids with exponential dependence of viscosity on the pressure induced by the lower plate that moves in its plane with the constant velocity W will be denoted by $w_C(x,t)$ and $\eta_C(x,t)$. Taking $\omega = 0$ in Eqs. (27) and (29) one obtains the long time components $w_{Cp}(x)$ and $\tau_{Cp}(x)$ of these two entities, namely

$$w_{Cp}(x) = \sqrt{\mathrm{e}^{\alpha x}} \times \mathrm{Re}\left\{ \frac{Y_1\left(\frac{2i}{\alpha}\sqrt{M}\right)J_1\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{\alpha(x-1)}}\right) - J_1\left(\frac{2i}{\alpha}\sqrt{M}\right)Y_1\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{\alpha(x-1)}}\right)}{Y_1\left(\frac{2i}{\alpha}\sqrt{M}\right)J_1\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{-\alpha}}\right) - J_1\left(\frac{2i}{\alpha}\sqrt{M}\right)Y_1\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{-\alpha}}\right)} \right\}, \tag{35}$$

$$\eta_{Cp}(x) = \sqrt{M\mathrm{e}^{\alpha}}$$
$$\times \mathrm{Re}\left\{ i\, \frac{Y_1\left(\frac{2i}{\alpha}\sqrt{M}\right)J_0\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{\alpha(x-1)}}\right) - J_1\left(\frac{2i}{\alpha}\sqrt{M}\right)Y_0\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{\alpha(x-1)}}\right)}{Y_1\left(\frac{2i}{\alpha}\sqrt{M}\right)J_1\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{-\alpha}}\right) - J_1\left(\frac{2i}{\alpha}\sqrt{M}\right)Y_1\left(\frac{2i}{\alpha}\sqrt{M\mathrm{e}^{-\alpha}}\right)} \right\}, \tag{36}$$

These solutions are the identical for electrical conducting incompressible Newtonian and Maxwell fluids with viscosity exponentially depending on the pressure. This is not a surprise because the governing equations for steady motions of these fluids are identical.

Now, we use the next asymptotic approximations

$$J_0(z) \approx 1,\ J_1(z) \approx \frac{z}{2},\ Y_0(z) \approx \frac{2}{\pi}\left[\ln\left(\frac{z}{2}\right) + \delta\right],\ Y_1(z) \approx -\frac{2\pi}{z};\ |z| << 1, \tag{37}$$

for the Bessel functions in order to recover some known results from the literature. In Eq. (37) $\delta \approx 0.5772$ is the Euler Mascheroni constant. On the base of these approximations that are valid for small values of the magnetic parameter M, we recover the results

$$\lim_{M\to 0} w_{Cp}(x) = \frac{\mathrm{e}^{\alpha x} - \mathrm{e}^{\alpha}}{1 - \mathrm{e}^{\alpha}} = w_{Cp0}(x),\ \lim_{M\to 0} \eta_{Cp}(x) = \frac{\alpha \mathrm{e}^{\alpha}}{1 - \mathrm{e}^{\alpha}} = \eta_{Cp0}, \tag{38}$$

obtained by Fetecau et al. [9] in a different way. The expression of $w_{Cp0}(x)$ can be also obtained solving the corresponding boundary value problem. Now, it is interesting to observe that the shear stress η_{Cp0} is constant on the entire flow domain although the fluid velocity $w_{Cp0}(x)$ is a function of the spatial variable x. As a check of the last results, Figure 1 is below included.

3.3 Limiting case $\alpha \to 0$ (modified Stokes problems for ordinary Maxwell fluids)

Using the well known asymptotic approximations

$$J_1(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left[z - \frac{3\pi}{4}\right], Y_1(z) \approx \sqrt{\frac{2}{\pi z}} \sin\left[z - \frac{3\pi}{4}\right] \text{ for } |z| >> 1, \tag{39}$$

it is not difficult to show that for small values of the parameter α

$$w_{cp}(x,t) \approx \sqrt[4]{\mathrm{e}^{\alpha x}}\, \mathrm{Re}\left\{ \frac{\sin\left\{2\sqrt{\beta}\left[1 - \exp\left(\frac{\alpha(x-1)}{2}\right)\right]\right\}}{\sin\left\{2\sqrt{\beta}\left[1 - \exp\left(-\frac{\alpha}{2}\right)\right]\right\}} \mathrm{e}^{i\omega t} \right\}, \tag{40}$$

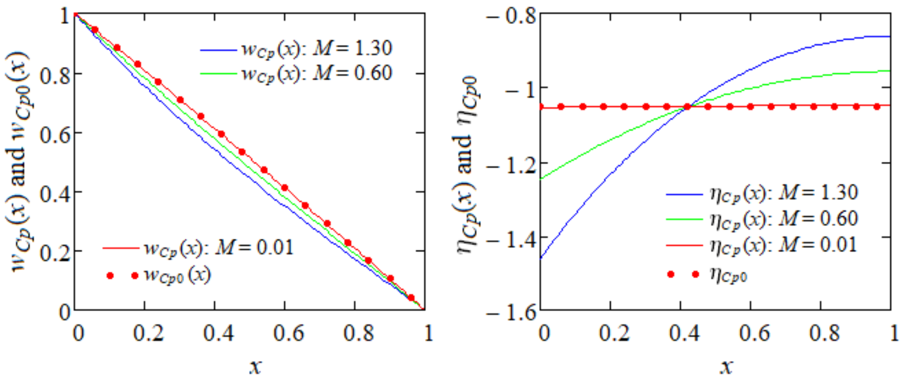

Figure 1: Convergence of $w_{Cp}(x)$ and $\eta_{Cp}(x)$ from Eqs. (35) and (36) to $w_{Cp0}(x)$ and η_{Cp0}, from Eqs. (38) when $\alpha = 0.1$ and $M \to 0$.

$$\eta_{cp}(x,t) \approx \frac{\sqrt{\mathrm{e}^{\alpha}}}{\sqrt[4]{\mathbf{e}^{\alpha x}}}\,\mathrm{Re}\left\{\frac{\cos\left\{2\sqrt{\beta}\left[1-\exp\left(\frac{\alpha(x-1)}{2}\right)\right]\right\}}{\sin\left\{2\sqrt{\beta}\left[1-\exp\left(-\frac{\alpha}{2}\right)\right]\right\}}\,\frac{i\sqrt{i\omega+M}}{\sqrt{i\omega\mathrm{We}+1}}\,\mathrm{e}^{i\omega t}\right\}, \quad (41)$$

Developing in Maclaurin series the functions $\exp[\alpha(x-1)/2]$ and $\exp(-\alpha/2)$, taking the limits of the obtained results when $\alpha \to 0$ and bearing in mind the fact that $\sin(iz) = i\sinh(z)$ and $\cos(iz) = i\cosh(z)$, one finds the dimensionless long time velocity $w_{Ocp}(x,t)$ and the shear stress $\eta_{Ccp}(x,t)$ corresponding to same motions of ordinary incompressible Maxwell. Their expressions are given by the relations

$$\begin{aligned} w_{Ocp}(x,t) &= \lim_{\alpha\to 0} w_{cp}(x,t) \\ &= \mathrm{Re}\left\{\frac{\sinh[(1-x)\sqrt{(i\omega+M)(i\omega\mathrm{We}+1)}]}{\sinh[\sqrt{(i\omega+M)(i\omega\mathrm{We}+1)}]}\,\mathrm{e}^{i\omega t}\right\}, \end{aligned} \quad (42)$$

$$\begin{aligned} &\eta_{Ocp}(x,t) = \lim_{\alpha\to 0}\eta_{cp}(x,t) \\ &= -\mathrm{Re}\left\{\frac{\cosh[(1-x)\sqrt{(i\omega+M)(i\omega\mathrm{We}+1)}]}{\sinh[\sqrt{(i\omega+M)(i\omega\mathrm{We}+1)}]}\,\frac{\sqrt{i\omega+M}}{\sqrt{i\omega\mathrm{We}+1}}\,\mathrm{e}^{i\omega t}\right\}. \end{aligned} \quad (43)$$

Of course, in the same way one finds that

$$\begin{aligned} w_{Osp}(x,t) &= \lim_{\alpha\to 0} w_{sp}(x,t) \\ &= \operatorname{Im}\left\{ \frac{\sinh[(1-x)\sqrt{(i\omega+M)(i\omega \mathrm{We}+1)}]}{\sinh[\sqrt{(i\omega+M)(i\omega \mathrm{We}+1)}]}\, \mathrm{e}^{i\omega t} \right\}, \end{aligned} \tag{44}$$

$$\begin{aligned} &\eta_{Osp}(x,t) = \lim_{\alpha\to 0} \eta_{sp}(x,t) \\ &= -\operatorname{Im}\left\{ \frac{\cosh[(1-x)\sqrt{(i\omega+M)(i\omega \mathrm{We}+1)}]}{\sinh[\sqrt{(i\omega+M)(i\omega \mathrm{We}+1)}]}\, \frac{\sqrt{i\omega+M}}{\sqrt{i\omega \mathrm{We}+1}}\, \mathrm{e}^{i\omega t} \right\}. \end{aligned} \tag{45}$$

Making $\mathrm{We} = 0$ in Eqs. (42)–(45) one obtains the dimensionless long time solutions $w_{NOcp}(x,t)$, $\eta_{NOcp}(x,t)$, $w_{NOsp}(x,t)$ and $\eta_{NOsp}(x,t)$ corresponding to the MHD modified Stokes second problem for ordinary incompressible Newtonian fluids. Their expressions are given by the relations:

$$w_{NOcp}(x,t) = \operatorname{Re}\left\{ \frac{\sinh[(1-x)\sqrt{i\omega+M}]}{\sinh[\sqrt{i\omega+M}]}\, \mathrm{e}^{i\omega t} \right\}, \tag{46}$$

$$w_{NOsp}(x,t) = \operatorname{Im}\left\{ \frac{\sinh[(1-x)\sqrt{i\omega+M}]}{\sinh[\sqrt{i\omega+M}]}\, \mathrm{e}^{i\omega t} \right\}, \tag{47}$$

$$\eta_{NOcp}(x,t) = -\operatorname{Re}\left\{ \frac{\cosh[(1-x)\sqrt{i\omega+M}]}{\sinh[\sqrt{i\omega+M}]}\, \sqrt{i\omega+M}\, \mathrm{e}^{i\omega t} \right\}, \tag{48}$$

$$\eta_{NOsp}(x,t) = -\operatorname{Im}\left\{ \frac{\cosh[(1-x)\sqrt{i\omega+M}]}{\sinh[\sqrt{i\omega+M}]}\, \sqrt{i\omega+M}\, \mathrm{e}^{i\omega t} \right\}. \tag{49}$$

Finally, taking $\omega = 0$ in Eqs. (42), (43) or (46), (48) one obtains the dimensionless long time solutions corresponding to the MHD modified first problem of Stokes for electrical conducting incompressible Newtonian or upper-convected Maxwell fluids. Their expressions are given by the relations

$$w_{NCp}(x) = \frac{\sinh[(1-x)\sqrt{M}]}{\sinh(\sqrt{M})}, \quad \eta_{NCp}(x) = -\sqrt{M}\, \frac{\cosh[(1-x)\sqrt{M}]}{\sinh(\sqrt{M})}, \tag{50}$$

Now, taking the limits of the relations (50) when the magnetic parameter $M \to 0$ we recover the solutions (65) of the reference [9], namely

$$\lim_{M\to 0} w_{NCp}(x) = 1 - x, \quad \lim_{M\to 0} \eta_{NCp}(x) = -1. \tag{51}$$

The Reynolds number Re does not appear in Eq. $(51)_2$ because a different normalization has been considered in the present relations (10).

4 Some numerical results and conclusions

In this note analytical expressions are derived for the dimensionless long time velocities and the associated shear stresses corresponding to the MHD modified Stokes problems for incompressible electrical conducting upper-convected Maxwell fluids whose viscosity exponentially depends on pressure. The obtained solutions can be used to determine the necessary time to get the steady state. Our main purpose here is to bring to light the oscillatory behavior of the dimensionless long time velocities $w_{cp}(x,t)$, $w_{sp}(x,t)$ and of the shear stresses $\eta_{cp}(x,t)$, $\eta_{sp}(x,t)$ corresponding to the modified second problem of Stokes and to show the influence of the magnetic field and the pressure viscosity coefficient on the fluid velocity. In order to do that, Figures 2-4 are prepared for fixed values of physical parameters α, ω and decreasing values of M or We.

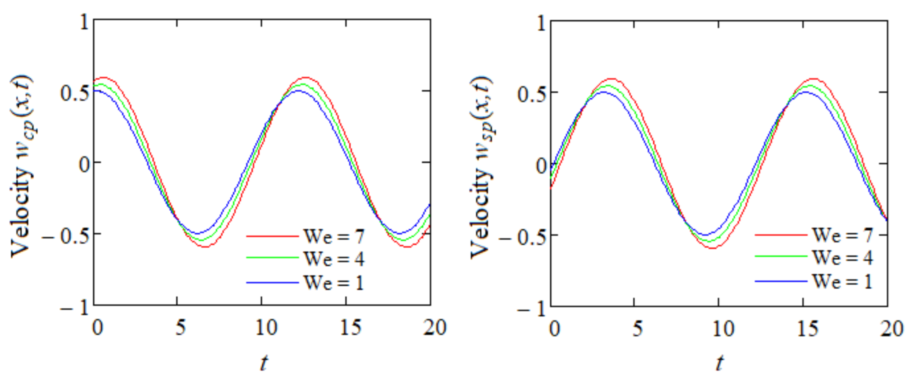

Figure 2: Variations in time of $w_{cp}(x,t)$ and $w_{sp}(x,t)$ given by Eqs. (27) and (28) for $\alpha = 0.1$, $\omega = \pi/6$, $M = 0.5$, $x = 0.5$ and decreasing values of the Weissenberg number We.

In Figures 2 and 3 are presented the variations in time of the long time velocities $w_{cp}(x,t)$, $w_{sp}(x,t)$ and of the corresponding shear stresses $\eta_{cp}(x,t)$, $\eta_{sp}(x,t)$ at the middle of the channel for decreasing values of Weissenberg number We and fixed values for the other parameters. The phase difference between the two motions induced by cosine or sine oscillations of the lower plate and their oscillatory characteristics are clearly visualized. In addition, as it was to be expected, the oscillation amplitudes which bring up for

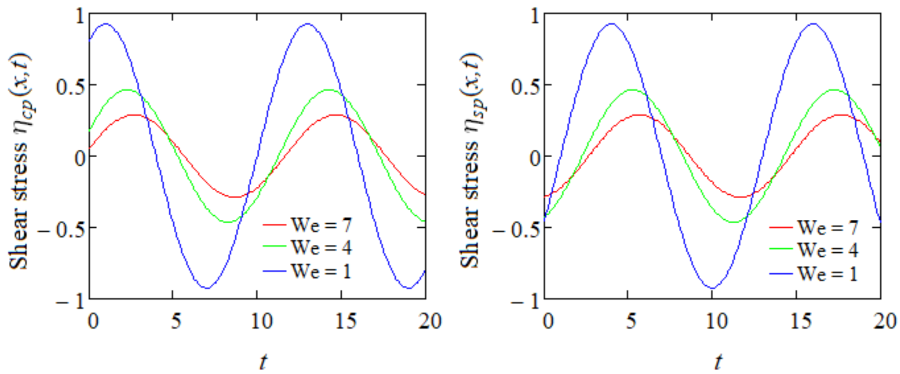

Figure 3: Variations in time of $\eta_{cp}(x,t)$ and $\eta_{sp}(x,t)$ given by Eqs. (29) and (30) for $\alpha = 0.1$, $\omega = \pi/6$, $M = 0.5$, $x = 0.5$ and decreasing values of Weissenberg number We.

increasing values of We and diminish for increasing values of the magnetic parameter M are identical for the two motions at the same values of physical parameters.

Last Figure 4 presents the profiles of the dimensionless long time velocity $w_{Cp}(x)$ and of the shear stress $\eta_{Cp}(x)$ at decreasing values of the pressure-viscosity parameter α. From this figure, as it was to be expected, it clearly results that the fluid velocity is a decreasing function with respect to α while the shear stress is a decreasing one. Consequently, the fluids with pressure dependent viscosity flow faster in comparison with ordinary fluids.

The main findings that have been obtained here are:

- Modified Stokes problems for incompressible Maxwell fluids with viscosity depending exponentially on pressure were analytically investigated when magnetic effects and gravitational acceleration are taken into consideration.

- Exact expressions were derived for dimensionless long time velocity fields and shear stresses. These expressions are important for the experimental researchers who want to know the transition moment of the motion to the steady state.

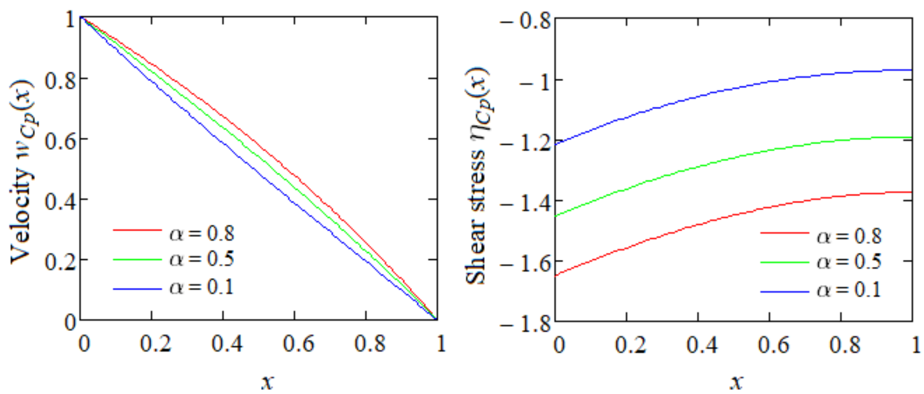

Figure 4: Variations in time of $w_{Cp}(x)$ and $\eta_{Cp}(x)$ given by Eqs. (35) and (36) for $M = 0.5$, and decreasing values of the parameter α.

- Results validation has been analytically and graphically investigated or showing that known results from the literature are recovered as limiting cases.
- The fluids with pressure dependent viscosity flow quickly than ordinary fluids.

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