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TOPOLOGY OPTIMIZATION PROBLEMS FOR CLAMPED KIRCHHOFF-LOVE PLATES*

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This paper is dedicated to the memory of Prof. Haim Brezis

Abstract

Fixed domain approaches are used for some topology optimization problems of clamped Kirchhoff-Love plates: detection of a damaged zone using pointwise observation, respectively compliance minimization and the first eigenvalue maximization. We discuss the derivability with respect to functional variations of the geometry and some descent directions are proposed. Numerical tests are presented.

Keywords: topology optimization, compliance minimization, maximization of the first eigenvalue.

MSC: 49Q10, 65K10.

1 Introduction

Shape and topology optimization of structures were analyzed in the monographs [31] (speed method), [5], [3] (homogenization), [29] (topological derivative) or in the paper [2] (level-set). In [4], the thickness optimization of simply supported and clamped plates are expressed as distributed optimal control problems governed by second order boundary value problems.

Fixed domain approaches based on a parametrization function for the geometry are used in [27], [28] for shape and topology optimization. The

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problem can be rewritten as an optimal control problem studied in [26]. The state equations written in a unknown domain Ω are rewritten in a fixed hold-all domain D. For simply supported plate this technique was used in [17], for linear elasticity in [19], for Stokes in [20]. All the papers cited in this paragraph, excepting [19], use penalization in the state equation in $D \setminus \Omega$. The approach can be also employed for non-linear problems as Navier-Stokes [24] or variational inequalities [25].

The boundary of the unknown domain can be parameterized as well using the solution of a Hamiltonian system. The analysis of shape and topology optimization for elliptic boundary value problems are presented in [18] for Dirichlet boundary condition, in [21] for boundary observation and in [22] for Neumann boundary condition. This technique was used for clamped plate in [23].

Here, we investigate some problems concerning topology optimization of clamped Kirchhoff-Love plates. We discuss the detection of a damaged zone using pointwise observation [30] in Section 2, compliance minimization [2] in Section 3 and maximization of the first eigenvalue [8] in Section 4. Differentiability with respect to the shape parametrization is studied and descent directions for the different cost functionals are proposed, using just functional variations [27], [28]. For other geometric variation types, see [15]. Since the original state equations are written directly in the hold-all domain (the plate is composed by two materials), then penalization in the complementary set or the Hamiltonian parametrization of the boundary of the unknown domain are not necessary. Numerical tests are presented in the last section.

2 Pointwise observation for Kirchhoff-Love plate with non-constant thickness

Let $D \subset \mathbb{R}^2$ be a given bounded Lipschitz domain and $\Omega \subset\subset D$ an open set. We consider a Kirchhoff-Love plate of thickness

$$e(\mathbf{x}) = \begin{cases} e_0, & \mathbf{x} \in D \setminus \overline{\Omega} \\ e_1, & \mathbf{x} \in \Omega \end{cases} \tag{1}$$

with $0 < e_1 < e_0$. The set Ω represents the damaged zone. Denoting $V = H_0^2(D)$, let us introduce $a : V \times V \to \mathbb{R}$ given by

$$\begin{aligned} a(w,v) &= \int_D \frac{Ee^3(\mathbf{x})}{12(1-\nu^2)}\Big[\nu\Delta w\Delta v \\ &+(1-\nu)\left(\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + 2\frac{\partial^2 w}{\partial x_1\partial x_2}\frac{\partial^2 v}{\partial x_1\partial x_2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2}\right)\Big]d\mathbf{x} \end{aligned} \quad (2)$$

where $E > 0$ the Young modulus and $0 < \nu < 1/2$ the Poisson ratio are constants. Also, we introduce $\ell : V \to \mathbb{R}$ given by

$$\ell(v) = \langle f, v\rangle \quad (3)$$

where $f \in V'$ (the dual of V and $\langle\cdot,\cdot\rangle$ is the duality bracket between V' and V). Concentrated transverse loads are applied at the points $\boldsymbol{\xi}_m \in D$, $m = 1,\ldots,M$ and we take $f = b_0\sum_{m=1}^M \delta(\boldsymbol{\xi}_m)$, where $b_0 \in \mathbb{R}$ and $\delta(\boldsymbol{\xi}_m)$ is the Dirac function of pole $\boldsymbol{\xi}_m$. We have $\langle\delta(\boldsymbol{\xi}_m), v\rangle = v(\boldsymbol{\xi}_m)$ since $V \subset \mathcal{C}(\overline{D})$. The scalar product and the norm of the Hilbert space $H^k(D)$, $k \in \mathbb{N}$ are denoted by $(\cdot,\cdot)_{k,D}$ and $\|\cdot\|_{k,D}$, respectively and $H^0(D) = L^2(D)$.

The transverse displacement of the clamped plate is $y \in V$, such that

$$a(y,v) = \ell(v),\ \forall v \in V. \quad (4)$$

The plate is clamped, i.e. $y = \frac{\partial y}{\partial \mathbf{n}} = 0$ on ∂D. The problem has a unique solution by Lax-Milgram Theorem. The ellipticity is based on the fact that

$$\begin{aligned} & a(v,v) \\ \geq\ & \frac{Ee_1^3}{12(1-\nu^2)}\int_D \Big[\nu(\Delta v)^2 \\ & +(1-\nu)\left(\left(\frac{\partial^2 v}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 v}{\partial x_1\partial x_2}\right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2}\right)^2\right)\Big]d\mathbf{x} \\ \geq\ & \frac{Ee_1^3(1-\nu)}{12(1-\nu^2)}\int_D \sum_{i,j=1}^2 \left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 d\mathbf{x} = \frac{Ee_1^3}{12(1+\nu)}|v|_{2,D}^2 \end{aligned}$$

and $|v|_{2,D}$ is a norm equivalent to $\|v\|_{2,D}$ in $H_0^2(D)$, see [12], p. 34-35.

We consider a family $\mathcal{F} \subset \mathcal{C}(\overline{D})$ of admissible controls, $\mathcal{F}$ is an open cone and

$$\Omega = \Omega_g = int\{\mathbf{x} \in D;\ g(\mathbf{x}) \leq 0\},\ g \in \mathcal{F}. \quad (5)$$

We point out that Ω_g defined by (5) is not necessarily connected.

The thickness can be written using $H : \mathbb{R} \to \mathbb{R}$ the Heaviside function

$$e(g) = e_0 H(g) + e_1 (1 - H(g)).$$

Let $H_\eta : \mathbb{R} \to \mathbb{R}$ be a regularization of H, where η is a positive parameter, such that

$$H_\eta \in \mathcal{C}^2(\mathbb{R}),\ 0 \leq H_\eta \leq 1,\ 0 \leq H'_\eta, \quad \text{for } \eta \text{ fixed } H'_\eta, H''_\eta \text{ bounded in } \mathbb{R}. \quad (6)$$

We put

$$e_\eta(g) = e_0 H_\eta(g) + e_1 (1 - H_\eta(g))$$

and let $a_\eta : V \times V \to \mathbb{R}$ be obtained from (2) by replacing e by $e_\eta(g)$.

The regularized problem is: find $y_g \in V$ such that

$$a_\eta(y_g, v) = \ell(v),\ \forall v \in V. \quad (7)$$

Proposition 1. *The regularized problem has a unique solution $y_g \in V$ and*

$$\|y_g\|_{2,D} \leq C \|f\|_{V'} \quad (8)$$

Proof. Since $H_\eta \geq 0$, $e_0 - e_1 > 0$ we get $e(g) = e_1 + H_\eta(g)(e_0 - e_1) \geq e_1$ and

$$a_\eta(v, v) \geq \frac{E e_1^3}{12(1+\nu)} |v|^2_{2,D} \geq C_1 \|v\|^2_{2,D}$$

then a_η is elliptic. Since $H_\eta \leq 1$ and using Cauchy-Schwarz inequality, we get

$$a_\eta(w, v) \leq C_2 |w|_{2,D} |v|_{2,D}.$$

From the Lax-Milgram Theorem we get the conclusion. □

For $w, v \in V$ and $r \in \mathcal{F}$, we put

$$\begin{aligned}\frac{\partial a_\eta}{\partial g}(w, v) r \quad &= \quad \int_D \frac{E}{12(1-\nu^2)} 3 e_\eta^2(g)(e_0 - e_1) H'_\eta(g) r \Big[\nu \Delta w \Delta v \\ &+ (1-\nu) \left(\frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + 2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 w}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right) \Big] d\mathbf{x}.\end{aligned}$$

Proposition 2. *When $\lambda \to 0$, then:*
i) $y_{(g+\lambda r)}$ converges to y_g strongly in V,
ii) $z_\lambda = \frac{y_{(g+\lambda r)} - y_g}{\lambda} \in V$ converges to q strongly in V, where q is the unique solution of:

$$a_\eta(q, v) = -\frac{\partial a_\eta}{\partial g}(y_g, v) r,\ \forall v \in V. \quad (9)$$

Proof. To simplify the calculus, we note

$$
\begin{aligned}
[y,v]_{\#} &= \frac{E}{12(1-\nu^2)}\Bigg[\nu\Delta y\Delta v \\
&\quad +(1-\nu)\left(\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2}+2\frac{\partial^2 y}{\partial x_1\partial x_2}\frac{\partial^2 v}{\partial x_1\partial x_2}+\frac{\partial^2 y}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2}\right)\Bigg]
\end{aligned}
$$

we get: $[y,v]_{\#}\in L^1(D)$, $[v,v]_{\#}\geq 0$ a.e. in D, for all $v\in V$ and

$$a_\eta(y,v)=\int_D e_\eta^3(g)[y,v]_{\#}\,d\mathbf{x}.$$

i) Subtracting (7) written for g from (7) written for $(g+\lambda r)$, we get

$$
\begin{aligned}
0 &= \int_D e_\eta^3(g+\lambda r)[y_{(g+\lambda r)},v]_{\#}-e_\eta^3(g)[y_g,v]_{\#}\,d\mathbf{x} \\
&= \int_D e_\eta^3(g+\lambda r)[y_{(g+\lambda r)},v]_{\#}-e_\eta^3(g)[y_{(g+\lambda r)},v]_{\#}\,d\mathbf{x} \\
&\quad +\int_D e_\eta^3(g)[y_{(g+\lambda r)},v]_{\#}-e_\eta^3(g)[y_g,v]_{\#}\,d\mathbf{x} \\
&= \int_D (e_\eta^3(g+\lambda r)-e_\eta^3(g))[y_{(g+\lambda r)},v]_{\#}\,d\mathbf{x} \\
&\quad +\int_D e_\eta^3(g)\left[y_{(g+\lambda r)}-y_g,v\right]_{\#}\,d\mathbf{x}.
\end{aligned}
$$

For $v=y_{(g+\lambda r)}-y_g$, we get

$$
\begin{aligned}
&\int_D e_\eta^3(g)\left[y_{(g+\lambda r)}-y_g,y_{(g+\lambda r)}-y_g\right]_{\#}\,d\mathbf{x} \\
=&\ -\int_D (e_\eta^3(g+\lambda r)-e_\eta^3(g))[y_{(g+\lambda r)},y_{(g+\lambda r)}-y_g]_{\#}\,d\mathbf{x}.
\end{aligned}
$$

It follows from the coercivity of a_η and Cauchy-Schwarz inequality

$$
\begin{aligned}
&C_1\left\|y_{(g+\lambda r)}-y_g\right\|_{2,D}^2 \\
\leq&\ \max_{\mathbf{x}\in\overline{D}}\left|e_\eta^3(g+\lambda r)-e_\eta^3(g)\right|\left\|y_{(g+\lambda r)}\right\|_{2,D}\left\|y_{(g+\lambda r)}-y_g\right\|_{2,D}C_3.
\end{aligned}
$$

But $e_1\leq e_\eta(g)\leq e_0$, for all g and from (9), we obtain

$$C_1\left\|y_{(g+\lambda r)}-y_g\right\|_{2,D}\leq\max_{\mathbf{x}\in\overline{D}}\left|e_\eta(g+\lambda r)-e_\eta(g)\right|3e_0^2\,C\|f\|_{V'}C_3.$$

Since H'_η is bounded in $\mathbb{R}$ and g is continuous in $\overline{D}$, we have

$$\begin{aligned}|e_\eta(g+\lambda r)-e_\eta(g)| &= (e_0-e_1)\,|H_\eta(g+\lambda r)-H_\eta(g)| \\ &\leq |\lambda|(e_0-e_1)\max_{\mathbb{R}}\left|H'_\eta\right|\max_{\mathbf{x}\in\overline{D}}|r(\mathbf{x})|\end{aligned}$$

and the conclusion of i) is proved. The boundedness of H'_η for η fixed from (6) was used at the end of the proof of i).

ii) *Weak convergence.* We have deduced at i) that

$$C_1\left\|\frac{y_{(g+\lambda r)}-y_g}{\lambda}\right\|_{2,D}\leq (e_0-e_1)\max_{\mathbb{R}}\left|H'_\eta\right|\max_{\mathbf{x}\in\overline{D}}|r(\mathbf{x})|3e_0^2\,C\|f\|_{V'}C_3$$

then there exits $\tilde{z}\in V$ and z_λ such that z_λ converges weakly to $\tilde{z}$ on a subsequence.

Subtracting (7) written for g from (7) written for $(g+\lambda r)$ and dividing by λ, we get

$$\begin{aligned}0 &= \int_D \frac{e_\eta^3(g+\lambda r)-e_\eta^3(g)}{\lambda}[y_{(g+\lambda r)},v]_{\#}\,d\mathbf{x}+\int_D e_\eta^3(g)\left[\frac{y_{(g+\lambda r)}-y_g}{\lambda},v\right]_{\#}d\mathbf{x}\\ &= \int_D \frac{e_\eta^3(g+\lambda r)-e_\eta^3(g)}{\lambda}[y_{(g+\lambda r)}-y_g,v]_{\#}\,d\mathbf{x}\\ &+\int_D \frac{e_\eta^3(g+\lambda r)-e_\eta^3(g)}{\lambda}[y_g,v]_{\#}\,d\mathbf{x}+\int_D e_\eta^3(g)\left[\frac{y_{(g+\lambda r)}-y_g}{\lambda},v\right]_{\#}d\mathbf{x}.\end{aligned}\tag{10}$$

We have the relations

$$\begin{aligned}&\frac{e_\eta^3(g+\lambda r)-e_\eta^3(g)}{\lambda}\\ =\;&(e_0-e_1)\frac{H_\eta(g+\lambda r)-H_\eta(g)}{\lambda}(e_\eta^2(g+\lambda r)+e_\eta(g+\lambda r)e_\eta(g)+e_\eta^2(g))\end{aligned}$$

and from the Lipschitz propriety of H_η we obtain

$$\forall\mathbf{x}\in\overline{D},\quad\left|\frac{H_\eta(g(\mathbf{x})+\lambda r(\mathbf{x}))-H_\eta(g(\mathbf{x}))}{\lambda}\right|\leq\max_{\mathbb{R}}\left|H'_\eta\right|\max_{\mathbf{x}\in\overline{D}}|r(\mathbf{x})|.\tag{11}$$

It follows using $e_\eta(g)\leq e_0$

$$\forall\mathbf{x}\in\overline{D},\quad\left|\frac{e_\eta^3(g(\mathbf{x})+\lambda r(\mathbf{x}))-e_\eta^3(g(\mathbf{x}))}{\lambda}\right|\leq(e_0-e_1)\max_{\mathbb{R}}\left|H'_\eta\right|\max_{\mathbf{x}\in\overline{D}}|r(\mathbf{x})|3e_0^2.$$

In addition, we have the pointwise convergence

$$\frac{H_\eta(g(\mathbf{x})+\lambda r(\mathbf{x}))-H_\eta(g(\mathbf{x}))}{\lambda} \to H'_\eta(g(\mathbf{x}))r(\mathbf{x}), \quad \forall \mathbf{x}\in \overline{D} \tag{12}$$

then

$$\frac{e_\eta(g(\mathbf{x})+\lambda r(\mathbf{x}))^3-e_\eta(g(\mathbf{x}))^3}{\lambda} \to (e_0-e_1)3e_0^2 H'_\eta(g(\mathbf{x}))r(\mathbf{x}), \quad \forall \mathbf{x}\in \overline{D}.$$

We get that

$$\left|\int_D \frac{e^3_\eta(g+\lambda r)-e^3_\eta(g)}{\lambda}[y_{(g+\lambda r)}-y_g, v]_\# \, d\mathbf{x}\right| \le C \left\|y_{(g+\lambda r)}-y_g\right\|_{2,D} \|v\|_{2,D}\,,$$

but $y_{(g+\lambda r)} \to y_g$ strongly in V, then

$$\int_D \frac{e^3_\eta(g+\lambda r)-e^3_\eta(g)}{\lambda}[y_{(g+\lambda r)}-y_g, v]_\# \, d\mathbf{x} \to 0.$$

From the Lebesgue's dominated convergence theorem, we obtain

$$\int_D \frac{e^3_\eta(g+\lambda r)-e^3_\eta(g)}{\lambda}[y_g, v]_\# \, d\mathbf{x} \to \int_D (e_0-e_1)3e_0^2\, H'_\eta(g)r\,[y_g, v]_\# \, d\mathbf{x}.$$

Since $z_\lambda \to \tilde{z}$ weakly on a subsequence, we obtain

$$\int_D e^3_\eta(g)\left[\frac{y_{(g+\lambda r)}-y_g}{\lambda}, v\right]_\# \, d\mathbf{x} \to \int_D e^3_\eta(g)\,[\tilde{z}, v]_\# \, d\mathbf{x}$$

Passing to the limit $\lambda \to 0$ in (10), we obtain

$$a_\eta(\tilde{z}, v) = -\frac{\partial a_\eta}{\partial g}(y_g, v)r, \;\forall v \in V$$

but the problem (9) has a unique solution denoted by $q\in V$. Then $\tilde{z}=q$ and the weak convergence of $z_\lambda \to q$ holds without taking subsequence.

Strong convergence. Putting $v = z_\lambda - q$ in (10) and using (9) we obtain

$$\begin{aligned}
& a_\eta(z_\lambda - q, z_\lambda - q) = a_\eta(z_\lambda, z_\lambda - q) - a_\eta(q, z_\lambda - q)\\
= & \int_D e^3_\eta(g)\left[\frac{y_{(g+\lambda r)}-y_g}{\lambda}, z_\lambda - q\right]_\# \, d\mathbf{x} - a_\eta(q, z_\lambda - q)\\
= & -\int_D \frac{e^3_\eta(g+\lambda r)-e^3_\eta(g)}{\lambda}[y_{(g+\lambda r)}-y_g, z_\lambda - q]_\# \, d\mathbf{x}\\
& -\int_D \frac{e^3_\eta(g+\lambda r)-e^3_\eta(g)}{\lambda}[y_g, z_\lambda - q]_\# \, d\mathbf{x} + \frac{\partial a_\eta}{\partial g}(y_g, z_\lambda - q)r
\end{aligned} \tag{13}$$

We have

$$\left| \int_D \frac{e_\eta^3(g+\lambda r) - e_\eta^3(g)}{\lambda} [y_{(g+\lambda r)} - y_g, z_\lambda - q]_\# \, d\mathbf{x} \right|$$
$$\leq \quad C \left\| y_{(g+\lambda r)} - y_g \right\|_{2,D} \| z_\lambda - q \|_{2,D}$$

and the last line of (13) is equal to

$$\int_D \left((e_0 - e_1) 3 e^2(g) \, H'_\eta(g) r - \frac{e_\eta^3(g+\lambda r) - e_\eta^3(g)}{\lambda} \right) [y_g, z_\lambda - q]_\# \, d\mathbf{x}.$$

We have $e_\eta(t) = e_1 + (e_0 - e_1) H_\eta(t)$, then $e'_\eta(t) = (e_0 - e_1) H'_\eta(t)$, $e''_\eta(t) = (e_0 - e_1) H''_\eta(t)$. Using Taylor formula of order two for the real function $t \to e_\eta^3(t)$ and the assumption H''_η bounded, we get

$$\left| e_\eta^3(t_1) - e_\eta^3(t_0) + (t_1 - t_0) 3 e_\eta^2(t_0)(e_0 - e_1) H'_\eta(t_0) \right| \leq (t_1 - t_0)^2 M, \ \forall t_0, t_1 \in \mathbb{R}.$$

For $t_0 = g(\mathbf{x})$ and $t_1 = g(\mathbf{x}) + \lambda r(\mathbf{x})$, we obtain

$$\left| (e_0 - e_1) 3 e^2(g(\mathbf{x})) \, H'_\eta(g(\mathbf{x})) r(\mathbf{x}) - \frac{e_\eta^3(g + (\mathbf{x}) \lambda r(\mathbf{x})) - e_\eta^3(g(\mathbf{x}))}{\lambda} \right|$$
$$\leq \quad \lambda r^2(\mathbf{x}) M \leq \lambda \max_{\mathbf{x} \in \overline{D}} |r^2(\mathbf{x})| M.$$

Finally, from the coercivity of a_η, (13) and the above estimations, we obtain

$$\begin{aligned} C_1 \| z_\lambda - q \|_{2,D}^2 \quad &\leq \quad C \left\| y_{(g+\lambda r)} - y_g \right\|_{2,D} \| z_\lambda - q \|_{2,D} \\ &\quad + \lambda \max_{\mathbf{x} \in \overline{D}} |r^2(\mathbf{x})| M \, \| y_g \|_{2,D} \, \| z_\lambda - q \|_{2,D} \, . \end{aligned}$$

But $\left\| y_{(g+\lambda r)} - y_g \right\|_{2,D} \to 0$ when $\lambda \to 0$, then we get that z_λ converges strongly to q in V. The boundedness of H''_η from (6) was used for the proof of the strong convergence in ii). $\square$

We consider N sensors located at $\mathbf{x}_n \in D$, $n = 1, \ldots, N$. Since $V \subset \mathcal{C}(\overline{D})$, we can define $y_g(\mathbf{x}_n)$, but not $\nabla y_g(\mathbf{x}_n)$. Formally, we want to solve the optimization problem:

$$\inf_{g \in \mathcal{F}} J(g) = \sum_{n=1}^{N} \left((y_g - y^*)^2 + \left(\frac{\partial y_g}{\partial x_1} - \frac{\partial y^*}{\partial x_1} \right)^2 + \left(\frac{\partial y_g}{\partial x_2} - \frac{\partial y^*}{\partial x_2} \right)^2 \right) (\mathbf{x}_n) \tag{14}$$

for given $y^* \in \mathcal{C}^1(D)$. This cost function is well adapted when the shape of the industrial deformed part y^* could be measured only pointwise by a finite number of sensors. The aim is to detect the damaged zone $\Omega^* \subset D$ using only pointwise observation of y^*, the displacement of the plate corresponding to Ω^*. It is an inverse problem.

For testing the descent direction algorithm in the last section, the given displacement y^* is computed by the Finite Element Method, corresponding to a given Ω^*. We point out that the algorithm does not "know" Ω^*. The continuous version of this optimization problem has a global minimum corresponding to Ω^* and the optimal value of the cost function is zero. We ignore if the global minimum is unique.

We can increase the regularity of y_g by using $\varphi_\varepsilon(\mathbf{x} - \boldsymbol{\xi}_m)$, a mollifier approximation of the Dirac functional $\delta(\boldsymbol{\xi}_m)$. We put $\varphi_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^2}\varphi\left(\frac{\mathbf{x}}{\varepsilon}\right)$, where $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2)$ with support in the unitary disk centered at $(0,0)$ and $\int_{\mathbb{R}^2} \varphi(\mathbf{x})\, d\mathbf{x} = 1$. In this case, we get $f_\varepsilon = \sum_{m=1}^M b_0 \varphi_\varepsilon(\mathbf{x} - \boldsymbol{\xi}_m) \in \mathcal{C}^\infty(D)$. If we use smooth coefficient $e_\eta(g) \in \mathcal{C}^2(D)$, for example $g \in \mathcal{F} \subset \mathcal{C}^2(D)$, from [1], Ch. 9, for $f_\varepsilon \in L^2(D)$ we obtain $y_g \in H^4_{loc}(D)$ and

$$\|y_g\|_{4,D_1} \leq C \, \|f_\varepsilon\|_{0,D}$$

where D_1 is an open set with $\mathcal{C}^1$ boundary, such that $\{\mathbf{x}_n\}_{1\leq n\leq N} \subset D_1 \subset \overline{D}_1 \subset D$.

From the Sobolev inequalities, see [13], Ch. 5 for example, we have $H^4(D_1) \subset \mathcal{C}^{2,\gamma}(\overline{D}_1)$, with $0 < \gamma < 1$ and in addition, we have

$$\|y_g\|_{\mathcal{C}^{2,\gamma}(\overline{D}_1)} \leq C \, \|y_g\|_{4,D_1} .$$

For the interior regularity, it is not necessary supplementary smoothness of ∂D. In the case where some sensors $\mathbf{x}_n$ are on ∂D, we have to assume then ∂D is at least $\mathcal{C}^4$.

We introduce the adjoint problem: find $p_g \in V$, such that

$$a_\eta(v, p_g) = 2 \sum_{n=1}^N \left((y - y^*)v + \frac{\partial (y_g - y^*)}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial (y_g - y^*)}{\partial x_2} \frac{\partial v}{\partial x_2} \right) (\mathbf{x}_n) \tag{15}$$

for all $v \in V$.

Proposition 3. *The directional derivative of J given by (14) at g in the direction r is*

$$J'(g)r = -\frac{\partial a_\eta}{\partial g}(y_g, p_g) r. \tag{16}$$

Proof. We have for $\lambda \neq 0$

$$
\begin{aligned}
& \frac{(y_{(g+\lambda r)} - y^*)^2(\mathbf{x}_n) - (y_g - y^*)^2(\mathbf{x}_n)}{\lambda} \\
= \quad & \frac{(y_{(g+\lambda r)} - y_g)(\mathbf{x}_n)}{\lambda} \left(y_{(g+\lambda r)} - 2y^* + y_g\right)(\mathbf{x}_n).
\end{aligned}
$$

We have proved Prop. 2, that $y_{(g+\lambda r)}$ converges to y_g and z_λ converges to q strongly in $V = H_0^2(D)$. We have from the Sobolev inequalities

$$
\begin{aligned}
\left\|y_{(g+\lambda r)} - y_g\right\|_{\mathcal{C}^{0,\gamma}(\overline{D})} & \leq C\left\|y_{(g+\lambda r)} - y_g\right\|_{2,D}, \\
\|z_\lambda - q\|_{\mathcal{C}^{0,\gamma}(\overline{D})} & \leq C\|z_\lambda - q\|_{2,D},
\end{aligned}
$$

it follows $\lim_{\lambda\to 0} y_{(g+\lambda r)}(\mathbf{x}_n) = y_g(\mathbf{x}_n)$ and $\lim_{\lambda\to 0} \frac{(y_{(g+\lambda r)} - y_g)(\mathbf{x}_n)}{\lambda} = q(\mathbf{x}_n)$ then

$$
\lim_{\lambda\to 0} \frac{(y_{(g+\lambda r)} - y^*)^2(\mathbf{x}_n) - (y_g - y^*)^2(\mathbf{x}_n)}{\lambda} = q(\mathbf{x}_n)2(y_g - y^*)(\mathbf{x}_n).
$$

Also, we have

$$
\begin{aligned}
& \frac{\partial}{\partial x_1}\left(\frac{(y_{(g+\lambda r)} - y^*)^2 - (y_g - y^*)^2}{\lambda}\right) \\
= \quad & \frac{\partial}{\partial x_1}\left(\frac{(y_{(g+\lambda r)} - y_g)}{\lambda}\left(y_{(g+\lambda r)} - 2y^* + y_g\right)\right) \\
= \quad & \left(y_{(g+\lambda r)} - 2y^* + y_g\right)\frac{\partial}{\partial x_1}\left(\frac{y_{(g+\lambda r)} - y_g}{\lambda}\right) \\
& + \left(\frac{y_{(g+\lambda r)} - y_g}{\lambda}\right)\frac{\partial}{\partial x_1}\left(y_{(g+\lambda r)} - 2y^* + y_g\right).
\end{aligned}
$$

Since the restriction to $\overline{D}_1$ of $y_{(g+\lambda r)}, y_g, y^*$ are in $\mathcal{C}^2(\overline{D}_1)$, then for $\lambda \to 0$

$$
\begin{aligned}
\left(y_{(g+\lambda r)} - 2y^* + y_g\right) & \rightarrow 2\left(y_g - y^*\right), \text{ in } \mathcal{C}^2(\overline{D}_1) \\
\frac{\partial}{\partial x_1}\left(y_{(g+\lambda r)} - 2y^* + y_g\right) & \rightarrow 2\frac{\partial}{\partial x_1}\left(y_g - y^*\right), \text{ in } \mathcal{C}^1(\overline{D}_1).
\end{aligned}
$$

Moreover, we have

$$
z_\lambda \rightarrow q, \text{ in } \mathcal{C}^0(\overline{D}_1)
$$

then $z_\lambda(\mathbf{x}_n) \to q(\mathbf{x}_n)$.

Now we study the convergence of $\frac{\partial z_\lambda}{\partial x_1}(\mathbf{x}_n)$. In the beginning of the proof of Prop. 2, we have deduced

$$\int_D e_\eta^3(g)\,[z_\lambda, v]_\#\,d\mathbf{x} = -\frac{1}{\lambda}\int_D (e_\eta^3(g+\lambda r) - e_\eta^3(g))[y_{(g+\lambda r)}, v]_\#\,d\mathbf{x},\ \forall v \in V.$$

But the coefficients $e_\eta^3(g+\lambda r)$ are in $\mathcal{C}^2(D)$ and $y_{(g+\lambda r)} \in H_0^2(D)$, then the right-hand side is in the dual of $L^2(D)$. Consequently, $z_\lambda \in H^4(D_1) \subset \mathcal{C}^{2,\gamma}(\overline{D}_1)$ and $\|z_\lambda\|_{4,D_1}$ bounded. There exists $\hat{z} \in H^4(D_1)$ such that z_λ converges in a subsequence to $\hat{z}$ weakly in $H^4(D_1)$ and strongly in $H^2(D_1)$. Then $\hat{z} = q$ in D_1 and the convergence holds without taking subsequences. It follows that $\frac{\partial z_\lambda}{\partial x_1}(\mathbf{x}_n) \to \frac{\partial q}{\partial x_1}(\mathbf{x}_n)$.

Finally, we get

$$J'(g)r = 2\sum_{n=1}^{N}\left((y-y^*)q + \frac{\partial(y_g-y^*)}{\partial x_1}\frac{\partial q}{\partial x_1} + \frac{\partial(y_g-y^*)}{\partial x_2}\frac{\partial q}{\partial x_2}\right)(\mathbf{x}_n). \quad (17)$$

Putting $v = p_g$ in (9) and $v = q$ in (15), we get the conclusion. □

Remark 1. *Putting*

$$\begin{aligned} d_g &= \nu\Delta y_g \Delta p_g \\ &\quad +(1-\nu)\left(\frac{\partial^2 y_g}{\partial x_1^2}\frac{\partial^2 p_g}{\partial x_1^2} + 2\frac{\partial^2 y_g}{\partial x_1\partial x_2}\frac{\partial^2 p_g}{\partial x_1\partial x_2} + \frac{\partial^2 y_g}{\partial x_2^2}\frac{\partial^2 p_g}{\partial x_2^2}\right), \end{aligned} \quad (18)$$

since $0 < \nu < \frac{1}{2}$, $e_0 - e_1 > 0$, $H'_\eta \geq 0$, *it follows*

$$-\frac{\partial a_\eta}{\partial g}(y_g, p_g)d_g = -\int_D \frac{E}{12(1-\nu^2)}3e_\eta^2(g)(e_0-e_1)H'_\eta(g)d_g^2 d\mathbf{x} \leq 0,$$

then $r_g = d_g$ *is a descent direction for* J *at* g, *i.e.* $J'(g)r_g < 0$, *if* $J'(g)r_g \neq 0$. *The condition* $J'(g)r_g = 0$ *could be used as stopping criteria for the descent direction algorithm. For an introduction to descent direction methods, see [9]. Similarly, we have that* $-\frac{\partial a_\eta}{\partial g}(y_g, p_g)r_g$ *is equal to*

$$\begin{aligned} &-\int_D \frac{E}{12(1-\nu^2)}3e_\eta^2(g)(e_0-e_1)\left(H'_\eta(g)d_g\right)^2 d\mathbf{x} \leq 0,\ \text{for } r_g = H'_\eta(g)d_g \\ &-\int_D \frac{E}{12(1-\nu^2)}3e_\eta^3(g)(e_0-e_1)H'_\eta(g)d_g^2 d\mathbf{x} \leq 0,\ \text{for } r_g = e_\eta(g)d_g \\ &-\int_D \frac{E}{12(1-\nu^2)}3e_\eta^3(g)(e_0-e_1)\left(H'_\eta(g)d_g\right)^2 d\mathbf{x} \leq 0,\ \text{for } r_g = e_\eta(g)H'_\eta(g)d_g. \end{aligned}$$

Then, $r_g = H'_\eta(g)d_g$, $r_g = e_\eta(g)d_g$ *and* $r_g = e_\eta(g)H'_\eta(g)d_g$ *are descent directions, too, if* $J'(g)r_g \neq 0$.

3 Compliance minimization of Kirchhoff-Love plate with elastic support

Let $D \subset \mathbb{R}^2$ be a given bounded Lipschitz domain and $\Omega \subset D$ an open set. We consider clamped boundary condition and we put $V = H_0^2(D)$. We consider $\ell \in V'$ as in (3) with $f = b_0 \sum_{m=1}^{M} \delta(\boldsymbol{\xi}_m)$, where $b_0 \in \mathbb{R}$ and $\delta(\boldsymbol{\xi}_m)$ is the Dirac function of pole $\boldsymbol{\xi}_m$. We follow [8] and we introduce $a^c : V \times V \to \mathbb{R}$ given by

$$
\begin{aligned}
a^c(w, v) = \int_D \alpha(\mathbf{x}) \frac{Ee^3}{12(1-\nu^2)} \Big[& \nu \Delta w \Delta v \\
& + (1-\nu) \left(\frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + 2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 w}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right) \Big] d\mathbf{x} \\
& + \int_D s(\mathbf{x})\, wv \, d\mathbf{x} \qquad (19)
\end{aligned}
$$

where $E > 0$ the Young modulus, $0 < \nu < 1/2$ the Poisson ratio, e the thickness are constants, but $\alpha, s : D \to \mathbb{R}$

$$
\alpha(\mathbf{x}) = \begin{cases} \alpha_0, & \mathbf{x} \in \Omega \\ \alpha_1, & \mathbf{x} \in D \setminus \overline{\Omega} \end{cases} \qquad s(\mathbf{x}) = \begin{cases} s_0, & \mathbf{x} \in \Omega \\ s_1, & \mathbf{x} \in D \setminus \overline{\Omega} \end{cases} \qquad (20)
$$

where $\alpha_0, \alpha_1 > 0$, $s_0, s_1 \geq 0$.

The problem is to find $y \in V$, such that

$$
a^c(y, v) = \ell(v), \ \forall v \in V. \qquad (21)
$$

and it can be interpreted as a clamped Kirchhoff-Love plate with elastic support, see [8]. The term containing s in (19) is modeling the elastic support, like springs, which reduces the displacement of the plate. We can interpret s as the stiffness of the elastic support. The problem (21) has a unique solution.

The elipticity of a^c is based on $\min(\alpha_0, \alpha_1) > 0$, in other words, there are two materials, one in Ω with coefficients depending on α_0, s_0 and the other in $D \setminus \overline{\Omega}$ with coefficients depending on α_1, s_1. The clamped boundary conditions all over ∂D are used, too. In [19], the minimization of the compliance for linear elasticity equation is analyzed using similar technique. But there is only one material. The elipticity in this case is obtained if the material "touches" a part of ∂D, where the homogeneous Dirichlet boundary condition is imposed.

Let $a^c_\eta : V \times V \to \mathbb{R}$ be obtained from (19) by replacing α, s by

$$\alpha_\eta(g) = \alpha_0(1 - H_\eta(g)) + \alpha_1 H_\eta(g) \quad s_\eta(g) = s_0(1 - H_\eta(g)) + s_1 H_\eta(g)$$

for $g \in \mathcal{F} \subset \mathcal{C}(\overline{D})$ a parametrization of Ω as in (5) and $H_\eta \in \mathcal{C}^1(\mathbb{R})$, $0 \leq H_\eta \leq 1$, $0 \leq H'_\eta$, H'_η bounded. The regularized problem is: find $y_g \in V$ such that

$$a^c_\eta(y_g, v) = \ell(v),\ \forall v \in V. \tag{22}$$

Since $0 \leq H_\eta(g) \leq 1$, we can get $\alpha(g) = \alpha_1 + (1 - H_\eta(g))(\alpha_0 - \alpha_1) \geq \alpha_1$ if $\alpha_0 - \alpha_1 \geq 0$ or $\alpha_\eta(g) = \alpha_0 + H_\eta(g)(\alpha_1 - \alpha_0) \geq \alpha_0$, if $\alpha_1 - \alpha_0 \geq 0$ and similarly we get $s_\eta(g) \geq \min(s_0, s_1) \geq 0$, then a^c_η is elliptic. It follows that the regularized problem has a unique solution $y_g \in V$.

We define the compliance $j : \mathcal{F} \to \mathbb{R}$ by

$$j(g) = \ell(y_g) \tag{23}$$

(the work done by the load, see [3], p. 5) and we observe that $j(g) = a^c_\eta(y_g, y_g)$ from (22).

Let $j_D = j(g)$, for a g such that $\max_{\mathbf{x}\in\overline{D}} g(\mathbf{x}) < 0$ and $|\Omega|$ the area of set Ω. Also, we introduce $\mathcal{A} : \mathcal{F} \to \mathbb{R}$ by

$$\mathcal{A}(g) = \int_D (1 - H_\eta(g)) d\mathbf{x} \tag{24}$$

and $\mathcal{A}(g)$ is an approximation of $|\Omega|$. The cost functional to be minimized adapted from [8] is

$$J(g) = \frac{j(g)}{j_D} + \mu \frac{\mathcal{A}(g)}{|D|} \tag{25}$$

where $\mu > 0$ is a penalization term for the area of Ω. This cost function combines the relative compliance $\frac{j(g)}{j_D}$ and an approximation of the volume fraction $\frac{|\Omega|}{|D|}$. It is more appropriate to compare the performance of different numerical methods.

For $w, v \in V$ and $r \in \mathcal{F}$, we put

$$\begin{aligned}
\frac{\partial a^c_\eta}{\partial g}(w, v) r \quad = \quad & \int_D H'_\eta(g) r \frac{(\alpha_1 - \alpha_0) E e^3}{12(1 - \nu^2)} \Bigg[\nu \Delta w \Delta v \\
& + (1 - \nu) \left(\frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + 2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 w}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right) \Bigg] d\mathbf{x} \\
& + \int_D H'_\eta(g) r (s_1 - s_0) w v \, d\mathbf{x}.
\end{aligned}$$

We have similar results as in the precedent Section. The proof of the Proposition 4 follows the same techniques as for the Proposition 2.

Proposition 4. *When* $\lambda \to 0$, $z_\lambda = \frac{y_{(g+\lambda r)} - y_g}{\lambda}$ *converges to* q *strongly in* V*, where* q *is the unique solution of*

$$a_\eta^c(q, v) = -\frac{\partial a_\eta^c}{\partial g}(y_g, v)r, \ \forall v \in V. \tag{26}$$

Proposition 5. *The directional derivative of the cost function (25) is*

$$J'(g)r = -\frac{1}{j_D}\frac{\partial a_\eta^c}{\partial g}(y_g, y_g)r - \frac{\mu}{|D|}\int_D H_\eta'(g)r \, d\mathbf{x}. \tag{27}$$

Proof. From the Lipschitz propriety of H_η (11), the pointwise convergence (12) and the Lebesgue's dominated convergence theorem, we get the directional derivative of $\mathcal{A}$ exists and

$$\mathcal{A}'(g)r = -\int_D H_\eta'(g)r \, d\mathbf{x}.$$

Since $\ell \in V'$, and from the Proposition 4, we have

$$\begin{aligned} &\frac{\ell(y_{(g+\lambda r)}) - \ell(y_g)}{\lambda} = \ell\left(\frac{y_{(g+\lambda r)} - y_g}{\lambda}\right) = \ell(z_\lambda) \\ \to \quad &\ell(q) = a_\eta^c(y_g, q) = a_\eta^c(q, y_g) = -\frac{\partial a_\eta^c}{\partial g}(y_g, y_g)r \end{aligned}$$

which gives the conclusion. □

Remark 2. *Putting*

$$\begin{aligned} d_g^c &= \frac{1}{j_D}\frac{(\alpha_1 - \alpha_0)Ee^3}{12(1-\nu^2)}\Bigg[\nu(\Delta y_g)^2 \\ &\quad +(1-\nu)\left(\left(\frac{\partial^2 y_g}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 y_g}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 y_g}{\partial x_2^2}\right)^2\right)\Bigg] \\ &\quad + \frac{1}{j_D}(s_1 - s_0)(y_g)^2 + \frac{\mu}{|D|} \end{aligned} \tag{28}$$

it follows that $J'(g)r = -\int_D H_\eta'(g) r d_g^c \, d\mathbf{x}$*, consequently for* $r_g = H_\eta'(g)d_g^c$ *we get* $J'(g)r_g = -\int_D \left(H_\eta'(g)d_g^c\right)^2 \, d\mathbf{x} \le 0$*, then* r_g *is a descent direction for* J *at* g*, if* $J'(g)r_g \neq 0$*. Since* $H_\eta'(g) \ge 0$, $J'(g)d_g = -\int_D H_\eta'(g)\left(d_g^c\right)^2 \, d\mathbf{x} \le 0$*, then* $r_g = d_g^c$ *is a descent direction, too, if* $J'(g)r_g \neq 0$*.*

4 Maximization of the first eigenvalue for Kirchhoff-Love plates

Let $D \subset \mathbb{R}^2$ be a given bounded Lipschitz domain and $\Omega \subset\subset D$ an open set. We set $V = H_0^2(D)$. For $g \in \mathcal{F} \subset \mathcal{C}(\overline{D})$, we introduce $a_\eta^e : V \times V \to \mathbb{R}$ given by

$$a_\eta^e(w,v) = \int_D \alpha_\eta(g) \frac{Ee^3}{12(1-\nu^2)} \Big[\nu \Delta w \Delta v + (1-\nu) \left(\frac{\partial^2 w}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + 2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 w}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right) \Big] d\mathbf{x} \qquad (29)$$

where $E > 0$ the Young modulus, $0 < \nu < 1/2$ the Poisson ratio, e the thickness are constants, but $\alpha_\eta(g) = \alpha_0(1 - H_\eta(g)) + \alpha_1 H_\eta(g)$ in D with $0 < \alpha_1 < \alpha_0$, H_η verifiyng (6). Also, we define $b_\eta^e : V \times V \to \mathbb{R}$ given by

$$b_\eta^e(w,v) = \int_D \rho_\eta(g) e\, wv\, d\mathbf{x} + \int_D \left(\sum_{m=1}^M b_m \varphi_\varepsilon(\mathbf{x} - \boldsymbol{\xi}_m) \right) e\, wv\, d\mathbf{x} \qquad (30)$$

where $b_m > 0$ are constants, $\boldsymbol{\xi}_m \in D$ are the concentrated mass locations, $m = 1, \dots, M$ and $\rho_\eta(g) = \rho_0(1 - H_\eta(g)) + \rho_1 H_\eta(g)$ in D with $0 < \rho_1 < \rho_0$. We set

$$f_\varepsilon(g) = \rho_\eta(g) e + \sum_{m=1}^M b_m \varphi_\varepsilon(\mathbf{x} - \boldsymbol{\xi}_m) e$$

then $b_\eta^e(w,v) = \int_D f_\varepsilon(g) wv\, d\mathbf{x}$ with $f \in L^2(D)$.

As in the precedent sections, we can prove that a_η^e is elliptic, continuous, symmetric and b_η^e is continuous, symmetric and $b_\eta^e(v,v) \geq \rho_1 e \|v\|_{0,D}^2$, for all $v \in V$, based on $b_m > 0$. In addition, the continuous inclusion $H_0^2(D) \subset L^2(D)$ is compact, $L^2(D)$ is separable and $H_0^2(D)$ is dense in $L^2(D)$. In fact, $b_\eta^e(\cdot,\cdot)$ is a scalar product in $L^2(D)$ and the induced norm is equivalent to $\|\cdot\|_{0,D}$.

The eigenvalue $\lambda_i(g) \in \mathbb{R}$ and its associated eigenfunction $\phi_i(g) \in V$ are verifying

$$a_\eta^e(\phi_i(g), v) = \lambda_i(g)\, b_\eta^e(\phi_i(g), v), \quad \forall v \in V \qquad (31)$$

$$b_\eta^e(\phi_i(g), \phi_i(g)) = 1 \qquad (32)$$

for all $i \in \mathbb{N} \setminus \{0\}$. We have from [10], vol. 3, Ch. VIII, Sect. 2.6, Th. 7 that

$$0 < \lambda_1(g) \leq \lambda_2(g) \leq \dots \leq \lambda_i(g) \leq \dots$$

with $\lim_{i\to\infty}\lambda_i(g) = +\infty$.

We intend to maximize $\lambda_1(g)$ or equivalently to minimize $\frac{1}{\lambda_1(g)}$. Optimization of the first eigenvalue under constant area of Ω is studied in many works, see for example [15], [7] or [16]. Here, we relax this constraint and we introduce

$$J(g) = \frac{\lambda_{1,D}}{\lambda_1(g)} + \mu\frac{\mathcal{A}(g)}{|D|} \tag{33}$$

where $\lambda_{1,D} > 0$ is a reference first eigenvalue obtained for some $g < 0$ in D, $\mu > 0$ is a penalization term for the area of Ω and $\mathcal{A}(g)$ is given by (24).

The derivability of a simple eigenvalue with respect to the variation of domain by deformation field is presented in [15], Ch. 5, Sect. 7 for the second order PDE and in [7] for the fourth order problem. The proof is based on the Implicit Function Theorem. For the second-order elliptic operator, the first eigenvalue is simple, see [13], Sect. 6.5, Th. 2. The proof is based on the positivity of the eigenfunction. For the clamped plate in general domains, positivity preserving property does not hold, see [32], [11].

To simplify the calculus, we note

$$[w,v] = \nu\Delta w\Delta v + (1-\nu)\left(\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + 2\frac{\partial^2 w}{\partial x_1\partial x_2}\frac{\partial^2 v}{\partial x_1\partial x_2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2}\right).$$

For $w, v \in V$ fixed, $r \in \mathcal{F}$ and g an interior element of $\mathcal{F}$ for the topology of $\mathcal{C}(\overline{D})$, the applications $g \to a_\eta^e(w,v)$ and $g \to b_\eta^e(w,v)$ are Fréchet differentiable and

$$\begin{aligned}
\frac{\partial a_\eta^e}{\partial g}(w,v)r &= \int_D H_\eta'(g)r\,\frac{(\alpha_1-\alpha_0)E\,e^3}{12(1-\nu^2)}\,[w,v]\,d\mathbf{x}\\
\frac{\partial b_\eta^e}{\partial g}(w,v) &= \int_D H_\eta'(g)r\,(\rho_1-\rho_0)ewv\,d\mathbf{x}.
\end{aligned}$$

We recall that $\mathcal{F}$ is supposed an open cone.

Proposition 6. *Assuming that $\lambda_1(g)$ is simple, the application $g \in \mathcal{F} \to \lambda_1(g) \in \mathbb{R}$ is Fréchet differentiable and for all $r \in \mathcal{F}$ we have*

$$\lambda_1'(g)r = \frac{\partial a_\eta^e}{\partial g}(\phi_1(g),\phi_1(g))r - \lambda_1(g)\frac{\partial b_\eta^e}{\partial g}(\phi_1(g),\phi_1(g))r. \tag{34}$$

Also, the application $g \in \mathcal{F} \to \phi_1(g) \in V$ is Fréchet differentiable and

$q = \phi_1'(g)r \in V$ *is the unique solution of*

$$a_\eta^e(q,v) - \lambda_1(g)\, b_\eta^e(q,v) = \lambda_1'(g) r\, b_\eta^e(\phi_1(g),v) - \frac{\partial a_\eta^e}{\partial g}(\phi_1(g),v) r + \lambda_1(g)\, \frac{\partial b_\eta^e}{\partial g}(\phi_1(g),v), \forall v \in V, \tag{35}$$

$$2 b_\eta^e(q,\phi_1(g)) = -\frac{\partial b_\eta^e}{\partial g}(\phi_1(g),\phi_1(g)). \tag{36}$$

Proof. We adapt the proofs from [15], Ch. 5, Sect. 7 and [7].

We can define $A(g) : V \to V$ and $B(g) : V \to V$ by

$$a_\eta^e(w,v) = (A(g)w,v)_{2,D}, \quad b_\eta^e(w,v) = (B(g)w,v)_{2,D}$$

and introduce $\mathcal{H} : \mathcal{F} \times V \times \mathbb{R} \to V \times \mathbb{R}$

$$\mathcal{H}(g,v,\lambda) = \left(A(g)v - \lambda B(g)v, (B(g)v,v)_{2,D} - 1\right).$$

We have $\mathcal{H}(g,\phi_1(g),\lambda_1(g)) = (0,0)$ and the partial derivative of $\mathcal{H}$ with respect to (v,λ) exists in $\mathcal{L}(V\times\mathbb{R}, V\times\mathbb{R})$ and for all $(\hat{v},\hat{\lambda}) \in V\times\mathbb{R}$, it has the form

$$\frac{\partial \mathcal{H}}{\partial(v,\lambda)}(g,v,\lambda)(\hat{v},\hat{\lambda}) = \Big(A(g)\hat{v} - \lambda B(g)\hat{v} - \hat{\lambda}B(g)v, 2(B(g)v,\hat{v})_{2,D}\Big) \in V\times\mathbb{R}.$$

We want to prove that $\frac{\partial \mathcal{H}}{\partial(v,\lambda)}(g,\phi_1(g),\lambda_1(g))$ is invertible. Let (Z,Λ) be in $V\times\mathbb{R}$, we will prove that the system

$$A(g)\hat{v} - \lambda_1(g)B(g)\hat{v} - \hat{\lambda}B(g)\phi_1(g) = Z \tag{37}$$

$$2(B(g)\phi_1(g),\hat{v})_{2,D} = \Lambda \tag{38}$$

has a unique solution $(\hat{v},\hat{\lambda}) \in V\times\mathbb{R}$. Let $V_1 \subset V$ be the eigenspace associated to $\lambda_1(g)$. We have that $V_1 = \{t\,\phi_1(g),\ t\in\mathbb{R}\}$ and it is closed in V.

In the following, we look for $\hat{v} \in V$ and $\hat{\lambda}\in\mathbb{R}$ such that

$$\begin{aligned} & \left(A(g) - \lambda_1(g)B(g)\right)\hat{v} - \hat{\lambda}B(g)\phi_1(g) = Z \\ \Leftrightarrow\ & \left(A(g) - \lambda_1(g)B(g)\right)\hat{v} = Z + \hat{\lambda}B(g)\phi_1(g) \\ \Leftrightarrow\ & \left(\frac{1}{\lambda_1(g)}I - A^{-1}(g)B(g)\right)\hat{v} = \frac{1}{\lambda_1(g)}A^{-1}(g)\left(Z + \hat{\lambda}B(g)\phi_1(g)\right). \end{aligned} \tag{39}$$

From the Fredholm alternative, [6], Ch. VI, applied to $\mu_1 I - T$ with $\mu_1 = \frac{1}{\lambda_1(g)}$ and the compact operator $T = A^{-1}(g)B(g) : V \to V$ see also [7], Lemma 2, there exists $\overline{v}\in V$ a solution of (39), if and only if

$$\left(A^{-1}(g)\left(Z + \hat{\lambda}B(g)\phi_1(g)\right), u\right)_{2,D} = 0, \quad \forall u \in \mathrm{Ker}(\mu_1 I - T^*) \tag{40}$$

where $T^* = (A^{-1}(g)B(g))^* = B(g)A^{-1}(g)$ is the adjoint of T. We obtain that

$$\begin{aligned} & u \in \mathrm{Ker}(\mu_1 I - T^*) \Leftrightarrow B(g)A^{-1}(g)u = \mu_1 u \\ \Leftrightarrow \quad & \lambda_1(g)B(g)v_1 = A(g)v_1 \text{ and } A^{-1}(g)u = v_1 \Leftrightarrow v_1 \in V_1 \text{ and } u = A(g)v_1. \end{aligned}$$

Then (40) is equivalent to

$$\begin{aligned} & \left(A^{-1}(g)\left(Z + \hat{\lambda}B(g)\phi_1(g)\right), A(g)v_1\right)_{2,D} = 0, \quad \forall v_1 \in V_1 \\ \Leftrightarrow \quad & \left(Z + \hat{\lambda}B(g)\phi_1(g), \left(A^{-1}(g)\right)^* A(g)v_1\right)_{2,D} = 0, \quad \forall v_1 \in V_1 \\ \Leftrightarrow \quad & \left(Z + \hat{\lambda}B(g)\phi_1(g), v_1\right)_{2,D} = 0, \quad \forall v_1 \in V_1 \end{aligned} \tag{41}$$

since $\left(A^{-1}(g)\right)^* = A^{-1}(g)$.

But V_1 is of dimension one, so we look for $\hat{\lambda}$ such that

$$\begin{aligned} & 0 = \left(Z + \hat{\lambda}B(g)\phi_1(g), \phi_1(g)\right)_{2,D} \\ \Leftrightarrow \quad & 0 = (Z, \phi_1(g))_{2,D} + \hat{\lambda}\left(B(g)\phi_1(g), \phi_1(g)\right)_{2,D} \\ \Leftrightarrow \quad & 0 = (Z, \phi_1(g))_{2,D} + \hat{\lambda} \end{aligned}$$

since $(B(g)\phi_1(g), \phi_1(g))_{2,D} = b^e_\eta\left(\phi_1(g), \phi_1(g)\right) = 1$. We put $\hat{\lambda} = -\left(Z, \phi_1(g)\right)_{2,D}$.

If $\overline{v} \in V$ is a particular solution of (39) then all the solutions of the same equation has the form $\hat{v} = \gamma\phi_1(g) + \overline{v}$, $\gamma \in \mathbb{R}$. Using (32), we get

$$\begin{aligned} & 2(B(g)\phi_1(g), \hat{v})_{2,D} = \Lambda \Leftrightarrow 2(B(g)\phi_1(g), \gamma\phi_1(g) + \overline{v})_{2,D} = \Lambda \\ \Leftrightarrow \quad & \gamma = \frac{1}{2}\left(\Lambda - (B(g)\phi_1(g), \overline{v})_{2,D}\right). \end{aligned}$$

So, we have proved that the system (37)-(38) has at least a solution.

Now, we prove the uniqueness. Supposing, $(\hat{v}, \hat{\lambda})$, $(\tilde{v}, \tilde{\lambda})$ in $V \times \mathbb{R}$ are two solutions of (37)-(38). From the orthogonality condition (41), it follows that $\hat{\lambda} = \tilde{\lambda}$ and from (39) we obtain $\hat{v} - \tilde{v} \in V_1$. But $0 = (B(g)\phi_1(g), \hat{v} - \tilde{v})_{2,D} = (B(g)\phi_1(g), t\phi_1(g))_{2,D} = t$.

Finally, $\frac{\partial \mathcal{H}}{\partial(v,\lambda)}(g, \phi_1(g), \lambda_1(g))$ is invertible. Its inverse is continuous by Open Mapping Theorem, see [6], Sect. II.3. Then, we can apply the Implicit Function Theorem. Since $\mathcal{H}$ is of classe $\mathcal{C}^1$, we get that $g \to \lambda_1(g)$ and $g \to \phi_1(g)$ are Fréchet differentiable.

Deriving in (31) we obtain

$$\begin{aligned}
&\frac{\partial a_\eta^e}{\partial g}(\phi_1(g),v)r + a_\eta^e(\phi_1'(g)r,v) \\
= \;& \lambda_1'(g)r\, b_\eta^e(\phi_1(g),v) + \lambda_1(g)\,\frac{\partial b_\eta^e}{\partial g}(\phi_1(g),v) + \lambda_1(g)\, b_\eta^e(\phi_1'(g)r,v).
\end{aligned}$$

Putting $v = \phi_1(g)$, using (31), (32) and the symmetry of a_η^e, b_η^e, we get (34).

Deriving in (32), we obtain that $q = \phi_1'(g)r \in V$ is solution of (35)-(36). The uniqueness is as before, similar to (37)-(38). □

We set

$$\begin{aligned}
d_g^e \;=\; & \frac{\lambda_{1,D}}{\lambda_1^2(g)}\left(\frac{(\alpha_1-\alpha_0)E\,e^3}{12(1-\nu^2)}\,[\phi_1(g),\phi_1(g)] - \lambda_1(g)(\rho_1-\rho_0)e(\phi_1(g))^2\right) \\
& + \frac{\mu}{|D|}
\end{aligned} \tag{42}$$

and we can deduce that the directional derivative of J given by (33) is

$$J'(g)r = -\int_D H_\eta'(g)r\, d_g^e\, d\mathbf{x}. \tag{43}$$

Remark 3. *Consequently, for $r_g = H_\eta'(g)d_g^e$, $J'(g)r_g = -\int_D \left(H_\eta'(g)d_g^e\right)^2 d\mathbf{x} \leq 0$, then r_g is a descent direction for J at g, if $J'(g)r_g \neq 0$. Since $H_\eta'(g) \geq 0$, $J'(g)d_g^e = -\int_D H_\eta'(g)\left(d_g^e\right)^2 d\mathbf{x} \leq 0$, then $r_g = d_g^e$ is a descent direction, too, if $J'(g)r_g \neq 0$.*

5 Numerical tests

For $\mathcal{T}_h$ a triangulation of D with mesh size $h > 0$, let $\mathbb{V}_h \subset H_0^2(D)$ be the vectorial space based on the Hsieh, Clough and Tocher (HCT) finite element, globally of class $\mathcal{C}^1$, see [10], vol. 4, Ch. XII, Sect. 4.

The finite element approximation of (7) is: find $y_h \in \mathbb{V}_h$, such that

$$a_\eta(y_h,v_h) = \ell(v_h),\ \forall v_h \in \mathbb{V}_h$$

and similarly for the adjoint problem, compliance minimization, etc.

We use the software FreeFem++, [14] and we show that our approach based on functional variations works efficiently. In some examples the numerical minimization gives very good results.

Pointwise observation

Test 1.

The test is inspired from [30], where the topological derivative was used (see also [29]). We consider a square plate of sides 1 m, $D =]0, 1[\times]0, 1[$, background thickness $h_0 = 0.01$ m, Young Modulus $E = 210$ GPa, Poisson ratio $\nu = 0.3$.

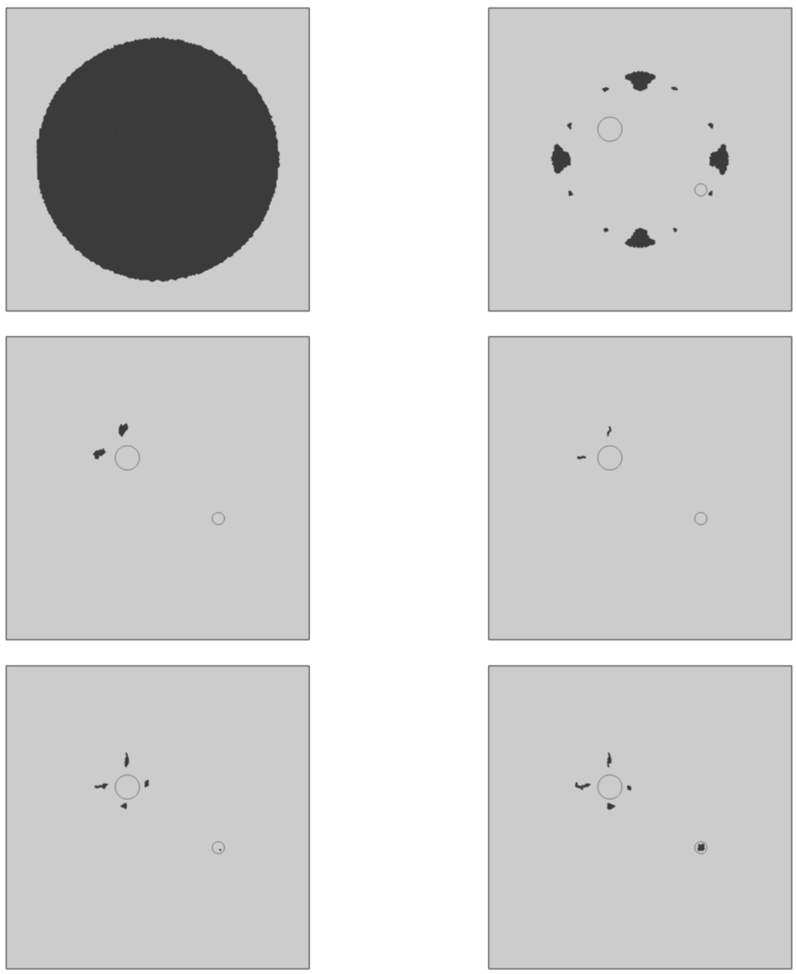

Figure 1: Test 1. Ω at iterations: 0, 1, 2, 3, 6, 39 (from left to right and from top to the bottom). The damaged zone Ω^* to be detected is in interior of the two circles.

The coefficient for the pointwise transverse load is $b_0 = 10^4$ N and the $M = 64$ source points $\boldsymbol{\xi}_m$, $m = 1, \ldots, M$ are located at $(i/9, j/9)$ for $i, j = 1, \ldots, 8$. We use mollifier approximation of Dirac function

$$f_\varepsilon = \sum_{m=1}^{M} b_0 \varphi_\varepsilon(\mathbf{x} - \boldsymbol{\xi}_m)$$

with $\varepsilon = 0.05$. There are $N = 69$ sensors $\mathbf{x}_n$, $n = 1, \ldots, N$ located on the grid $(i/10, j/10)$ for $i, j = 1, \ldots, 9$, but excluding 3 nodes in each corner.

We use a descent direction algorithm, with the direction given by (18). The stopping criterion is $|J_h(g_h^{k+1}) - J_h(g_h^k)| \leq tol = 10^{-8}$. To simplify the notation, we use $J_k = J_h(g_h^k)$. The mesh for D has 23603 vertices and 46644 triangles.

The damaged zone Ω^* to be detected is composed by two disks: one of radius 0.04 m centered at $(0.40, 0.60)$, the second of radius 0.02 m centered at $(0.70, 0.40)$, the contrast $\frac{e_1}{e_0} = 0.5$ for both disks.

The algorithm stops after 39 iterations, starting from $g_0(x_1, x_2) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.4^2$. The domains change the topology during the iterations, see Figure 1. The initial value of the discrete objective function is 39.3798, the final value is 4.94866×10^{-5}. The history of the discrete objective function is plotted in Figure 2 The computing time for 39 iterations was 454 minutes on a PC with 64 Go RAM.

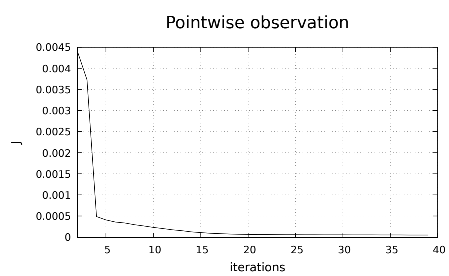

Figure 2: Test 1. The history of the discrete objective function J_k for iterations $k \geq 2$. $J_0 = 39.3798$ and $J_1 = 0.045051$.

Compliance minimization

Test 2.

The tests are inspired from [8], where the topological derivative was used. We consider a square plate of sides 1 m, $D =]0,1[\times]0,1[$, thickness $e = 0.05$ m, Young Modulus $E = 210$ GPa, Poisson ratio $\nu = 0.3$.

The coefficient for the pointwise transverse load is $b_0 = -10^6\ N$ and the $M = 4$ source points $\boldsymbol{\xi}_m$, $m = 1, \ldots, M$ are located at $(0.25, 0.25)$, $(0.75, 0.25)$, $(0.25, 0.75)$, $(0.75, 0.75)$. We use mollifier approximation of Dirac function $f_\varepsilon = \sum_{m=1}^{M} b_0 \varphi_\varepsilon(\mathbf{x} - \boldsymbol{\xi}_m)$ with $\varepsilon = 0.05$.

The coefficients are $\alpha_0 = 1$, $\alpha_1 = 10^{-4}\alpha_0$, $s_0 = 10^{-2}E$, $s_1 = 10^{-4}s_0$ and for penalization $\mu = 1.7$. The mesh for D has 23603 vertices and 46644 triangles.

The compliance j, the area $\mathcal{A}$ and the objective function J are given by the formulas (23), (24), (25), respectively. We use the descent direction given by (28).

Case k1. We start from

$$g_0(x_1, x_2) = -0.1\left((x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.2^2\right).$$

The initial value of the objective function is $J_0 = 2.69573$ and at the iteration 100 is $J_{100} = 2.29042$. In [8], the optimal value of the objective function is about 2.35. Some Ω are plotted in Figure 5. It is not necessary to have Ω connected, the plate occupies the domain D, but the material is not homogeneous.

Case k2. We start from $g_0(x_1, x_2) = -0.01$. The initial value of the objective function is $J_0 = 2.7$ and at the iteration 100 is $J_{100} = 2.28547$. The computing time for 100 iterations was about 1143 minutes for k1 as well as for k2. The history of J is plotted in Figure 3. The optimal value of the objective function in [8] is similarly, less than 2.3. We observe in Figure 4 that the compliance increases and area decreases. This behavior was observed in [8], too.

Some Ω are plotted in Figure 6 and we observe that at the iteration 100, the images are similar in both cases k1 and k2, despite the initial domains are different. In [19], for different initial domains we have obtained different local optimal solutions since this kind of optimization problem is highly non convex.

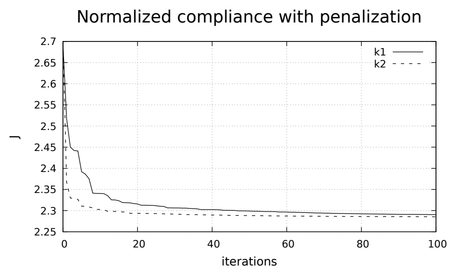

Figure 3: Test 2. The history of the objective function J given by (25) for k1 and k2.

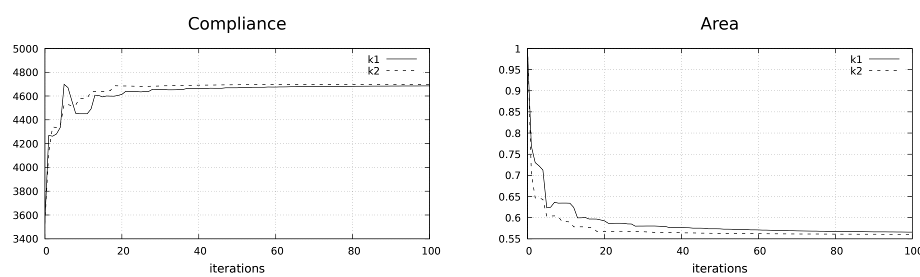

Figure 4: Test 2. The history of the compliance and area for k1 and k2.

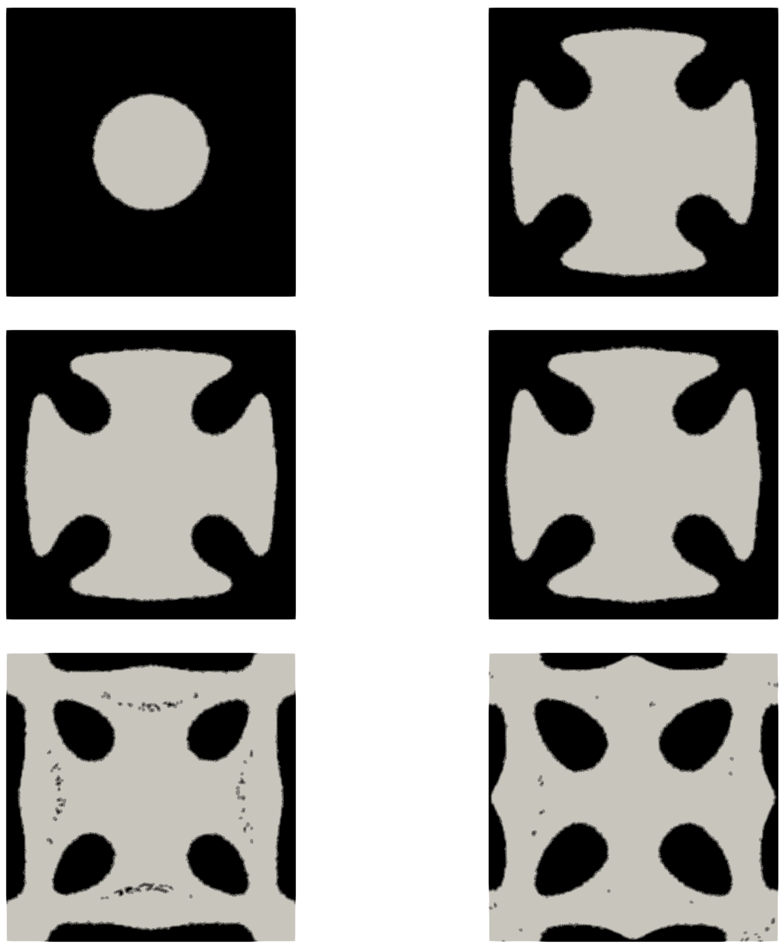

Figure 5: Test 2, Case k1. Ω at iterations: 0, 1, 2, 3, 4, 100 (from left to right and from top to the bottom).

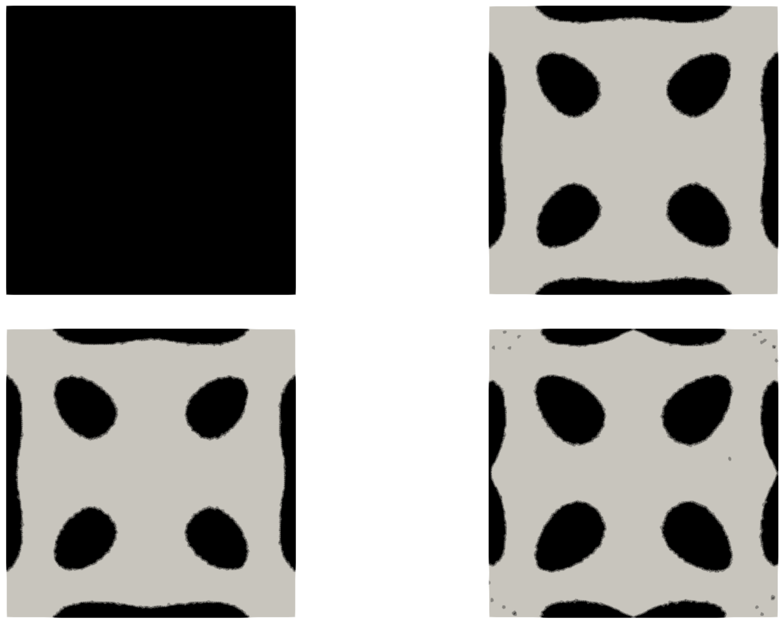

Figure 6: Test 2, Case k2. Ω at iterations: 0, 1, 2, 100 (from left to right and from top to the bottom).

Maximization of the first eigenvalue

Test 3.

The tests are inspired from [8]. We consider a square plate of sides $1\ m$, $D =]0,1[\times]0,1[$, thickness $e = 0.05\ m$, Young Modulus $E = 210\ GPa$, Poisson ratio $\nu = 0.3$, mass density $\rho_0 = 7800\ kg/m^3$.

The coefficients are $\alpha_0 = 1$, $\alpha_1 = 10^{-3}\alpha_0$, $\rho_1 = 10^{-3}\rho_0$. We use the descent direction given by (42), starting from $g_0 = -0.01$. The mesh for D has 23603 vertices and 46644 triangles.

Case k1. $M = 1$. The coefficient for the concentrated mass is $b_1 = 10^3\rho_0$ and the location is $\boldsymbol{\xi}_1 = (0.5, 0.5)$. The penalization parameter is $\mu = 1.2$.

Case k2. $M = 4$, $b_m = 10^3\rho_0$, $m = 1,\ldots,M$ and the location of $\boldsymbol{\xi}_m$ are $(0.25, 0.25)$, $(0.75, 0.25)$, $(0.25, 0.75)$, $(0.75, 0.75)$. The penalization parameter is $\mu = 1.4$.

The history of J is plotted in Figure 7. As in [8], we observe in Figure 8 that $\lambda_1/\lambda_{1,D}$ and area decrease, with little bit exceptions. The final Ω are plotted in Figure 9. The computing time for 100 iterations was about 2723 minutes for k1 and 2730 minutes for k2.

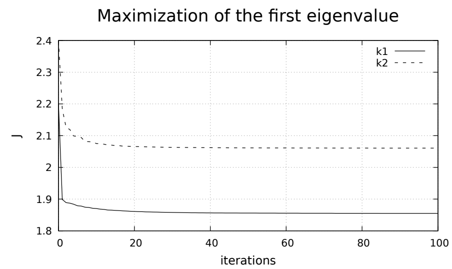

Figure 7: Test 3. The history of the objective function J given by (33) for k1 and k2.

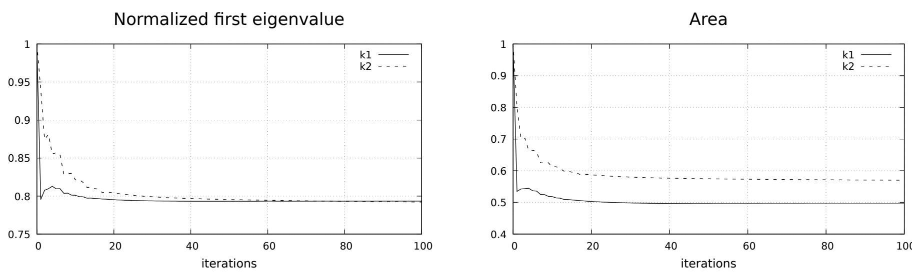

Figure 8: Test 3. The history of the $\lambda_1/\lambda_{1,D}$ (left) and area (right) for k1 and k2.

Figure 9: Test 3. Ω final in the case k1 (left) and k2 (right).

References

[1] S. Agmon, *Lectures on elliptic boundary value problems,* AMS Chelsea Publishing, 2010.

[2] G. Allaire, F. Jouve and A.-M. Toader, Structural optimization using sensitivity analysis and a level-set method, *J. Comput. Phys.* 194 (2004), 363-393.

[3] G. Allaire, *Conception optimale de structures,* vol. 58, Math. & Appl., Springer-Verlag, Berlin, 2007.

[4] V. Arnautu, H. Langmach, J. Sprekels and D. Tiba, On the approximation and the optimization of plates, *Numer. Funct. Anal. Optim.* 21 (2000), 337-354.

[5] M. Bendsoe and O. Sigmund, *Topology optimization: Theory, Methods and Applications,* second edition, Springer-Verlag, Berlin, 2003.

[6] H. Brezis, *Analyse fonctionnelle. Théorie et applications,* Dunod, 2005.

[7] D. Bucur, A. Ferrero and F. Gazzola, On the first eigenvalue of a fourth order Steklov problem, *Calc. Var.* 35 (2009), 103-131.

[8] F.S. Carvalho, D. Ruscheinsky, S.M. Giusti, C.T.M. Anflor and A.A. Novotny, Topological derivative-based topology optimization of plate structures under bending effects, *Struct. Multidisc. Optim.* 63 (2021), 617-630.

[9] P.G. Ciarlet, *Introduction to Numerical Linear Algebra and Optimization,* Cambridge University Press, 2018.

[10] R. Dautray and J.L. Lions, *Mathematical analysis and numerical methods for science and technology*, Vol. 3, 4. Springer-Verlag, Berlin, 1990.

[11] A. Dall'Acqua, *Higher order elliptic problems and positivity*, PhD Thesis, Technische Universiteit Delft, 2005.

[12] Ph. Destuynder and M. Salaun, *Mathematical analysis of thin plate models,* Springer-Verlag, Berlin, 1996.

[13] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.

[14] F. Hecht, New development in FreeFem++, *J. Numer. Math.* 20 (2012), 251–265.

[15] A. Henrot and M. Pierre, *Variations et optimisation de formes. Une analyse géométrique*, Springer, 2005.

[16] A. Henrot, *Shape optimization and spectral theory,* De Gruyter Open, 2017.

[17] C.M. Murea and D. Tiba, Optimization of a plate with holes, *Comput. Math. Appl.* 77 (2019), 3010-3020.

[18] C.M. Murea and D. Tiba, Topological optimization via cost penalization, *Topol. Methods Nonlinear Anal.* 54 (2019), 1023-1050.

[19] C.M. Murea and D. Tiba, Topological optimization and minimal compliance in linear elasticity, *Evol. Equ. Control Theory* 9 (2020), 1115-1131.

[20] C.M. Murea and D. Tiba, Topology optimization for the Stokes system, *Math. Rep.* 24 (2022), 301-317.

[21] C.M. Murea, D. Tiba, Implicit parametrization in shape optimization: boundary observation, *Pure Appl. Funct. Anal.* 7 (2022), 1835-1857.

[22] C.M. Murea and D. Tiba, Periodic Hamiltonian systems in shape optimization problems with Neumann boundary conditions, *J. Differ. Equations* 321 (2022), 1-39.

[23] C.M. Murea and D. Tiba, Topology optimization and boundary observation for clamped plates, arXiv:2310.11905.

[24] C.M. Murea, and D. Tiba, Penalization of stationary Navier-Stokes equations and applications, *Jpn. J. Ind. Appl. Math.* 42 (2025) 289-329.

[25] C.M. Murea and D. Tiba, Topological optimization with nonlinear state equation, *SeMA J.* (2024), https://doi.org/10.1007/s40324-024-00371-7

[26] P. Neittaanmäki, J. Sprekels and D. Tiba, *Optimization of elliptic systems. Theory and applications.* Springer, New York, 2006.

[27] P. Neittaanmäki, A. Pennanen and D. Tiba, Fixed domain approaches in shape optimization problems with Dirichlet boundary conditions, *Inverse Probl.* 25 (2009), 1-18.

[28] P. Neittaanmäki and D. Tiba, Fixed domain approaches in shape optimization problems, *Inverse Probl.* 28 (2012), 1-35.

[29] A. Novotny and J. Sokolowski, *Topological derivatives in shape optimization*, Springer, Berlin, 2013.

[30] A.A.M. Silva and A. Novotny, Damage identification in plate structures based on the topological derivative method, *Struct. Multidiscip. Optim.* 65 (2022), 7.

[31] J. Sokolowski and J.P. Zolesio, *Introduction to Shape Optimization. Shape Sensitivity Analysis*, Springer, Berlin, 1992.

[32] M. Tegmark and H.S. Shapiro, An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign, *SIAM Rev.* 36 (1994), 99-101.