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FIXED POINT RESULTS FOR MULTI-VALUED GRAPH CONTRACTIONS ON A SET ENDOWED WITH TWO METRICS*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

In this paper we will study existence, uniqueness and data dependence of the fixed points of multi-valued operators on a set endowed with two metrics. The case of multi-valued graph contractions is considered. Then, an extension to a more general contraction type condition is also given.

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keywords: metric space, fixed point, multi-valued graph contraction, data dependence.

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1 Introduction

Throughout this paper, the standard notations and terminologies in nonlinear analysis (see [17], [18], [8]) are used. For the convenience of the reader we recall some of them.

Let (X, d) be a metric space. The following notations are used through the paper:

$P(X) := \{Y \subset X|\ Y \text{ is nonempty}\}$, $P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\}$.

If $Y, Z \in P(X)$, then the gap between these sets is defined by

$$D_d(Y, Z) = \inf\{d(y, z)|\ y \in Y,\ z \in Z\}.$$

In particular, for for $y \in X$, the value $D_d(y, Z) = D_d(\{y\}, Z)$ is called the distance from the point y to the set Z.

The Pompeiu-Hausdorff generalized distance between $Y, Z \in P(X)$ is defined by the following formula:

$$H_d(Y, Z) := \max\{\sup_{y\in Y} \inf_{z\in Z} d(y, z), \sup_{z\in Z} \inf_{y\in Y} d(y, z)\}.$$

If $T : X \to P(X)$ is a multivalued operator, then we denote by $Graph(T) := \{(x, y) \in X \times Y|y \in T(x)\}$ the graph of T and by $Fix(T) := \{x \in X|\ x \in T(x)\}$ the fixed point set of T. In particular, $SFix(T) := \{x \in X|\ \{x\} = T(x)\}$ is the strict fixed point set of T. Also, for $x \in X$, we denote $F^n(x) := F(F^{n-1}(x))$, $n \in \mathbb{N}^*$, where $F^0(x) := \{x\}$.

Let (X, d) be a metric space and $T : X \to P(X)$. Then T is said to be a multi-valued α-contraction if $\alpha \in [0, 1[$ and

$$H_d(F(x), F(y)) \le \alpha d(x, y),\ \text{for every } (x, y) \in X \times X.$$

In particular, if the above relation holds for any $(x, y) \in Graph(T)$, then T is called a multi-valued graph α-contraction.

The aim of this paper is to give some fixed point theorems for multi-valued graph α-contractions on a set endowed with two metrics. Our results extend and generalize some results given in [3], [5], [12].

For the single-valued case, see R. P. Agarwal, D. O'Regan [1], I. A. Rus, A. Petruşel, G. Petruşel [17] and the references therein. For the multi-valued case see [12]. For related results see [2], [4], [6], [9], [10], [14], [15].

2 Multivalued graph contractions on a set with two metrics

The following notions are very important in our approach.

Definition 1. *Let (X, d) be a metric space, and $T : X \to P(X)$ be a multi-valued operator. By definition, T is a multi-valued weakly Picard operator with respect to d if for each $(x, y) \in Graph(T)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:*
(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in (X, d) to $x^*(x, y) \in Fix(T)$.

Remark 1. *A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the conditions (i) and (ii) of Definition 1 is called an iterative sequence of Picard type for T starting from $(x, y) \in Graph(T)$.*

Let (X, d) be a metric space. If $T : X \to P(X)$ is a multi-valued weakly Picard operator with respect to d, then we can define the multi-valued operator $T^\infty : Graph(F) \to P(FixF)$, by $T^\infty(x, y) := \{\ x^* \in Fix(T) \mid$ there exists an iterative sequence of Picard type T starting from (x, y) that converges in (X, d) to $x^*\ \}$.

Definition 2. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued weakly Picard operator with respect to d. Then T is called a c-multi-valued weakly Picard operator with respect to d if $c > 0$ and there exists a mapping $t^\infty : Graph(T) \to Fix(T)$, such that:*
(a) $t^\infty(x, y) \in T^\infty(x, y)$, for each $(x, y) \in Graph(T)$;
(b) the following relation takes place

$$d(x, t^\infty(x, y)) \le cd(x, y),\ \text{for all } (x, y) \in Graph(T). \tag{1}$$

The relation (1) is called the retraction-displacement condition for T with respect to the set retraction $t^\infty : Graph(T) \to Fix(T)$ in (X, d).

Notice that, by the Multi-valued Contraction Principle proved by Covitz and Nadler (see [3], [7]) we have that any multi-valued α-contraction on a complete metric space (X, d) is a $\frac{1}{1-\alpha}$-multi-valued weakly Picard operator with respect to d.

Our first main result is a multi-valued version of Maia's fixed point theorem for multi-valued graph α-contractions.

Theorem 1. *Let X be a nonempty set, d and ρ be two metrics on X and $T : X \to P(X)$ be a multi-valued operator. We suppose that:*
(i) (X, d) is a complete metric space;
(ii) there exists $\beta > 0$ such that $d(x, y) \le \beta\rho(x, y)$, for each $x, y \in X$;
(iii) $Graph(T)$ is closed in the metric topology of (X, d);

(iv) there exists $\alpha \in [0,1[$ *such that* $H_\rho(T(x), T(y)) \leq \alpha\rho(x,y)$, *for each* $(x,y) \in Graph(T)$.

Then we have:

(a) T *is a multi-valued weakly Picard operator with respect to* d;

(b) If we denote by $x^* := x^*(x_0, x_1)$ *the fixed point of* T *which is the limit in* (X,d) *of an iterative sequence of Picard type for* T *starting from* $(x_0, x_1) \in Graph(T)$, *then*

$$\rho(x_0, x^*) \leq \frac{1}{1-\alpha}\rho(x_0, x_1).$$

Proof. As in the proof of Theorem 4.1 in [13], by hypothesis (iv) we obtain an iterative sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type for T starting from arbitrary $(x,y) \in Graph(T)$ which is a Cauchy sequence in (X,ρ). From (ii) it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d) too. By (i) we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in (X,d). Let us denote by $x^* \in X$ the limit of this sequence in (X,d). By (iii) we get that $x^* \in T(x^*)$. The conclusion (b) follows by a similar approach as that given in the proof of Theorem 4.1 in [13]. The proof is complete. □

Remark 2. *The conclusion (b) means that a retraction-displacement condition for* T *with respect to* ρ *holds.*

The second main result of this section is the following extension of the above result.

Theorem 2. *Let* X *be a nonempty set,* d *and* ρ *be two metrics on* X *and* $T : X \to P(X)$ *be a multi-valued operator. We suppose that:*

(i) (X,d) *is a complete metric space;*

(ii) there exists $\beta > 0$ *such that* $d(x,y) \leq \beta\rho(x,y)$, *for each* $x, y \in X$;

(iii) $Graph(T)$ *is closed in the metric topology of* (X,d);

(iv) there exists $\alpha \in [0,1[$ *such that* $D_\rho(y, T(y)) \leq \alpha\rho(x,y)$, *for each* $(x,y) \in Graph(T)$.

Then, the following conclusions hold:

(a) T *is a multi-valued weakly Picard operator with respect to* d;

(b) If we denote by $x^* := x^*(x_0, x_1)$ *the fixed point of* T *which is the limit in* (X,d) *of an iterative sequence of Picard type for* T *starting from* $(x_0, x_1) \in Graph(T)$, *then*

$$\rho(x_0, x^*) \leq \frac{1}{1-\alpha}\rho(x_0, x_1).$$

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$ be arbitrary. Then, for arbitrary $q > 1$ there exists $x_2 \in T(x_1)$ such that $\rho(x_1, x_2) \leq qD_\rho(x_1, T(x_1))$. Then, we have $\rho(x_1, x_2) \leq q\alpha\rho(x_0, x_1)$. By mathematical induction, we obtain an iterative sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type for T starting from $(x_0, x_1) \in Graph(T)$ satisfying

$$\rho(x_n, x_{n+1}) \leq (q\alpha)^n \rho(x_0, x_1), \text{ for each } n \in \mathbb{N}. \tag{2}$$

Let us choose now $1 < q < \frac{1}{\alpha}$. Then, by the above relation we immediately get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, ρ). By (ii) we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). By (i) it follows that there exists $x^* := x^*(x_0, x_1) \in X$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x^* in (X, d). By (iii) the element x^* belongs to $T(x^*)$. For the conclusion (b) notice first that by (2), we also get that

$$\rho(x_n, x_{n+p}) \leq \frac{(q\alpha)^n}{1 - q\alpha}\rho(x_0, x_1), \text{ for each } n, p \in \mathbb{N}, p \geq 1. \tag{3}$$

Letting $p \to \infty$ in (3) we get that

$$\rho(x_n, x^*) \leq \frac{(q\alpha)^n}{1 - q\alpha}\rho(x_0, x_1), \text{ for each } n \in \mathbb{N}. \tag{4}$$

Taking $n = 0$ in (4) and letting then $q \searrow 1$, we obtain

$$\rho(x_0, x^*) \leq \frac{1}{1 - \alpha}\rho(x_0, x_1),$$

which proves that a retraction-displacement condition for T with respect to ρ holds. □

A data dependence result is the following theorem.

Theorem 3. *Let X be a nonempty set, d and ρ two metrics on X and $T, S : X \to P(X)$ be two multivalued operators. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) there exists $\beta > 0$ such that $d(x, y) \leq \beta\rho(x, y)$, for each $x, y \in X$;

(iii) $Graph(T)$ is closed in the metric topology of (X, d);

(iv) there exists $\alpha \in [0, 1[$ such that $H_\rho(T(x), T(y)) \leq \alpha\rho(x, y)$, for each $(x, y) \in Graph(T)$;

(v) $F_S \neq \emptyset$;

(vi) there exists $\eta > 0$ such that $H_\rho(T(x), S(x)) \leq \eta$, for each $x \in X$.

Then, for each $s^* \in Fix(S)$ there exists $x^* \in Fix(T)$ such that

$$\rho(s^*, x^*) \leq \frac{\eta}{1 - \alpha} \text{ and } d(s^*, x^*) \leq \frac{\beta\eta}{1 - \alpha}$$

Proof. Let $s^* \in Fix(S)$ and $t \in T(s^*)$ be arbitrary chosen. From the conclusion (b) of Theorem 1 we have that $\rho(s^*, x^*) \leq \frac{1}{1-\alpha}\rho(s^*, t)$, where x^* is the fixed point of T which is the limit of an iterative sequence of Picard type starting from $(s^*, t) \in Graph(T)$. Now, for any $q > 1$ and $s^* \in S(s^*)$ there exists $t^* \in T(s^*)$ such that $\rho(s^*, t^*) \leq qH_\rho(S(s^*), T(s^*)) \leq q\eta$. Thus, we get that

$$\rho(s^*, x^*) \leq \frac{q\eta}{1-\alpha}.$$

Letting $q \searrow 1$ we get the conclusion. □

Remark 3. *A similar data dependence result can be given in association with the assumptions of Theorem 2.*

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