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ON THE HEAT DISSIPATION FUNCTION FOR MAGNETIC RELAXATION PHENOMENA IN ANISOTROPIC MEDIA*

L. Restuccia†

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

Using the methods of classical irreversible thermodynamics with internal variables, the heat dissipation function for magnetizable anisotropic media, in which phenomena of magnetic relaxation occur, is derived. It is assumed that if different types of irreversible microscopic phenomena give rise to magnetic relaxation, it is possible to describe these microscopic phenomena splitting the total specific magnetization in two irreversible parts and introducing one of these partial specific magnetizations as internal variable in the thermodynamic state space. It is seen that, when the theory is linearized, the heat dissipation function is due to the electric conduction, magnetic relaxation, viscous, magnetic irreversible phenomena. This is the case of complex media, where different kinds of molecules have different magnetic susceptibilities and relaxation times, present magnetic relaxation phenomena and contribute to the total magnetization. These situations arise in nuclear magnetic resonance in medicine and biology and in other fields of the applied sciences. Also, the heat conduction equation for these media is worked out and the special cases of anisotropic Snoek media and

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†lrestuccia@unime.it University of Messina, Department of Mathematical and Computer Sciences, Physical Sciences and Earth Sciences , Viale F. Stagno d'Alcontres, Salita Sperone, 31, 98166 Messina, Italy

anisotropic De-Groot-Mazur media are treated.

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1 Introduction

In some previous papers [1]-[8] a linear theory for magnetic relaxation phenomena in magnetizable continuous media was developed, that is based on thermodynamics of irreversible processes [9]-[15] with internal variables (see [16]). In [5]-[7] it was shown that, if an arbitrary number n of microscopic phenomena give rise to magnetic relaxation, it is possible to describe these microscopic phenomena introducing n macroscopic axial vectorial internal variables in the thermodynamic state vector and assuming that the specific magnetization axial vector $\mathbf{m}$ can be split in n+1 irreversible contributions, i.e.,

$$\mathbf{m} = \mathbf{m}^{(0)} + \sum_{k=1}^{n} \mathbf{m}^{(k)}, \tag{1}$$

where $\mathbf{m}^{(0)}$ and $\mathbf{m}^{(k)}$ $(k = 1, ..., n)$ are called partial specific magnetizations.

In the isotropic case (see [5], [6]) the following magnetic relaxation equation generalizing Snoek equation was obtained having the form of a linear relation among the magnetic field $\mathbf{B}$, the first n time derivatives of this field, the total magnetization $\mathbf{M}$ and the first n + 1 time derivatives of $\mathbf{M}$

$$\chi^{(0)}_{(BM)}\mathbf{B} + \chi^{(1)}_{(BM)}\frac{d\mathbf{B}}{dt} + ... + \chi^{(n-1)}_{(BM)}\frac{d^{n-1}\mathbf{B}}{dt^{n-1}} + \frac{d^n\mathbf{B}}{dt^n} =$$

$$\chi^{(0)}_{(MB)}\mathbf{M} + \chi^{(1)}_{(MB)}\frac{d\mathbf{M}}{dt} + ... + \chi^{(n)}_{(MB)}\frac{d^n\mathbf{M}}{dt^n} + \chi^{(n+1)}_{(MB)}\frac{d^{n+1}\mathbf{M}}{dt^{n+1}}, \tag{2}$$

where n is the number of phenomena that give rise to the total magnetization $\mathbf{M}$, with $\mathbf{M} = \rho\mathbf{m}$, being ρ the mass density of the medium, supposed a constant quantity, and $\chi^{(k)}_{(BM)}(k = 0, 1, ..., n-1)$ and $\chi^{(k)}_{(MB)}$ $(k = 0, 1, ..., n+1)$ are constant quantities. In particular, they are algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state.

This is the case of media where different kinds of molecules (for instance of water, of proteins, etc.) contribute to the total magnetization, have different magnetic susceptibilities and relaxation times, and present particular magnetic relaxation phenomena. These physical situations arise in nuclear magnetic resonance in medicine, in biology and also in different fields of appplied sciences, where complex media are used. In [17]-[21] Maugin gives a continuous description of the magnetization field in a deformable crystal, below its magnetic phase-transition temperature T_{cr}, assuming that the magnetization field per unit mass is the sum of n partial magnetization fields, arising from n different ionic species. According to microscopic considerations, a spin density per unit mass is associated with each partial magnetization per unit mass, so that the total spin intrinsic momentum per unit mass is given by the sum of n partial spin densities. Following Maugin, n interactions arise between each magnetic sublattice and the crystal lattice, called *spin-lattice interactions*, having dimensions of magnetic fields. This fact is known as *magnetic anisotropy* (since each magnetic interaction is linked to the orientation of the relative corresponding partial magnetization density with respect to the crystal lattice) [7]. It is seen that magnetic interactions also account for the *intra and intermagnetic-sublattice interactions* describing the short-range *spin-interactions*. The continuum theory for magnetizable bodies developed by Maugin gives an explanation to internal mechanisms in magnetizable bodies with internal variables. For this multimagnetic-sublattice approach used for the description of magnetic properties of complex media see also [22]-[28].

In the present paper we consider anisotropic magnetizable media in which the magnetization axial vector $\mathbf{m}$ is additively composed of two irreversible contributions $\mathbf{m}^{(0)}$ and $\mathbf{m}^{(1)}$, i.e.,

$$\mathbf{m} = \mathbf{m}^{(0)} + \mathbf{m}^{(1)}. \tag{3}$$

The explicit expression for the heat dissipation function is worked out. It is seen that this function is due to the electric conduction, magnetic relaxation, viscous, magnetic irreversible processes. Also the heat conduction equation for these media is worked out. The methods of classical irreversible thermodynamics with internal variables are used, where the local equilibrium hypothesis is assumed (see [29]-[37]). Each point in the considered medium can be considered as an elementary volume, where the reversible thermodynamics is valid. The values of the fields differ from elementary volume to elementary volume describing a non-equilibrium situation. The size d of this elementary volume should be bigger than the average distance traveled by the (charges, heat) carriers between two successive collisions, defined mean free path l $(d > l)$, see [31]. In Section 2 the governing equations for magnetizable media are introduced. In Section 3 the entropy balance equation and the phenomenological equations are derived. In Section 4 a reference state and a thermodynamic equilibrium state are introduced. In Sections 5 and 6 the linear equations of state and the magnetic relaxation equations are considered, in the linear case and when all cross effects are disregarded, except for possible cross effects among different types of magnetic relaxation phenomena. In Section 7 the heat conduction equation is derived and it is seen that the heat dissipated per unit of volume and per unit of time due to the presence of irreversible phenomena inside the considered media is described by the heat dissipation function, that has the form of a quadratic function of the components of the strain tensor, the components of the electric field, the components of the magnetic field, the components of the total magnetization, the components of the time derivatives of this last axial vector and of the temperature. In Sections 8 and 9 the results are applied to the particular cases of anisotropic Snoek media (see [38]) and anisotropic De Groot-Mazur media.

2 Governing equations for magnetizable media

The standard Cartesian tensor notation in a rectangular coordinate system is used and the equations governing the behavior of magnetizable media are considered in a current configuration $\mathcal{K}_t$. In Galilean approximation the physical processes occurring inside the magnetizable media under consideration are governed by the following laws (see [14] and [5]-[7]):

Maxwell's equations (in rationalized Gauss system), keeping the form

$$rot\mathbf{H}-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}=\frac{1}{c}\mathbf{J}^{(el)}, \quad div\mathbf{E}=\rho^{(el)}, \quad rot\mathbf{E}+\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}=0, \quad div\mathbf{B}=0, \tag{4}$$

where c is the modulus of the light velocity, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$, denote the electric field, the magnetic induction and the magnetic displacement, respectively, and $\mathbf{J}^{(el)}$ and $\rho^{(el)}$ are the electric current density per unit volume and the electric charge density per unit volume, satisfying the conservation law

$$\frac{\partial \rho^{(el)}}{\partial t}=-div\mathbf{J}^{(el)}; \tag{5}$$

the *mass conservation law*, having the form

$$\frac{\partial \rho}{\partial t}=-div\rho\mathbf{v}, \quad \text{or} \quad \rho\frac{dv}{dt}=div\mathbf{v}, \tag{6}$$

with d/dt the material derivative, defined by $\frac{d}{dt} = \frac{\partial}{\partial t} + v_\gamma \frac{\partial}{\partial x_\gamma}$ (where Einstein convention on repeating indices is used), ρ the density mass of the medium under consideration, that is assumed a constant quantity, v the specific volume, defined by $v = \rho^{-1}$, and $\mathbf{v}$ the velocity field, given by $\mathbf{v} = \frac{d\mathbf{u}}{dt}$, where $\mathbf{u}$ is the displacement field;

the *equation of motion*, taking the form [14]

$$\rho \frac{d\mathbf{v}}{dt} = div\boldsymbol{\tau} + \rho^{(el)}\mathbf{E} + \frac{1}{c}\mathbf{J}^{(el)} \times \mathbf{B} + (grad\mathbf{B}) \cdot \mathbf{M} - \frac{1}{c}\frac{d}{dt}(\mathbf{M} \times \mathbf{E}) + \rho\mathbf{F}, \quad (7)$$

where $\boldsymbol{\tau}$ is the symmetric stress tensor, $\mathbf{F}$ is the volume force per unit of mass and $\mathbf{M}$ is the magnetization axial vector defined by

$$\mathbf{M} = \mathbf{B} - \mathbf{H}, \quad \mathbf{M} = \rho\mathbf{m}, \quad (8)$$

with $\mathbf{m}$ the magnetization per unit volume, called specific magnetization;

the *first law of thermodynamics*, having the form

$$\rho \frac{du}{dt} = -div\mathbf{J}^{(q)} + \tau_{\alpha\beta}\frac{d\epsilon_{\alpha\beta}}{dt} + \mathbf{J}^{(el)} \cdot \mathbf{E} + \rho\mathbf{B} \cdot \frac{d\mathbf{m}}{dt}, \quad (9)$$

where u is the specific internal energy (energy per unit of mass), $\mathbf{J}^{(q)}$ is the heat flux vector and the small strain tensor $\epsilon_{\alpha\beta}$ (assuming that the deformations and rotations of the considered media are small from a kinematical point of view) is defined by $\epsilon_{\alpha\beta} = \frac{1}{2}(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}) \quad (\alpha, \beta = 1, 2, 3)$, i.e., as the symmetric part of the gradient of the displacement field $\mathbf{u}$.

3 A description of anisotropic media with magnetic relaxation

In [1]-[8] within the framework of the classical irreversible thermodynamics with internal variables, a theory was developed for magnetizable media with relaxation phenomena. In this paper we consider anisotropic magnetizable media in which the contributions of microscopic phenomena to the macroscopic magnetization axial vector can be described by introducing one internal variable in the expression of the entropy. Then, we suppose that the total specific magnetization $\mathbf{m}$ is additively composed of two irreversible parts, $\mathbf{m} = \mathbf{m}^{(0)} + \mathbf{m}^{(1)}$, and we introduce the partial specific magnetization $\mathbf{m}^{(1)}$ as internal variable in the state space C

$$C = C\left(u, \epsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}\right). \quad (10)$$

Following the general philosophy of TIP (see [29]-[37]) the local equilibrium hypothesis is assumed and dissipative fluxes, gradients and time derivatives of the considered physical fields are not included in the thermodynamic state space (10).

Now, let us assume that the specific entropy (the entropy per unit of mass) is a constitutive function of the state space C

$$s = s\left(u, \epsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}\right). \tag{11}$$

According to the reversible thermodynamics, we shall define the equilibrium temperature (the absolute temperature) T, the equilibrium stress tensor $\tau^{(eq)}_{\alpha\beta}$, the equilibrium magnetic field $\mathbf{B}^{(eq)}$ and the axial vector field $\mathbf{B}^{(1)}$ (that represents the thermodynamic affinity conjugate to the internal variable $\mathbf{m}^{(1)}$) by

$$T^{-1} = \frac{\partial}{\partial u} s\left(u, \epsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}\right), \tag{12}$$

$$\tau^{(eq)}_{\alpha\beta} = -\rho T \frac{\partial}{\partial \epsilon_{\alpha\beta}} s\left(u, \epsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}\right), \tag{13}$$

$$\mathbf{B}^{(eq)} = -T \frac{\partial}{\partial \mathbf{m}} s\left(u, \epsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}\right), \tag{14}$$

$$\mathbf{B}^{(1)} = T \frac{\partial}{\partial \mathbf{m}^{(1)}} s\left(u, \epsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}\right). \tag{15}$$

We expand the entropy (11) into Taylor's series with respect to an equilibrium state, considering very small deviations with respect to this state. Confining our considerations to the linear terms, we obtain the differential of the entropy in a point of the thermodynamic phase space in thermodynamic equilibrium (see [39], [40]), corresponding to a local position $\mathbf{x}$ in a current configuration $\mathcal{K}_t$, having the following form

$$s = s^{(eq)} + \frac{\partial s}{\partial u}\left(u - u^{(eq)}\right) + \frac{\partial s}{\partial \epsilon_{\alpha\beta}}\left(\epsilon_{\alpha\beta} - \epsilon^{(eq)}_{\alpha\beta}\right) + \frac{\partial s}{\partial \mathbf{m}}\left(\mathbf{m} - \mathbf{m}^{(eq)}\right) + \frac{\partial s}{\partial \mathbf{m}^{(1)}}\left(\mathbf{m}^{(1)} - \mathbf{m}^{(1eq)}\right). \tag{16}$$

Multiplying the obtained expression (16) by the temperature T and taking into account (12)-(15) and the definition $\rho = \frac{1}{v}$, we obtain the following Gibbs relation

$$T ds = du - v\tau^{(eq)}_{\alpha\beta} d\epsilon_{\alpha\beta} - \mathbf{B}^{(eq)} \cdot d\mathbf{m} + \mathbf{B}^{(1)} \cdot d\mathbf{m}^{(1)}. \tag{17}$$

Thus, from (17) it follows that the time derivative of the entropy (11) in the point $\mathbf{x}$ keeps the form

$$T\frac{ds}{dt} = \frac{du}{dt} - v\tau_{\alpha\beta}^{(eq)}\frac{d\epsilon_{\alpha\beta}}{dt} - \mathbf{B}^{(eq)} \cdot \frac{d\mathbf{m}}{dt} + \mathbf{B}^{(1)} \cdot \frac{d\mathbf{m}^{(1)}}{dt}. \tag{18}$$

Inserting in (18) the expression (9) for the time derivative of the specific internal energy we obtain the following entropy balance valid for small deviations from a thermodynamical equilibrium state

$$\rho\frac{ds}{dt} = -\, divJ^{(s)} + \sigma^{(s)}, \tag{19}$$

where $J^{(s)} = T^{-1}\mathbf{J}^{(q)}$ is the entropy flux and $\sigma^{(s)}$ is the entropy production per unit volume and per unit time given by

$$\sigma^{(s)} = T^{-1}\left(\tau_{\alpha\beta}^{(vi)}\frac{d\epsilon_{\alpha\beta}}{dt} + \mathbf{J}^{(q)} \cdot \mathbf{X}^{(q)} + \mathbf{J}^{(el)} \cdot \mathbf{E} + \right.$$

$$\left. \rho\mathbf{B}^{(ir)} \cdot \frac{d\mathbf{m}}{dt} + \rho\mathbf{B}^{(1)} \cdot \frac{d\mathbf{m}^{(1)}}{dt}\right). \tag{20}$$

In equation (20) the viscous stress tensor (called also irreversible stress tensor) $\tau_{\alpha\beta}^{(vi)}$, the irreversible magnetic field $\mathbf{B}^{(ir)}$ and the field $\mathbf{X}^{(q)}$ are defined, respectively, by

$$\tau_{\alpha\beta}^{(vi)} = \tau_{\alpha\beta} - \tau_{\alpha\beta}^{(eq)}, \quad \mathbf{B}^{(ir)} = \mathbf{B} - \mathbf{B}^{(eq)}, \quad \mathbf{X}^{(q)} = -T^{-1}gradT. \tag{21}$$

The theory for relaxation magnetizable phenomena developed in [1]-[8] is a linear theory. The deviations with respect to the thermodynamical equilibrium states are very small, and all the considered equations (Maxwell's equations, balance equations, phenomenological equations, constitutive relations) have to be taken in "perturbation" around the same thermodynamical equilibrium state.

If we assume that the specific entropy s is a constitutive function of the state space $C = C\left(u, \epsilon_{\alpha\beta}, \mathbf{m}^{(0)}, \mathbf{m}^{(1)}\right)$

$$s = s\left(u, \epsilon_{\alpha\beta}, \mathbf{m}^{(0)}, \mathbf{m}^{(1)}\right), \tag{22}$$

in equation (19) $\sigma^{(s)}$ assumes the following form

$$\sigma^{(s)} = T^{-1}\left(\tau_{\alpha\beta}^{(vi)}\frac{d\epsilon_{\alpha\beta}}{dt} + \mathbf{J}^{(q)} \cdot \mathbf{X}^{(q)} + \mathbf{J}^{(el)} \cdot \mathbf{E} + \right.$$

$$\rho \mathbf{B}^{(ir)} \cdot \frac{d\mathbf{m}^{(0)}}{dt} + \rho(\mathbf{B}^{(1)} + \mathbf{B}^{(ir)}) \cdot \frac{d\mathbf{m}^{(1)}}{dt} \Bigg). \tag{23}$$

It is seen from (23) that if the magnetic field $\mathbf{B}$ of Maxwell's equations equals the equilibrium magnetic field $\mathbf{B}^{(eq)}$, $\mathbf{B}^{(ir)}$ vanishes (see equation $(21)_2$), the specific partial magnetization $\mathbf{m}^{(1)}$ only contributes to the entropy production and changes in $\mathbf{m}^{(0)}$ become reversible (see in Section 8 Snoek media). $\mathbf{m}^{(0)}$ is an irreversible part of the total magnetization only when the magnetizable medium is not in a thermodynamic equilibrium state. Furthermore, if $\mathbf{m} = \mathbf{m}^{(0)}$, there is no internal variable $\mathbf{m}^{(1)}$ and one obtains for $\sigma^{(s)}$ the expression derived by De Groot-Mazur (see Section 9). It is seen from (20) that the entropy production is a sum of terms, where each term is the inner product of two vectors or two second order tensors of which one is a flux and the other is the thermodynamic force or " affinity " conjugate to the flux. Hence, according to the usual procedure of non equilibrium thermodynamics (see for instance [29]), from (20) we obtain the following phenomenological equations, in which the irreversible fluxes are *linear functions* of the thermodynamic forces

$$B_\alpha^{(ir)} = \rho L^{(0,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(0,1)}_{(M)\alpha\beta} B^{(1)}_\beta + L^{(0,el)}_{(M)\alpha\beta} E_\beta +$$

$$L^{(0,q)}_{(M)\alpha\beta} X^{(q)}_\beta + L^{(0,vi)}_{(M)\alpha\beta\gamma} \frac{d\epsilon_{\beta\gamma}}{dt}, \tag{24}$$

$$\rho \frac{dm_\alpha^{(1)}}{dt} = \rho L^{(1,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta} B^{(1)}_\beta + L^{(1,el)}_{(M)\alpha\beta} E_\beta +$$

$$L^{(1,q)}_{(M)\alpha\beta} X^{(q)}_\beta + L^{(1,vi)}_{(M)\alpha\beta\gamma} \frac{d\epsilon_{\beta\gamma}}{dt}, \tag{25}$$

$$J_\alpha^{(el)} = \rho L^{(el,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(el,1)}_{(M)\alpha\beta} B^{(1)}_\beta + L^{(el,el)}_{\alpha\beta} E_\beta +$$

$$L^{(el,q)}_{\alpha\beta} X^{(q)}_\beta + L^{(el,vi)}_{\alpha\beta\gamma} \frac{d\epsilon_{\beta\gamma}}{dt}, \tag{26}$$

$$J_\alpha^{(q)} = \rho L^{(q,0)}_{(M)\alpha\beta} \frac{dm_\beta}{dt} + L^{(q,1)}_{(M)\alpha\beta} B^{(1)}_\beta + L^{(q,el)}_{\alpha\beta} E_\beta +$$

$$L^{(q,q)}_{\alpha\beta} X^{(q)}_\beta + L^{(q,vi)}_{\alpha\beta\gamma} \frac{d\epsilon_{\beta\gamma}}{dt}, \tag{27}$$

$$\tau_{\alpha\beta}^{(vi)} = \rho L^{(vi,0)}_{(M)\alpha\beta\gamma} \frac{dm_\gamma}{dt} + L^{(vi,1)}_{(M)\alpha\beta\gamma} B^{(1)}_\gamma + L^{(vi,el)}_{\alpha\beta\gamma} E_\gamma +$$

$$L^{(vi,q)}_{\alpha\beta\gamma} X^{(q)}_\gamma + L^{(vi,vi)}_{\alpha\beta\gamma\zeta} \frac{d\epsilon_{\gamma\zeta}}{dt}. \tag{28}$$

Equations (24) and (25) are the phenomenological equations describing the irreversible process of magnetic relaxation. Eqs. (26), (27) and (28) represent the generalizations of Ohm's law, Fourier's law and Newton's law, respectively. Furthermore, in (24) and (25) (for instance) the third, the fourth and the fifth terms on the right-hand side describe the cross effects among electric conduction, heat conduction, mechanical viscosity and magnetic relaxation. The quantities $L^{(0,0)}_{(M)\alpha\beta}$, $L^{(0,1)}_{(M)\alpha\beta}$, etc., which occur in the above equations, are constant phenomenological tensors (polar or axial). For instance $L^{(vi,vi)}_{\alpha\beta\gamma\zeta}$ is the polar viscosity tensor of order four, $L^{(q,q)}_{\alpha\beta}$ is the polar tensor of heat conductivity of order two, $L^{(0,el)}_{(M)\alpha\beta}$ and $L^{(0,q)}_{(M)\alpha\beta}$ are axial tensors of order two connected with the influence of the electric conduction and the heat flux on the magnetic relaxation, respectively. From experiments (see for instance [41]) it is possible to know the values of these phenomenological coefficients. By virtue of the symmetry of $\epsilon_{\alpha\beta}$ from (13) it follows that also $\tau^{(eq)}_{\alpha\beta}$ is a symmetric tensor. Moreover, if we assume that the mechanical stress tensor $\tau_{\alpha\beta}$ is symmetric, from $(21)_1$ it follows that also the viscous stress tensor $\tau^{(vi)}_{\alpha\beta}$ is a symmetric tensor. Therefore, one gets the following symmetry relations for the phenomenological tensors

$$L^{(0,vi)}_{(M)\alpha\beta\gamma} = L^{(0,vi)}_{(M)\alpha\gamma\beta}, \quad L^{(vi,0)}_{(M)\alpha\beta\gamma} = L^{(vi,0)}_{(M)\beta\alpha\gamma}, \tag{29}$$

$$L^{(1,vi)}_{(M)\alpha\beta\gamma} = L^{(1,vi)}_{(M)\alpha\gamma\beta}, \quad L^{(vi,1)}_{(M)\alpha\beta\gamma} = L^{(vi,1)}_{(M)\beta\alpha\gamma}, \tag{30}$$

$$L^{(el,vi)}_{\alpha\beta\gamma} = L^{(el,vi)}_{\alpha\gamma\beta}, \quad L^{(vi,el)}_{\alpha\beta\gamma} = L^{(vi,el)}_{\beta\alpha\gamma}, \quad L^{(q,vi)}_{\alpha\beta\gamma} = L^{(q,vi)}_{\alpha\gamma\beta}, \quad L^{(vi,q)}_{\alpha\beta\gamma} = L^{(vi,q)}_{\beta\alpha\gamma}, \tag{31}$$

$$L^{(vi,vi)}_{\alpha\beta\gamma\zeta} = L^{(vi,vi)}_{\beta\alpha\gamma\zeta} = L^{(vi,vi)}_{\alpha\beta\zeta\gamma} = L^{(vi,vi)}_{\beta\alpha\zeta\gamma}. \tag{32}$$

Since $\mathbf{B}^{(ir)}$, $\mathbf{B}^{(1)}$, $\mathbf{j}^{(el)}$, $\mathbf{J}^{(q)}$ and $\frac{d\epsilon_{\alpha\beta}}{dt}$ are odd functions under time reversal and $\rho\frac{d\mathbf{m}}{dt}$, $\rho\frac{d\mathbf{m}^{(1)}}{dt}$, $\mathbf{E}$, $\mathbf{X}^{(q)}$ and $\tau^{(vi)}_{\alpha\beta}$ are even functions under time reversal the Onsager-Casimir reciprocity relations read (see [14], [15], [29], [42], [43])

$$L^{(0,0)}_{(M)\alpha\beta} = L^{(0,0)}_{(M)\beta\alpha}, \quad L^{(1,1)}_{(M)\alpha\beta} = L^{(1,1)}_{(M)\beta\alpha}, \tag{33}$$

$$L^{(el,el)}_{\alpha\beta} = L^{(el,el)}_{\beta\alpha}, \quad L^{(q,q)}_{\alpha\beta} = L^{(q,q)}_{\beta\alpha}, \tag{34}$$

$$L^{(vi,vi)}_{\alpha\beta\gamma\zeta} = L^{(vi,vi)}_{\gamma\zeta\alpha\beta}, \quad L^{(0,1)}_{(M)\alpha\beta} = -L^{(1,0)}_{(M)\beta\alpha}, \tag{35}$$

$$L^{(0,el)}_{(M)\alpha\beta} = L^{(el,0)}_{(M)\beta\alpha}, \;\; L^{(0,q)}_{(M)\alpha\beta} = L^{(q,0)}_{(M)\beta\alpha}, \;\; L^{(1,el)}_{(M)\alpha\beta} = -L^{(el,1)}_{(M)\beta\alpha}, \tag{36}$$

$$L^{(1,vi)}_{(M)\alpha\beta\gamma} = L^{(vi,1)}_{(M)\beta\gamma\alpha}, \quad L^{(1,q)}_{(M)\alpha\beta} = -L^{(q,1)}_{(M)\beta\alpha}, \tag{37}$$

$$L^{(0,vi)}_{(M)\alpha\beta\gamma} = -L^{(vi,0)}_{(M)\beta\gamma\alpha}, \quad L^{(el,q)}_{\alpha\beta} = L^{(q,el)}_{\beta\alpha},$$

$$L^{(el,vi)}_{\alpha\beta\gamma} = -L^{(vi,el)}_{\beta\gamma\alpha}, \quad L^{(q,vi)}_{\alpha\beta\gamma} = -L^{(vi,q)}_{\beta\gamma\alpha}. \tag{38}$$

Equations (29)-(38) reduce the number of independent components of the phenomenological tensors. Furthermore, reductions may occur as a consequence of physical properties (as for instance isotropy properties) of the medium. Now, introducing the expressions (24)-(28) for $\mathbf{B}^{(ir)}$, $\rho\frac{d\mathbf{m}^{(1)}}{dt}$, $\mathbf{j}^{(el)}$, $\mathbf{J}^{(q)}$, and $\tau^{(vi)}_{\alpha\beta}$ in (20) and by virtue of symmetry and Onsager-Casimir reciprocity relations, we obtain for the entropy production the following form

$$\sigma^{(s)} = T^{-1}\left\{\rho^2 L^{(0,0)}_{(M)\alpha\beta}\frac{dm_\alpha}{dt}\frac{dm_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\alpha B^{(1)}_\beta + L^{(el,el)}_{\alpha\beta}E_\alpha E_\beta + \right.$$
$$L^{(q,q)}_{\alpha\beta}X^{(q)}_\alpha X^{(q)}_\beta + L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\frac{d\epsilon_{\alpha\beta}}{dt}\frac{d\epsilon_{\gamma\zeta}}{dt} + 2L^{(1,vi)}_{(M)\alpha\beta\gamma}B^{(1)}_\alpha\frac{d\epsilon_{\beta\gamma}}{dt} + \tag{39}$$
$$\left. 2\rho L^{(0,el)}_{(M)\alpha\beta}\frac{dm_\alpha}{dt}E_\beta + 2L^{(el,q)}_{\alpha\beta}E_\alpha X^{(q)}_\beta + 2\rho L^{(0,q)}_{(M)\alpha\beta}\frac{dm_\alpha}{dt}X^{(q)}_\beta\right\}.$$

Relation (39) shows that the entropy production is a quadratic form in the components of the time derivative of the total specific magnetization axial vector, the components of the thermodynamic force conjugate to the partial specific magnetization axial vector, $B^{(1)}$, the components of the electric field, the components of the temperature gradient and the components of the time derivative of the strain tensor. The entropy production is a positive definite quadratic form, i.e.

$$\sigma^{(s)} \geq 0. \tag{40}$$

A medium is in a state of thermodynamic equilibrium if the entropy production vanishes. It follows from the positive definite character of this quantity that it vanishes if (see (39) and (21))

$$\frac{d\mathbf{m}}{dt} = 0, \quad \mathbf{E} = 0, \quad gradT = 0, \quad \frac{d\epsilon_{\alpha\beta}}{dt} = 0, \tag{41}$$

$$\mathbf{B}^{(1)}\left(T, \epsilon_{\alpha\beta}, \mathbf{m}, \mathbf{m}^{(1)}\right) = 0. \tag{42}$$

It is seen from (42) that at thermodynamic equilibrium the internal variable $\mathbf{m}^{(1)}$ depends on independent variables and it is determined if T, $\epsilon_{\alpha\beta}$ and $\mathbf{m}$ are given. From the positive definite character of the entropy production several inequalities may be derived for the components of the phenomenological coefficients, resulting from the fact that all the elements of the main diagonal of the matrix associated to the quadratic form (39) must be non-negative and all principal minors of this matrix must be non-negative (see [29], [44], [45] as examples in the case of isotropic magnetizable media and of three-dimensional isotropic and anisotropic rigid media). For instance, we have

$$L^{(0,0)}_{(M)\alpha\alpha} \geq 0, \quad L^{(el,el)}_{\alpha\alpha} \geq 0, \tag{43}$$

$$L^{(1,1)}_{(M)\alpha\alpha} \geq 0, \quad L^{(q,q)}_{\alpha\alpha} \geq 0, \quad L^{(vi,vi)}_{\alpha\beta\alpha\beta} \geq 0.$$

Also, from the fifth of the inequalities we obtain by virtue of symmetry and Onsager-Casimir relations

$$L^{(vi,vi)}_{\alpha\beta\beta\alpha} \geq 0, \quad L^{(vi,vi)}_{\beta\alpha\alpha\beta} \geq 0, \quad L^{(vi,vi)}_{\beta\alpha\beta\alpha} \geq 0. \tag{44}$$

4 Reference state and thermodynamic equilibrium state

Let us consider a reference state of the medium, with an arbitrary (but fixed) uniform temperature $T_{(0)}$, in which the mechanical stress tensor $\tau_{\alpha\beta}$ and the magnetic field $\mathbf{B}$ vanish in the medium. We notice that $\tau^{(eq)}_{\alpha\beta}$, $\mathbf{B}^{(eq)}$ and $\mathbf{B}^{(1)}$ are functions of the temperature $T_{(0)}$, of the strain tensor $\epsilon_{\alpha\beta}$ and of the magnetizations $\mathbf{m}$ and $\mathbf{m}^{(1)}$. We require that in this reference state (indicated by the symbol "$_{(0)}$") the value $\epsilon_{(0)\alpha\beta}$ for the strain tensor and the values $\mathbf{m}_{(0)}$ and $\mathbf{m}^{(1)}_{(0)}$ for the magnetization axial vectors are such that

$$\tau^{(eq)}_{\alpha\beta}\left(T_{(0)}, \epsilon_{(0)\alpha\beta}, \mathbf{m}_{(0)}, \mathbf{m}^{(1)}_{(0)}\right) = 0, \tag{45}$$

$$\mathbf{B}^{(eq)}\left(T_{(0)}, \epsilon_{(0)\alpha\beta}, \mathbf{m}_{(0)}, \mathbf{m}^{(1)}_{(0)}\right) = 0, \tag{46}$$

and

$$\mathbf{B}^{(1)}\left(T_{(0)}, \epsilon_{(0)\alpha\beta}, \mathbf{m}_{(0)}, \mathbf{m}^{(1)}_{(0)}\right) = 0. \tag{47}$$

Being $\tau^{(eq)}_{\alpha\beta}$ a symmetric tensor, equations (45)-(47) form a set of 12 equations for the values of the 6 independent components of the symmetric strain tensor $\epsilon_{(0)\alpha\beta}$ and the values of the 6 components of the vectors $\mathbf{m}_{(0)}$ and $\mathbf{m}^{(1)}_{(0)}$. All strains will be measured with respect to this state and we choose the tensor $\epsilon_{\alpha\beta}$ and the axial vectors $\mathbf{m}, \mathbf{m}^{(1)}$, so that they vanish in the reference state. Then, $\epsilon_{(0)\alpha\beta}=0$; $m_{(0)\alpha}=0$; $m^{(1)}_{(0)\alpha}=0$, from which we have

$$\tau^{(eq)}_{\alpha\beta}=0, \quad \mathbf{B}^{(eq)}=0, \quad \mathbf{B}^{(1)}=0,$$

$$\text{for} \quad T=T_{(0)} \quad \text{and} \quad \epsilon_{(0)\alpha\beta}=m_{(0)\alpha}=m^{(1)}_{(0)\alpha}=0. \tag{48}$$

A medium is in a state of thermodynamic equilibrium if the entropy production vanishes. It follows from (41) and (42) that the reference state is a state of thermodynamic equilibrium, provided that $\epsilon_{\alpha\beta}$ and the vectors $\mathbf{m}$ and $\mathbf{m}^{(1)}$ (determined by (45)-(47)) are kept constant. Moreover, from (41) the electric field must be kept vanishing in this state of thermodynamic equilibrium. We note that in the reference state the medium has the uniform temperature $T_{(0)}$ and hence grad T vanishes in this state. Moreover, from the conditions (41), (42) and the phenomenological equation (28) the viscous stress tensor $\tau^{(vi)}_{\alpha\beta}$ vanishes in the thermodynamic equilibrium and by virtue of $(21)_1$ it follows that

$$\tau_{\alpha\beta}=\tau^{(eq)}_{\alpha\beta}. \tag{49}$$

Then, the chosen reference state is a state of thermodynamic equilibrium.

5 Linear equations of state for anisotropic media with magnetic relaxation

Let us define the specific free energy f by $f=u-Ts$ and, by virtue of the local equilibrium hypothesis, it is assumed that in each point of the considered medium the reversible thermodynamics is applicable (see [29]-[37]). Therefore, the following definitions are valid

$$s=-\frac{\partial}{\partial T}f\left(T,\epsilon_{\alpha\beta},\mathbf{m},\mathbf{m}^{(1)}\right), \tag{50}$$

$$\tau^{(eq)}_{\alpha\beta}=\rho\frac{\partial}{\partial \epsilon_{\alpha\beta}}f\left(T,\epsilon_{\alpha\beta},\mathbf{m},\mathbf{m}^{(1)}\right), \tag{51}$$

$$\mathbf{B}^{(eq)}=\frac{\partial}{\partial \mathbf{m}}f\left(T,\epsilon_{\alpha\beta},\mathbf{m},\mathbf{m}^{(1)}\right), \tag{52}$$

$$\mathbf{B}^{(1)}=-\frac{\partial}{\partial \mathbf{m}^{(1)}}f\left(T,\epsilon_{\alpha\beta},\mathbf{m},\mathbf{m}^{(1)}\right). \tag{53}$$

Furthermore, from $f=u-Ts$, with the aid of Gibbs relation (17), one gets the following expression for the differential of f,

$$df=-sdT+v\tau^{(eq)}_{\alpha\beta}d\epsilon_{\alpha\beta}+\mathbf{B}^{(eq)}\cdot d\mathbf{m}-\mathbf{B}^{(1)}\cdot d\mathbf{m}^{(1)}. \tag{54}$$

Now, we assume the following form for the specific free energy f of an anisotropic medium with magnetic relaxation

$$f = f^{(1)} + f^{(2)}, \tag{55}$$

where $f^{(1)}$ is some function of the temperature and of the strain tensor and $f^{(2)}$ is some function of the temperature and of the specific magnetization axial vectors $\mathbf{m}$ and $\mathbf{m}^{(1)}$, i.e.

$$f^{(1)} = f^{(1)}(T, \epsilon_{\alpha\beta}) \quad \text{and} \quad f^{(2)} = f^{(2)}(T, \mathbf{m}, \mathbf{m}^{(1)}). \tag{56}$$

Now, we expand the free energy f into Taylor's series with respect to the considered reference state and we consider very small deviations with respect to this state. We assume the following form for the expansion of $f^{(1)}$ (characterizing an anisotropic Kelvin-Voigt medium)

$$f^{(1)} = v_{(0)} \left\{ \frac{1}{2} a_{\alpha\beta\gamma\zeta} \epsilon_{\alpha\beta} \epsilon_{\gamma\zeta} + (T - T_{(0)}) a_{\alpha\beta} \epsilon_{\alpha\beta} \right\} - \varphi(T), \tag{57}$$

where $v_{(0)}$ is the specific volume in the reference state, given by $v_{(0)} = \frac{1}{\rho_{(0)}}$. In the following, we shall replace it by $v = \frac{1}{\rho}$, which is supposed to be a constant. $\varphi(T)$ is some function of the temperature and $a_{\alpha\beta\gamma\zeta}$ and $a_{\alpha\beta}$ do not depend on the temperature and on the strain tensor. They are determined by the physical properties of the medium in the reference state and, furthermore, they are constant quantities and satisfy the following symmetry relations

$$a_{\alpha\beta\gamma\zeta} = a_{\beta\alpha\gamma\zeta} = a_{\alpha\beta\zeta\gamma} = a_{\beta\alpha\zeta\gamma} = a_{\gamma\zeta\alpha\beta} = a_{\gamma\zeta\beta\alpha} = a_{\zeta\gamma\alpha\beta} = a_{\zeta\gamma\beta\alpha}, \tag{58}$$

$$a_{\alpha\beta} = a_{\beta\alpha}.$$

Furthermore, we assume the following form for the expansion of $f^{(2)}$

$$f^{(2)} = \frac{1}{2} \rho \left\{ a^{(0,0)}_{(M)\alpha\beta}\, m_\alpha \left(m_\beta - 2 m^{(1)}_\beta \right) + a^{(1,1)}_{(M)\alpha\beta}\, m^{(1)}_\alpha m^{(1)}_\beta \right\} +$$

$$(T - T_{(0)}) \left(a^{(0)}_{(M)\alpha}\, m_\alpha - a^{(1)}_{(M)\alpha} m^{(1)}_\alpha \right). \tag{59}$$

In (59) the vectors $a^{(0)}_{(M)\alpha}$ and $a^{(1)}_{(M)\alpha}$ and the tensors $a^{(0,0)}_{(M)\alpha\beta}$, $a^{(1,1)}_{(M)\alpha\beta}$ are constant quantities (i.e. they do not depend on the temperature and on the specific magnetizations). They are determined by the physical properties of the medium in the reference state and, furthermore, they satisfy the following symmetry relations

$$a^{(0,0)}_{(M)\alpha\beta} = a^{(0,0)}_{(M)\beta\alpha}, \quad a^{(1,1)}_{(M)\alpha\beta} = a^{(1,1)}_{(M)\beta\alpha}. \tag{60}$$

The symmetry relations (58) and (60) are coming from the invariance properties with respect to the priority of derivation of $f^{(1)}$ and $f^{(2)}$ with respect their own independent variables (see[29] for some detailed calculations in the case of magnetizable media).

By virtue of (50) and (55)-(60) we have the following form for the specific entropy

$$s = -\left(a^{(0)}_{(M)\alpha} m_\alpha - a^{(1)}_{(M)\alpha} m^{(1)}_\alpha\right) - v\ a_{\alpha\beta}\epsilon_{\alpha\beta} + \frac{d\varphi}{dT}. \tag{61}$$

From (51) and (55)-(60) we obtain for the equilibrium stress tensor the expression form

$$\tau^{(eq)}_{\alpha\beta} = a_{\alpha\beta\gamma\zeta}\epsilon_{\alpha\beta} + a_{\alpha\beta}(T - T_{(0)}), \tag{62}$$

where $a_{\alpha\beta\gamma\zeta}$ are the elastic constants and $a_{\alpha\beta}$ are the thermoelastic constants. Now, we define the fields $\mathbf{M}^{(0)}$ and $\mathbf{M}^{(1)}$ by

$$\mathbf{M}^{(0)} = \rho\mathbf{m}^{(0)} \quad \text{and} \quad \mathbf{M}^{(1)} = \rho\mathbf{m}^{(1)}. \tag{63}$$

Finally, from (52), (53) and (55)-(60) we have the following equations of state

$$B^{(eq)}_\alpha = a^{(0,0)}_{(M)\alpha\beta}\left(M_\beta - M^{(1)}_\beta\right) + a^{(0)}_{(M)\alpha}(T - T_{(0)}), \tag{64}$$

$$B^{(1)}_\alpha = a^{(0,0)}_{(M)\alpha\beta}M_\beta - a^{(1,1)}_{(M)\alpha\beta}M^{(1)}_\beta + a^{(1)}_{(M)\alpha}(T - T_{(0)}). \tag{65}$$

Because of the mass density ρ is constant equations (24) and (25) for the irreversible magnetic relaxation phenomena may be written in the form

$$B_\alpha = B^{(eq)}_\alpha + L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} + L^{(0,1)}_{(M)\alpha\beta}B^{(1)}_\beta + \\ L^{(0,el)}_{(M)\alpha\beta}E_\beta + L^{(0,q)}_{(M)\alpha\beta}X^{(q)}_\beta + L^{(0,vi)}_{(M)\alpha\beta\gamma}\frac{d\epsilon_{\beta\gamma}}{dt}, \tag{66}$$

$$\frac{dM^{(1)}_\alpha}{dt} = L^{(1,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\beta + L^{(1,el)}_{(M)\alpha\beta}E_\beta + \\ L^{(1,q)}_{(M)\alpha\beta}X^{(q)}_\beta + L^{(1,vi)}_{(M)\alpha\beta\gamma}\frac{d\epsilon_{\beta\gamma}}{dt}, \tag{67}$$

where we have used the expression $(21)_2$. If all cross effects are neglected, except for possible cross effects among the different types of magnetic relaxation phenomena, we obtain the following equations for the irreversible magnetic relaxation phenomena, the stress tensor, the electric flux and the heat flux, respectively,

$$B_\alpha = B_\alpha^{(eq)} + L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} + L^{(0,1)}_{(M)\alpha\beta}B^{(1)}_\beta, \tag{68}$$

$$\frac{dM^{(1)}_\alpha}{dt} = L^{(1,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\beta, \tag{69}$$

$$\tau_{\alpha\beta} = a_{\alpha\beta\gamma\zeta}\epsilon_{\alpha\beta} + a_{\alpha\beta}(T - T_{(0)}) + L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\frac{d\epsilon_{\gamma\zeta}}{dt}, \tag{70}$$

$$J^{(q)}_\alpha = L^{(q,q)}_{\alpha\beta}X^{(q)}_\beta, \quad J^{(el)}_\alpha = L^{(el,el)}_{\alpha\beta}E_\beta. \tag{71}$$

See equations (26)-(28), (62), (66) and (67).

6 Magnetic relaxation equation for anisotropic media with magnetic relaxation

Taking into account (64) and (65), equations (68) and (69) may be written, respectively, in the form (see [7]):

$$c^{(1)}_{\alpha\beta}M^{(1)}_\beta = Q^{(1)}_{(0,0)\alpha}, \tag{72}$$

where

$$c^{(1)}_{\alpha\beta} = a^{(0,0)}_{(M)\alpha\beta} + L^{(0,1)}_{(M)\alpha\gamma}a^{(1,1)}_{(M)\gamma\beta}, \tag{73}$$

$$\begin{aligned} Q^{(1)}_{(0,0)\alpha} = &\left(a^{(0,0)}_{(M)\alpha\beta} + L^{(0,1)}_{(M)\alpha\gamma}a^{(0,0)}_{(M)\gamma\beta}\right)M_\beta + L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} - B_\alpha \\ &+ \left(a^{(0)}_{(M)\alpha} + L^{(0,1)}_{(M)\alpha\beta}a^{(1)}_{(M)\beta}\right)(T - T_{(0)}), \end{aligned} \tag{74}$$

and

$$\frac{dM^{(1)}_\beta}{dt} + h_{\beta\gamma}M^{(1)}_\gamma = Q_{(1,0)\beta}, \tag{75}$$

where

$$h_{\beta\gamma} = L^{(1,1)}_{(M)\beta\eta}a^{(1,1)}_{(M)\eta\gamma} \tag{76}$$

and

$$Q_{(1,0)\beta} = L^{(1,1)}_{(M)\beta\eta}a^{(0,0)}_{(M)\eta\gamma}M_\gamma + L^{(1,0)}_{(M)\beta\gamma}\frac{dM_\gamma}{dt} + L^{(1,1)}_{(M)\beta\gamma}a^{(1)}_{(M)\gamma}(T - T_{(0)}). \tag{77}$$

Assuming that it is possible to define the inverse matrix $(c^{(1)}_{\alpha\beta})^{-1}$, such that

$$(c^{(1)}_{\alpha\beta})^{-1}c^{(1)}_{\beta\gamma} = c^{(1)}_{\alpha\beta}(c^{(1)}_{\beta\gamma})^{-1} = \delta_{\alpha\gamma}, \tag{78}$$

eliminating the internal magnetic axial field $\mathbf{M}^{(1)}$, given by (see (72))

$$M^{(1)}_{\alpha} = (c^{(1)}_{\alpha\beta})^{-1} Q^{(1)}_{(0,0)\beta}, \tag{79}$$

one gets the following magnetic relaxation equation (derived in [7])

$$\chi^{(0)}_{(BM)\alpha\beta} B_\beta + \frac{dB_\alpha}{dt} = \chi^{(0)}_{(MB)\alpha\beta} M_\beta + \chi^{(1)}_{(MB)\alpha\beta} \frac{dM_\beta}{dt}$$

$$+ \chi^{(2)}_{(MB)\alpha\beta} \frac{d^2 M_\beta}{dt^2} + \chi^{(0)}_{(T)\alpha}(T - T_0) + \chi^{(1)}_{(T)\alpha} \frac{dT}{dt}, \tag{80}$$

where

$$\chi^{(0)}_{(BM)\alpha\beta} = c^{(1)}_{\alpha\gamma} h_{\gamma\zeta} (c^{(1)}_{\zeta\beta})^{-1}, \tag{81}$$

$$\chi^{(0)}_{(MB)\alpha\beta} = c^{(1)}_{\alpha\gamma} \left\{ h_{\gamma\zeta} (c^{(1)}_{\zeta\eta})^{-1} \left(a^{(0,0)}_{(M)\eta\beta} + L^{(0,1)}_{(M)\eta\mu} a^{(0,0)}_{(M)\mu\beta} \right) - L^{(1,1)}_{(M)_{\gamma\mu}} a^{(0,0)}_{(M)\mu\beta} \right\}, \tag{82}$$

$$\chi^{(1)}_{(MB)\alpha\beta} = c^{(1)}_{\alpha\gamma} \left\{ h_{\gamma\zeta} (c^{(1)}_{\zeta\eta})^{-1} L^{(0,0)}_{(M)\eta\beta} - L^{(1,0)}_{(M)\gamma\beta} \right\} + a^{(0,0)}_{(M)\alpha\beta} + L^{(0,1)}_{(M)\alpha\eta} a^{(0,0)}_{(M)\eta\beta}, \tag{83}$$

$$\chi^{(2)}_{(MB)\alpha\beta} = L^{(0,0)}_{(M)\alpha\beta}, \tag{84}$$

$$\chi^{(0)}_{(T)\alpha} = c^{(1)}_{\alpha\gamma} \left\{ h_{\gamma\zeta} (c^{(1)}_{\zeta\eta})^{-1} \left(a^{(0)}_{(M)\eta} + L^{(0,1)}_{(M)\eta\beta} a^{(1)}_{(M)\beta} \right) - L^{(1,1)}_{(M)\gamma\beta} a^{(1)}_{(M)\beta} \right\}, \tag{85}$$

$$\chi^{(1)}_{(T)\alpha} = a^{(0)}_{(M)\alpha} + L^{(0,1)}_{(M)\alpha\beta} a^{(1)}_{(M)\beta}. \tag{86}$$

Hence, it is seen that the linearization of the theory leads to a relaxation equation for anisotropic magnetizable media which has the form of a linear relation among the temperature, the magnetic field, the total magnetization field, the time derivative of the temperature, the first derivatives with respect to time of the of magnetic field and of the total magnetization field and the second derivative with respect to time of this last axial vector.

In [3], in the linear approximation, Kluitenberg derived, magnetizable media, isotropic with respect to all the rotations and inversions of the frame of axes, and having the total magnetization $\mathbf{M}$ composed of two irreversible parts, i. e. $\mathbf{M} = \mathbf{M}^{(0)} + \mathbf{M}^{(1)}$, the following magnetic relaxation equation, by eliminating the internal variable,

$$\chi^{(0)}_{(BM)} \mathbf{B} + \frac{d\mathbf{B}}{dt} = \chi^{(0)}_{(MB)} \mathbf{M} + \chi^{(1)}_{(MB)} \frac{d\mathbf{M}}{dt} + \chi^{(2)}_{(MB)} \frac{d^2\mathbf{M}}{dt^2}, \tag{87}$$

where $\chi^{(0)}_{(BM)}$ and $\chi^{(k)}_{(MB)}$ $(k = 0, 1, 2)$ are constant quantities, algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state. From (87), we see that in the isotropic case in the magnetic relaxation equation there is no the contribution due to the temperature field. Using (73), (74), (79), (84) and (86), equation (65) may be written in the form

$$B_\alpha^{(1)} = (c_{\beta\gamma}^{(1)})^{-1} \Big[a_{(M)\alpha\beta}^{(0,0)} \left(a_{(M)\gamma\zeta}^{(0,0)} + L_{(M)\gamma\xi}^{(0,1)} a_{(M)\xi\zeta}^{(1,1)} \right)$$

$$- a_{(M)\alpha\beta}^{(1,1)} \left(a_{(M)\gamma\zeta}^{(0,0)} + L_{(M)\gamma\xi}^{(0,1)} a_{(M)\xi\zeta}^{(0,0)} \right) \Big] M_\zeta - (c_{\beta\gamma}^{(1)})^{-1} \Big[a_{(M)\alpha\beta}^{(1,1)} L_{(M)\gamma\zeta}^{(0,0)} \frac{dM_\zeta}{dt} \tag{88}$$

$$+ a_{(M)\alpha\beta}^{(1,1)} \left(a_{(M)\gamma}^{(0)} + L_{(M)\gamma\zeta}^{(0,1)} a_{(M)\zeta}^{(1)} \right) (T - T_0) - a_{(M)\alpha\beta}^{(1,1)} B_\gamma \Big] + a_{(M)\alpha}^{(1)} (T - T_0),$$

where

$$c_{\beta\gamma}^{(1)} = a_{(M)\beta\gamma}^{(0,0)} + L_{(M)\beta\eta}^{(0,1)} a_{(M)\eta\gamma}^{(1,1)}. \tag{89}$$

Hence, using (84) one obtains

$$B_\alpha^{(1)} = D_{(M)\alpha\beta}^{(1)} M_\beta + D_{(M)\alpha\beta}^{(2)} \frac{dM_\beta}{dt} + D_{(M)\alpha\beta}^{(3)} B_\beta + D_{(M)\alpha}^{(4)} (T - T_0), \tag{90}$$

where

$$D_{(M)\alpha\beta}^{(1)} = (c_{\zeta\gamma}^{(1)})^{-1} \left[a_{(M)\alpha\zeta}^{(0,0)} c_{\gamma\beta}^{(1)} - a_{(M)\alpha\zeta}^{(1,1)} \left(a_{(M)\gamma\beta}^{(0,0)} + L_{(M)\gamma\xi}^{(0,1)} a_{(M)\xi\beta}^{(0,0)} \right) \right], \tag{91}$$

$$D_{(M)\alpha\beta}^{(2)} = -(c_{\zeta\gamma}^{(1)})^{-1} a_{(M)\alpha\zeta}^{(1,1)} L_{(M)\gamma\beta}^{(0,0)}, \tag{92}$$

$$D_{(M)\alpha\beta}^{(3)} = -(c_{\gamma\beta}^{(1)})^{-1} a_{(M)\alpha\gamma}^{(1,1)}, \tag{93}$$

$$D_{(M)\alpha}^{(4)} = -(c_{\beta\gamma}^{(1)})^{-1} a_{(M)\alpha\beta}^{(1,1)} \left(a_{(M)\gamma}^{(0,0)} + L_{(M)\gamma\xi}^{(0,1)} a_{(M)\xi}^{(1)} \right) + a_{(M)\alpha}^{(1)}. \tag{94}$$

From (90) $\mathbf{B}^{(1)}$ may be expressed as a linear function of the temperature, the magnetic field, the total magnetization field and the time derivative of this last axial vector field.

7 The heat dissipation function for anisotropic media with magnetic relaxation

Using (68)-(71), we obtain from (19) and (20) the following balance equation for the specific entropy s

$$\rho \frac{ds}{dt} = \frac{\partial}{\partial x_\alpha} \left(T^{-2} L_{\alpha\beta}^{(q,q)} \frac{\partial T}{\partial x_\beta} \right) + T^{-1} \left\{ T^{-2} L_{\alpha\beta}^{(q,q)} \frac{\partial T}{\partial x_\alpha} \frac{\partial T}{\partial x_\beta} \right\}$$

$$+T^{-1}\left\{\ L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\alpha}{dt}\frac{dM_\beta}{dt}+L^{(1,1)}_{(M)\alpha\beta}B^{(1)}_\alpha B^{(1)}_\beta+L^{(el,el)}_{\alpha\beta}E_\alpha E_\beta+\right. \tag{95}$$
$$\left.+L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\frac{d\epsilon_{\alpha\beta}}{dt}\frac{d\epsilon_{\gamma\zeta}}{dt}\right\}.$$

From the expression $u = f + Ts$ and equations (55)-(61) we obtain the following form for the specific internal energy

$$u=\frac{1}{2}\rho\left\{a^{(0,0)}_{(M)\alpha\beta}\ m_\alpha\left(m_\beta-2m^{(1)}_\beta\right)+a^{(1,1)}_{(M)\alpha\beta}\ m^{(1)}_\alpha m^{(1)}_\beta\right\}$$
$$-T_{(0)}\left(a^{(0)}_{(M)\alpha}\ m_\alpha-a^{(1)}_{(M)\alpha}m^{(1)}_\alpha\right) \tag{96}$$
$$+v\left\{\frac{1}{2}a_{\alpha\beta\gamma\zeta}\epsilon_{\alpha\beta}\epsilon_{\gamma\zeta}-T_{(0)}a_{\alpha\beta}\epsilon_{\alpha\beta}\right\}+T\frac{d\varphi}{dT}-\varphi(T).$$

The specific heat at constant deformation $c(\epsilon)$ may be defined by

$$c(\epsilon)=\frac{\partial}{\partial T}u\left(T,\epsilon_{\alpha\beta},\mathbf{m},\mathbf{m}^{(1)}\right). \tag{97}$$

Hence, we obtain

$$c(\epsilon)=T\frac{d^2\varphi}{dT^2}, \tag{98}$$

and, if $c(\epsilon)$ is constant, we have

$$\varphi=c(\epsilon)Tlog\frac{T}{T_{(0)}}+s_{(0)}T-c(\epsilon)(T-T_{(0)})-u_{(0)}, \tag{99}$$

where $s_{(0)}$ and $u_{(0)}$ are integration constants and represent the specific entropy and the specific internal energy in the reference state, respectively. Then, the first law of thermodynamics (9) becomes

$$\frac{1}{2}a^{(0,0)}_{(M)\alpha\beta}\ \frac{d}{dt}\left\{M_\alpha\left(M_\beta-2M^{(1)}_\beta\right)\right\}+a^{(1,1)}_{(M)\alpha\beta}\ M^{(1)}_\alpha\frac{d}{dt}M^{(1)}_\beta$$
$$-T_{(0)}\left(a^{(0)}_{(M)\alpha}\ \frac{d}{dt}M_\alpha-a^{(1)}_{(M)\alpha}\frac{d}{dt}M^{(1)}_\alpha\right)$$
$$+\frac{1}{2}a_{\alpha\beta\gamma\zeta}\frac{d}{dt}(\epsilon_{\alpha\beta}\epsilon_{\gamma\zeta})-T_{(0)}a_{\alpha\beta}\frac{d}{dt}\epsilon_{\alpha\beta}+\rho c_{(\epsilon)}\frac{d}{dt}T= \tag{100}$$
$$-div\mathbf{J}^{(q)}+\tau_{\alpha\beta}\frac{d\epsilon_{\alpha\beta}}{dt}+J^{(el)}_\alpha E_\alpha+B_\alpha\frac{dM_\alpha}{dt},$$

where we have supposed that $J^{(el)}_\alpha, J^{(q)}_\alpha, \tau^{(eq)}_{\alpha\beta}$ are given by (70) and (71).

From (61) and (98) it follows that

$$\rho \frac{ds}{dt} = -a^{(0)}_{(M)\alpha} \frac{dM_\alpha}{dt} + a^{(1)}_{(M)\alpha} \frac{dM^{(1)}_\alpha}{dt} - a_{\alpha\beta} \frac{d\epsilon_{\alpha\beta}}{dt} + \rho c_{(\epsilon)} T^{-1} \frac{dT}{dt}. \tag{101}$$

By virtue of (95) and (101) we obtain the following equation for the heat conduction

$$\rho c_{(\epsilon)} \frac{dT}{dt} = T \left\{ a^{(0)}_{(M)\alpha} \frac{dM_\alpha}{dt} - a^{(1)}_{(M)\alpha} \frac{dM^{(1)}_\alpha}{dt} + a_{\alpha\beta} \frac{d\epsilon_{\alpha\beta}}{dt} \right\}$$
$$+ \frac{\partial}{\partial x_\alpha} \left(T^{-1} L^{(q,q)}_{\alpha\beta} \frac{\partial T}{\partial x_\beta} \right) + \sigma^{(h)}, \tag{102}$$

where $\mathbf{M}^{(1)}$ is given by (see (79))

$$M^{(1)}_\alpha = \left(a^{(0,0)}_{(M)\alpha\beta} + L^{(0,1)}_{(M)\alpha\gamma} a^{(1,1)}_{(M)\gamma\beta} \right)^{-1} Q^{(1)}_{(0,0)\beta} \tag{103}$$

and $\sigma^{(h)}$ has the following form

$$\sigma^{(h)} = \sigma^{(M)} + \sigma^{(E)} + \sigma^{(R)}, \tag{104}$$

with

$$\sigma^{(M)} = L^{(0,0)}_{(M)\alpha\beta} \frac{dM_\alpha}{dt} \frac{dM_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta} B^{(1)}_\alpha B^{(1)}_\beta, \tag{105}$$

$$\sigma^{(E)} = L^{(el,el)}_{\alpha\beta} E_\alpha E_\beta, \tag{106}$$

$$\sigma^{(R)} = L^{(vi,vi)}_{\alpha\beta\gamma\zeta} \frac{d\epsilon_{\alpha\beta}}{dt} \frac{d\epsilon_{\gamma\zeta}}{dt}. \tag{107}$$

The physical quantity $\sigma^{(h)}$ is called heat dissipation function and $\sigma^{(M)}$, $\sigma^{(E)}$, and $\sigma^{(R)}$ represent the heat dissipated per unit of volume and per unit of time by the magnetic relaxation, electric conduction, viscous and magnetic irreversible phenomena, respectively. $\sigma^{(R)}$ is a generalized Rayleigh dissipation function. When $\sigma^{(M)}$ vanishes, no magnetic phenomena are present and $\sigma^{(h)}$ reduces to $\sigma^{(E)} + \sigma^{(R)}$.

Substituting in (105) the expression (90) of $\mathbf{B}^{(1)}$ conjugate to the internal variable, we eliminate the internal variable, obtaining the following form for $\sigma^{(M)}$:

$$\sigma^{(M)} = L^{(0,0)}_{(M)\alpha\beta} \frac{dM_\alpha}{dt} \frac{dM_\beta}{dt} + L^{(1,1)}_{(M)\alpha\beta} \Bigg[D^{(1)}_{(M)\alpha\zeta} M_\zeta + D^{(2)}_{(M)\alpha\zeta} \frac{dM_\zeta}{dt} + D^{(3)}_{(M)\alpha\zeta} B_\zeta +$$

$$D^{(4)}_{(M)\alpha}(T-T_0)\Big]\cdot\Big[D^{(1)}_{(M)\beta\eta}M_\eta+D^{(2)}_{(M)\beta\eta}\frac{dM_\eta}{dt}+D^{(3)}_{(M)\beta\eta}B_\eta+D^{(4)}_{(M)\beta}(T-T_0)\Big], \tag{108}$$

i.e.,

$$\sigma^{(M)}=G^{(1)}_{\alpha\beta}\frac{dM_\alpha}{dt}\frac{dM_\beta}{dt}+G^{(2)}_{\alpha\beta}M_\alpha M_\beta+G^{(3)}_{\alpha\beta}M_\alpha\frac{dM_\beta}{dt}+G^{(4)}_{\alpha\beta}\frac{dM_\alpha}{dt}M_\beta+$$
$$G^{(5)}_{\alpha\beta}B_\beta\frac{dM_\alpha}{dt}+G^{(6)}_{\alpha\beta}B_\alpha B_\beta+G^{(7)}_\alpha(T-T_0)M_\alpha+G^{(8)}_\alpha(T-T_0)\frac{dM_\alpha}{dt}+$$
$$G^{(9)}_\alpha(T-T_0)B_\alpha+G^{(10)}(T-T_0)^2, \tag{109}$$

where

$G^{(1)}_{\alpha\beta}=L^{(0,0)}_{(M)\alpha\beta}+L^{(1,1)}_{(M)\gamma\xi}D^{(2)}_{(M)\gamma\alpha}D^{(2)}_{(M)\xi\beta}$,
$G^{(2)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\xi}D^{(1)}_{(M)\gamma\alpha}D^{(1)}_{(M)\xi\beta}$,
$G^{(3)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\xi}\left(D^{(1)}_{(M)\gamma\alpha}D^{(2)}_{(M)\xi\beta}+D^{(1)}_{(M)\gamma\beta}D^{(2)}_{(M)\xi\alpha}\right)$,
$G^{(4)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\xi}\left(D^{(1)}_{(M)\gamma\alpha}D^{(3)}_{(M)\xi\beta}+D^{(1)}_{(M)\gamma\beta}D^{(3)}_{(M)\xi\alpha}\right)$,
$G^{(5)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\xi}\left(D^{(2)}_{(M)\gamma\alpha}D^{(3)}_{(M)\xi\beta}+D^{(2)}_{(M)\xi\alpha}D^{(3)}_{(M)\gamma\beta}\right)$,
$G^{(6)}_{\alpha\beta}=L^{(1,1)}_{(M)\gamma\xi}D^{(3)}_{(M)\gamma\alpha}D^{(3)}_{(M)\xi\beta}$,
$G^{(7)}_{\alpha}=L^{(1,1)}_{(M)\gamma\xi}\left(D^{(1)}_{(M)\gamma\alpha}D^{(4)}_{(M)\xi}+D^{(1)}_{(M)\xi\alpha}D^{(4)}_{(M)\gamma}\right)$,
$G^{(8)}_{\alpha}=L^{(1,1)}_{(M)\gamma\xi}D^{(2)}_{(M)\gamma\alpha}D^{(4)}_{(M)\xi}$,
$G^{(9)}_{\alpha}=L^{(1,1)}_{(M)\gamma\xi}\left(D^{(3)}_{(M)\gamma\alpha}D^{(4)}_{(M)\xi}+D^{(4)}_{(M)\gamma}D^{(3)}_{(M)\xi\alpha}\right)$,
$G^{(10)}=L^{(1,1)}_{(M)\alpha\beta}D^{(4)}_{(M)\alpha}D^{(4)}_{(M)\beta}$.

In (108) the coefficients $D^{(i)}_{(M)\alpha\beta}$ $(i=1,2,3,4)$ are given by (91)-(94). From (106), (107) and (109) it is seen that the heat dissipation function $\sigma^{(h)}$ has the form of a quadratic function of the components of the strain tensor, the components of the electric field, the components of the total magnetization field, the components of the time derivatives of this last axial vector, the magnetic field and of the temperature.

8 Heat dissipation function for anisotropic Snoek media, where $\mathbf{m}^{(0)}$ is reversible

In the case where $\mathbf{B}=\mathbf{B^{(eq)}}$, $\mathbf{B^{(ir)}}$ vanishes (see $(21)_2$) and from (68) and (69) one gets

$$L^{(0,0)}_{(M)\alpha\beta}=0,\quad\text{and}\quad L^{(0,1)}_{(M)\alpha\beta}=-L^{(1,0)}_{(M)\beta\alpha}=0. \tag{110}$$

Also from (23) it is seen that in Snoek media the specific partial magnetization $\mathbf{m}^{(0)}$ becomes a reversible part of the specific total magnetization $\mathbf{m}$.

Hence, the magnetic relaxation equation (80) becomes

$$\chi^{(0)}_{(BM)\alpha\beta}B_\beta+\frac{dB_\alpha}{dt}=\chi^{(0)}_{(MB)\alpha\beta}M_\beta+\chi^{(1)}_{(MB)\alpha\beta}\frac{dM_\beta}{dt}+\chi^{(0)}_{(T)\alpha}(T-T_0)+\chi^{(1)}_{(T)\alpha}\frac{dT}{dt}, \tag{111}$$

where

$$\chi^{(0)}_{(BM)\alpha\beta}=a^{(0,0)}_{(M)\alpha\gamma}L^{(1,1)}_{(M)\gamma\eta}a^{(1,1)}_{(M)\eta\zeta}\left(a^{(0,0)}_{(M)\zeta\beta}\right)^{-1}, \tag{112}$$

$$\chi^{(0)}_{(MB)\alpha\beta}=a^{(0,0)}_{(M)\alpha\eta}L^{(1,1)}_{(M)\eta\gamma}\left(a^{(1,1)}_{(M)\gamma\beta}-a^{(0,0)}_{(M)\gamma\beta}\right), \tag{113}$$

$$\chi^{(1)}_{MB)\alpha\beta}=a^{(0,0)}_{(M)\alpha\beta}, \tag{114}$$

$$\chi^{(0)}_{(T)\alpha}=a^{(0,0)}_{(M)\alpha\beta}L^{(1,1)}_{(M)\beta\gamma}\left\{a^{(1,1)}_{(M)\gamma\eta}\left(a^{(0,0)}_{(M)\eta\zeta}\right)^{-1}a^{(0)}_{(M)\zeta}-a^{(1)}_{(M)\gamma}\right\}, \tag{115}$$

$$\chi^{(1)}_{(T)\alpha}=a^{(0)}_{(M)\alpha}. \tag{116}$$

For the derivation of equation (111) see [7]. Taking into account (90)-(94) one gets

$$B^{(1)}_\alpha=D^{(1)}_{(M)\alpha\beta}M_\beta+D^{(3)}_{(M)\alpha\beta}B_\beta+D^{(4)}_{(M)\alpha}(T-T_0), \tag{117}$$

where

$$D^{(1)}_{(M)\alpha\beta}=(a^{(0,0)}_{(M)\zeta\gamma})^{-1}a^{(0,0)}_{(M)\gamma\beta}\left(a^{(0,0)}_{(M)\alpha\zeta}-a^{(1,1)}_{(M)\alpha\zeta}\right), \tag{118}$$

$$D^{(3)}_{(M)\alpha\beta}=-(a^{(0,0)}_{(M)\gamma\beta})^{-1}a^{(1,1)}_{(M)\alpha\gamma}, \tag{119}$$

$$D^{(4)}_{(M)\alpha}=-(a^{(0,0)}_{(M)\beta\gamma})^{-1}a^{(1,1)}_{(M)\alpha\beta}a^{(0)}_{(M)\gamma}+a^{(1)}_{(M)\alpha}, \tag{120}$$

being $D^{(2)}_{(M)\alpha\beta}=0$.

Thus, we obtain the heat conduction equation (102), where the heat dissipation function $\sigma^{(h)}$ has the form (104), $\sigma^{(h)}=\sigma^{(M)}+\sigma^{(E)}+\sigma^{(R)}$, with $\sigma^{(E)}$ and $\sigma^{(R)}$ holding the same expressions $\sigma^{(E)}=L^{(el,el)}_{\alpha\beta}E_\alpha E_\beta$, $\sigma^{(R)}=L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\frac{d\epsilon_{\alpha\beta}}{dt}\frac{d\epsilon_{\gamma\zeta}}{dt}$.

Substituting in (105) the expression (117) of $\mathbf{B}^{(1)}$ conjugate to the internal variable, we eliminate the internal variable, obtaining the following form for $\sigma^{(M)}$ $\sigma^{(M)} = L^{(1,1)}_{(M)\alpha\beta}\left[D^{(1)}_{(M)\alpha\zeta}M_\zeta + D^{(3)}_{(M)\alpha\zeta}B_\zeta + D^{(4)}_{(M)\alpha}(T-T_0)\right]\cdot$ $\left[D^{(1)}_{(M)\beta\eta}M_\eta + D^{(3)}_{(M)\beta\eta}B_\eta + D^{(4)}_{(M)\beta}(T-T_0)\right]$,

$$\sigma^{(M)} = G^{(2)}_{\alpha\beta}M_\alpha M_\beta + G^{(4)}_{\alpha\beta}\frac{dM_\alpha}{dt}M_\beta + G^{(6)}_{\alpha\beta}B_\alpha B_\beta + G^{(7)}_\alpha(T-T_0)M_\alpha + G^{(9)}_\alpha(T-T_0)B_\alpha + G^{(10)}(T-T_0)^2, \tag{121}$$

where
$G^{(2)}_{\alpha\beta} = L^{(1,1)}_{(M)\gamma\xi}D^{(1)}_{(M)\gamma\alpha}D^{(1)}_{(M)\xi\beta}$,
$G^{(4)}_{\alpha\beta} = L^{(1,1)}_{(M)\gamma\xi}\left(D^{(1)}_{(M)\gamma\alpha}D^{(3)}_{(M)\xi\beta} + D^{(1)}_{(M)\gamma\beta}D^{(3)}_{(M)\xi\alpha}\right)$,
$G^{(6)}_{\alpha\beta} = L^{(1,1)}_{(M)\gamma\xi}D^{(3)}_{(M)\gamma\alpha}D^{(3)}_{(M)\xi\beta}$,
$G^{(7)}_{\alpha} = L^{(1,1)}_{(M)\gamma\xi}\left(D^{(1)}_{(M)\gamma\alpha}D^{(4)}_{(M)\xi} + D^{(1)}_{(M)\xi\alpha}D^{(4)}_{(M)\gamma}\right)$,
$G^{(9)}_{\alpha} = L^{(1,1)}_{(M)\gamma\xi}\left(D^{(3)}_{(M)\gamma\alpha}D^{(4)}_{(M)\xi} + D^{(4)}_{(M)\gamma}D^{(3)}_{(M)\xi\alpha}\right)$,
$G^{(10)} = L^{(1,1)}_{(M)\alpha\beta}D^{(4)}_{(M)\alpha}D^{(4)}_{(M)\beta}$.
Furthermore, $\mathbf{M}^{(1)}$ has the form (see (103))

$$M^{(1)}_\alpha = \left(a^{(0,0)}_{(M)\alpha\beta}\right)^{-1}\left(a^{(0,0)}_{(M)\alpha\beta}M_\beta - B_\alpha + a^{(0)}_{(M)\alpha}(T-T_0)\right). \tag{122}$$

If the coefficients $L^{(1,1)}_{(M)\alpha\beta}$ vanish the relaxation equation (111) becomes

$$B_\beta = a^{(0,0)}_{(M)\alpha\beta}M_\beta + a^{(0)}_{(M)\alpha}T, \tag{123}$$

where we have continued to call T the quantity $T-T_0$. In this case changes in the magnetic field $\mathbf{B}$ are associated with changes in the magnetic axial vector $\mathbf{M}$ and in the temperature T. This result is well-known for media without magnetic relaxation [7].

9 Heat dissipation function for magnetizable anisotropic De Groot-Mazur media, where there is no $\mathbf{m}^{(1)}$

In the special case where

$$L^{(1,1)}_{(M)\alpha\beta} = 0, \quad \text{and} \quad L^{(1,0)}_{(M)\alpha\beta} = -L^{(0,1)}_{(M)\alpha\beta} = 0, \tag{124}$$

equations (68) and (69) become

$$B_{\alpha}^{(ir)} = L_{(M)\alpha\beta}^{(0,0)} \frac{dM_{\beta}}{dt} \tag{125}$$

and

$$\frac{dM_{\alpha}^{(1)}}{dt} = 0. \tag{126}$$

From (126) it is seen that $\mathbf{M}^{(1)}$ is constant and it can be supposed that $\mathbf{M}^{(1)} = 0$, i.e., there is no internal variable and from (23) $\mathbf{m} = \mathbf{m}^{(0)}$. Equation (80) reduces to magnetic relaxation equation for De Groot-Mazur media (see [7] and [1]-[3])

$$B_{\alpha} = \chi_{(MB)\alpha\beta}^{(1)} M_{\beta} + \chi_{(MB)\alpha\beta}^{(2)} \frac{dM_{\beta}}{dt} + \chi_{(T)\alpha}^{(1)} T, \tag{127}$$

where we have continued to call T the quantity $T - T_0$.

$$\chi_{(MB)\alpha\beta}^{(1)} = a_{(M)\alpha\beta}^{(0,0)}, \tag{128}$$

$$\chi_{(MB)\alpha\beta}^{(2)} = L_{(M)\alpha\beta}^{(0,0)}, \tag{129}$$

$$\chi_{T)\alpha}^{(1)} = a_{(M)\alpha}^{(0)}, \tag{130}$$

and in the heat conduction equation (102) the heat dissipation function $\sigma^{(M)}$ (105) reduces to the heat dissipation function for De Groot-Mazur media given by

$$\sigma^{(M)} = L_{(M)\alpha\beta}^{(0,0)} \frac{dM_{\alpha}}{dt} \frac{dM_{\beta}}{dt}, \tag{131}$$

and $\sigma^{(E)}$ and $\sigma^{(R)}$ take the expressions (106) and (107), respectively.

10 Conclusions

The paper deals with the study of the heat dissipation function in anisotropic magnetizable media, where different types of irreversible microscopic phenomena give rise to magnetic relaxation. The obtained results can be applied in several physical situations, in nuclear magnetic resonance in medicine and biology and other different fields of applied sciences, where complex media are used. A model for these magnetizable media was given by Maugin, that used a multimagnetic-sublattice approach. In this paper the standard procedures of irreversible thermodynamics with internal variables were used and, following a Kluitenberg theory, it was assumed that it is possible to describe the microscopic phenomena, giving rise the magnetic relaxation, splitting the total specific magnetization in two irreversible parts and introducing one of these partial specific magnetizations as internal variable in the thermodynamic state space. Linearizing the theory, the heat conduction equation and the heat dissipation function for these magnetizable anisotropic media were derived. It was seen that this last function is due to electric conduction, magnetic relaxation, viscous, and magnetic irreversible phenomena. The obtained results are applied to the special cases of anisotropic Snoek media, where the specific magnetization $\mathbf{m}^{(0)}$ becomes reversible part of $\mathbf{m}$ and De Groot-Mazur media, where there is no internal variable and $\mathbf{m} = \mathbf{m}^{(0)}$.

The author and Kluitenberg also studied the case of anisotropic polarizable media with relaxation, in which the total polarization vector is due to different microscopic irreversible dielectric phenomena and the total polarization vector is split in two irreversible parts (see [46]), and the case of anisotropic mechanical media with relaxation, in which the inelastic strain is due to different microscopic phenomena and the total small deformation tensor is split in two irreversible contributions (see [47]).

Dedication

I would like to dedicate this paper to Professor Dan Tiba, my dear friend, great teacher and eminent scientist, corresponding member of "Accademia Peloritana dei Pericolanti" di Messina, on the occasion of his seventieth birthday.

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