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ON THE EQUILIBRIUM EQUATIONS OF LINEAR 6-PARAMETER ELASTIC SHELLS*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

We consider the linearized theory of 6-parameter elastic shells with general anisotropy. We derive the equilibrium equations from the virtual power statement and formulate the corresponding variational problem in the suitable functional framework. Then, using a Korn-type inequality for the linearized strain measures we prove the existence and uniqueness of weak solutions. Finally, we show that our general theorem can be applied to obtain existence results in the case of isotropic elastic shells. We illustrate this procedure by investigating three different linear shell models established previously in the literature, namely the simplified isotropic 6-parameter shell, the Cosserat isotropic model, and the higher-order 6-parameter Cosserat model.

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1 Introduction

The theory of elastic shells is an important chapter of the mechanics of solids, since it has many significant applications in mechanical and civil engineering, in automotive and aerospace engineering etc. In the scientific literature there exist several linear or nonlinear models designed to describe the mechanical behaviour of thin elastic shells. This is due to the fact that any shell theory has an approximate character, since it attempts to describe a three-dimensional body by means of a two-dimensional model. Thus, a shell model should be simple enough, on the one hand, to be manageable in practical engineering problems but, on the other hand, it should be complex enough to account for significant curvature and three-dimensional effects. To overcome the limitations of the classical shell theory, the researchers have developed more refined shell models which can also be applied for shells made of advanced materials with microstructure.

From a mathematical point of view, one of the main tasks is to prove that the two-dimensional boundary-value problem for shells is well posed. This is still an open problem for many nonlinear models, since for instance the well posedness has not yet been proved for the Koiter or the Reissner–Mindlin models (see, e.g., [1] for an extensive account). In the linear theory, the classical models have been justified by means of asymptotic analysis, and the existence of solutions has been proved by Ciarlet and collaborators (see, e.g., [2]).

Concerning the refined linear shell models, we mention that the generalized Naghdi shells have been investigated mathematically by Sprekels and Tiba in [3]. For the model of Cosserat surfaces (two-dimensional continua endowed with a single deformable director), the existence theorems for equilibrium and dynamical equations has been established in [4]. Another Cosserat approach to linear elastic shells is the theory of simple shells (also called directed surfaces), which is a Reissner-type approach with 5 parameters. This model for elastic or thermoelastic shells has been studied in the papers [5, 6], where the existence and uniqueness of solutions have been proved.

In this paper, we consider the 6-parameter shell model, which is one of the most general approaches and has been shown to be very effective in solving complex shell problems. This kinematical model involves two independent fields: the translation vector field and the rotation tensor field (six independent variables in total). It was originally proposed by Reissner and was developed in the books [7, 8]. Subsequently, the nonlinear theory of 6-parameter shells has received considerable attention [9, 10]. The existence of minimizers under convexity assumptions has been proved in [11].

Concerning the linear theory of 6-parameter shells there are only few results available in the literature. Thus, the paper [12] investigates a specific (simplified) model of isotropic 6-parameter linear shells and shows the existence of weak solutions in the energy space. Recently, a higher-order nonlinear model for 6-parameter shells made of isotropic Cosserat material has been established and analyzed in [13, 14, 15]. Then, the linearized model for such isotropic 6-parameter Cosserat shells has been presented in [16, 17], where the existence and uniqueness of weak solutions is proved. In the present work, we derive the linearized equations of general (anisotropic) 6-parameter shells and show an existence result.

Outline of the paper. We present first the nonlinear kinematical model of general 6-parameter shells. Then, we linearize the governing equations of equilibrium in Section 3. Thus, we present the infinitesimal strain measures and deduce the equations of equilibrium from the virtual power statement. Under general assumptions on material symmetry (anisotropy) we introduce the suitable functional framework and formulate the variational problem in Section 4. Then, using Poincaré and Korn-type inequalities we prove a general theorem which states that the weak solution exists and is unique. In the last section, we show that this general existence theorem can be applied in the case of isotropic and higher-order 6-parameter shell models. In this way, we obtain existence results for isotropic linear 6-parameter shells which improve and complement the available results presented in the literature [12, 16, 17].

Summary of notations. Let us present some notations and conventions which are employed in this paper. The Latin indices $i, j, k, ...$ range over the set $\{1, 2, 3\}$, while the Greek indices $\alpha, \beta, \gamma, ...$ range over the set $\{1, 2\}$. The Einstein summation convention over repeated indices is used. A subscript comma preceding an index i (or α) designates partial differentiation with respect to the variable x_i (or x_α, respectively), e.g. $f_{,i} = \frac{\partial f}{\partial x_i}$. We denote by δ_i^j the Kronecker symbol and employ the direct tensor notation. Thus, $\otimes$ designates the dyadic product, while $\text{axl}(\mathbf{W})$ stands for the axial vector of any skew-symmetric tensor $\mathbf{W}$. Let $\text{tr}(\mathbf{X})$ denote the trace, $\text{sym}(\mathbf{X})$ the symmetric part, and $\text{skew}(\mathbf{X})$ the skew-symmetric part of any second order tensor $\mathbf{X}$. The scalar product between any second order tensors $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^T\mathbf{B})$. For any vector $\mathbf{v}$ and second order tensor $\mathbf{A}$ we write also $\mathbf{v}\mathbf{A} = \mathbf{A}^T\mathbf{v}$. Also, the cross product between a vector $\mathbf{v}$ and a second order tensor is defined by means of the relation $\mathbf{v} \times (\mathbf{u} \otimes \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) \otimes \mathbf{w}$, for any vectors $\mathbf{u}$ and $\mathbf{w}$.

2 Nonlinear equations of 6-parameter elastic shells

In this section we present briefly the nonlinear governing equations of general 6-parameter shells. Let $\mathcal{S}_c$ be the deformed (current) configuration of the shell and let $\mathcal{S}_\xi$ be its reference configuration. We designate by ω_ξ the midsurface of the reference configuration and by $\mathbf{y}_0(x_1, x_2)$ the position vector of the points on ω_ξ . The map $\mathbf{y}_0 : \omega \subset \mathbb{R}^2 \to \omega_\xi \subset \mathbb{R}^3$ is a parametric representation of the midsurface and the curvilinear coordinates (x_1, x_2) are assumed to be convected coordinates on the surface ω_ξ .

We present first some preliminaries concerning the differential geometry of the midsurface ω_ξ . Let $\mathbf{a}_\alpha$ be the covariant base vectors and $\mathbf{a}^\alpha$ the contravariant base vectors in the tangent plane, which are given by

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{y}_0}{\partial x_\alpha}\,, \qquad \mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta \qquad (\alpha, \beta = 1, 2). \tag{1}$$

We also introduce the vectors $\mathbf{a}_3 = \mathbf{a}^3 = \mathbf{n}_0$ which coincide to the unit normal given by $\mathbf{n}_0 = \mathbf{a}_1 \times \mathbf{a}_2 / \|\mathbf{a}_1 \times \mathbf{a}_2\|$. Let the tensor $\mathbf{a}$ be the first fundamental tensor of the midsurface

$$\mathbf{a} = \mathrm{Grad}_s\, \mathbf{y}_0 = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha = a_{\alpha\beta} \mathbf{a}^\alpha \otimes \mathbf{a}^\beta = a^{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta\,, \tag{2}$$

and let a denote the determinant $a = \det(a_{\alpha\beta})_{2\times 2} > 0$. Here, Grad_s is the surface gradient operator defined by $\mathrm{Grad}_s\, \mathbf{f} = \mathbf{f}_{,\alpha} \otimes \mathbf{a}^\alpha$, for any $\mathbf{f}$. We shall also employ the surface divergence operator defined by $\mathrm{Div}_s\, \mathbf{T} = \mathbf{T}_{,\alpha}\, \mathbf{a}^\alpha$, for any second order tensor $\mathbf{T}$.

We refer the shell to a Cartesian coordinate frame $Ox_1x_2x_3$ with orthonormal base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The reference configuration is characterized by the position vector $\mathbf{y}_0$ and the initial microrotation tensor $\mathbf{Q}_0$ as follows

$$\begin{aligned} &\mathbf{y}_0 : \omega \subset \mathbb{R}^2 \to \omega_\xi \subset \mathbb{R}^3, \qquad && \mathbf{y}_0 = \mathbf{y}_0(x_1, x_2), \\ &\mathbf{Q}_0 : \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3), && \mathbf{Q}_0 = \mathbf{d}_i^0(x_1, x_2) \otimes \mathbf{e}_i\,. \end{aligned} \tag{3}$$

The domain ω is assumed to be a bounded open connected domain with Lipschitz boundary $\partial\omega$ in the plane Ox_1x_2 . The vectors $\{\mathbf{d}_1^0, \mathbf{d}_2^0, \mathbf{d}_3^0\}$ designate the reference directors, which are orthonormal. In our model, the third director $\mathbf{d}_3^0$ is chosen to coincide with the unit normal in the reference configuration, i.e. $\mathbf{d}_3^0 = \mathbf{n}_0$. The deformation of the shell is characterized by two fields: the deformation function $\mathbf{m}$ and the microrotation tensor $\mathbf{Q}_e$

$$\begin{aligned} &\mathbf{m} : \omega \to \omega_c\,, \qquad && \mathbf{m} = \mathbf{m}(x_1, x_2), \\ &\mathbf{Q}_e : \omega \to \mathrm{SO}(3), && \mathbf{Q}_e = \mathbf{Q}_e(x_1, x_2) = \mathbf{d}_i \otimes \mathbf{d}_i^0\,. \end{aligned} \tag{4}$$

We have denoted by ω_c the deformed midsurface and by $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ the orthonormal triad of directors attached to the deformed configuration.

The nonlinear strain measures commonly used in the 6-parameter elastic shell models (see, e.g., [7, 8, 10, 11, 18]) are the *shell strain tensor*

$$\mathbf{E}^e = \mathbf{Q}_e^T \mathrm{Grad}_s \mathbf{m} - \mathbf{a} \tag{5}$$

and the *shell bending-curvature tensor*

$$\mathbf{K}^e = \mathrm{axl}(\mathbf{Q}_e^T \mathbf{Q}_{e,\alpha}) \otimes \mathbf{a}^\alpha. \tag{6}$$

The local equilibrium equations in the nonlinear theory of 6-parameter elastic shells have the following form (see, e.g., [10, 19])

$$\mathrm{Div}_s \mathbf{N} + \mathbf{f} = \mathbf{0}, \qquad \mathrm{Div}_s \mathbf{M} + \mathrm{axl}(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T) + \mathbf{l} = \mathbf{0}, \tag{7}$$

where $\mathbf{N}$ is the internal surface stress tensor and $\mathbf{M}$ the internal surface couple stress tensor (of the first Piola-Kirchhoff type). In equation (7), $\mathbf{F}$ represents the shell deformation gradient and is expressed by $\mathbf{F} = \mathrm{Grad}_s \mathbf{m} = \mathbf{m}_{,\alpha} \otimes \mathbf{a}^\alpha$. The vectors $\mathbf{f}$ and $\mathbf{l}$ are the external body forces and body couples, respectively. We consider the following boundary conditions of mixed type prescribed on the boundary curve $\partial\omega_\xi$ [9, 11]

$$\begin{aligned} &\mathbf{m} = \mathbf{m}^*, \qquad &&\mathbf{Q}_e = \mathbf{Q}^* \qquad &&\text{along } \partial\omega_d\,, \\ &\mathbf{N}\nu = \mathbf{N}^*, \qquad &&\mathbf{M}\nu = \mathbf{M}^* \qquad &&\text{along } \partial\omega_f\,, \end{aligned} \tag{8}$$

where $\partial\omega_d \cup \partial\omega_f = \partial\omega_\xi$ is a disjoint partition of the boundary curve $\partial\omega_\xi$. Here, $\mathbf{N}^*$ and $\mathbf{M}^*$ are, respectively, the external boundary force and couple force vectors applied along the deformed boundary curve, but measured per unit length of $\partial\omega_f$. The vector ν is the outer unit normal to the deformed boundary curve, lying in the tangent plane of ω_ξ.

Under hyperelasticity assumptions, the stress and couple stress tensors satisfy the following constitutive relations

$$\mathbf{Q}_e^T \mathbf{N} = \frac{\partial \mathcal{W}}{\partial \mathbf{E}^e}\,, \qquad \mathbf{Q}_e^T \mathbf{M} = \frac{\partial \mathcal{W}}{\partial \mathbf{K}^e}\,, \tag{9}$$

where $\mathcal{W}$ is the areal strain energy density for 6-parameter shells, which is given as a function of the strain measures in the form

$$\mathcal{W} = \mathcal{W}(\mathbf{E}^e, \mathbf{K}^e). \tag{10}$$

In the next section, we shall linearize the equations presented above to determine the governing equations of the linear theory of 6-parameter elastic shells.

3 Linearized equations for the equilibrium of 6-parameter shells

In the linear theory, we consider the displacement vector

$$\mathbf{u} = \mathbf{m} - \mathbf{y}_0 \tag{11}$$

and we assume that the displacement is infinitesimal, i.e. $\mathbf{u}$ is of the order $O(\epsilon)$, where ϵ is a *small* parameter such that all the terms of order $O(\epsilon^2)$ can be neglected.

For the microrotation tensor $\mathbf{Q}_e \in \mathrm{SO}(3)$ there exists a skew-symmetric tensor $\mathbf{W}$ such that $\mathbf{Q}_e = \exp(\mathbf{W})$. Let us denote by $\mathbf{w}$ the axial vector of the skew-symmetric tensor $\mathbf{W}$, i.e. $\mathbf{W} = \mathbf{w} \times \mathbf{1}_3$, where $\mathbf{1}_3$ is the unit tensor in the 3-space. The vector $\mathbf{w}$ is called the *rotation vector* and it is assumed to be infinitesimal, i.e. $\mathbf{w} = O(\epsilon)$. Then, neglecting the terms of order $O(\epsilon^2)$ we can write the microrotation tensor in the linear theory as follows

$$\mathbf{Q}_e \;=\; \exp(\mathbf{W}) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\,(\mathbf{W})^k \;=\; \mathbf{1}_3 + \mathbf{W}\,, \tag{12}$$

i.e.

$$\mathbf{Q}_e = \mathbf{1}_3 + \mathbf{w} \times \mathbf{1}_3\,. \tag{13}$$

Let us write next the linearized strain measures. By linearizing the shell strain tensor (5) we obtain

$$\begin{aligned} \mathbf{e} \;&=\; \Big[\mathbf{Q}_e^T \mathrm{Grad}_s \mathbf{m} - \mathbf{a}\Big]_{\mathrm{lin}} = \Big[(\mathbf{1}_3 - \mathbf{W})((\mathbf{u}_{,\alpha} + \mathbf{a}_\alpha) \otimes \mathbf{a}^\alpha) - \mathbf{a}\Big]_{\mathrm{lin}} \\ &=\; \Big[\mathbf{u}_{,\alpha} \otimes \mathbf{a}^\alpha - \mathbf{W}\mathbf{u}_{,\alpha} \otimes \mathbf{a}^\alpha - \mathbf{W}\mathbf{a}\Big]_{\mathrm{lin}} = \mathbf{u}_{,\alpha} \otimes \mathbf{a}^\alpha - \mathbf{W}\mathbf{a}\,, \end{aligned} \tag{14}$$

so the *infinitesimal strain tensor* $\mathbf{e}$ is given by

$$\mathbf{e} \;=\; \mathrm{Grad}_s \mathbf{u} - \mathbf{w} \times \mathbf{a} \;=\; (\mathbf{u}_{,\alpha} - \mathbf{w} \times \mathbf{a}_\alpha) \otimes \mathbf{a}^\alpha\,. \tag{15}$$

To obtain the infinitesimal bending-curvature tensor $\mathbf{k}$, we write first the linearization of the skew-symmetric tensor $\mathbf{Q}_e^T \mathbf{Q}_{e,\alpha}$ as follows

$$\begin{aligned} \Big[\mathbf{Q}_e^T \mathbf{Q}_{e,\alpha}\Big]_{\mathrm{lin}} \;&=\; \Big[(\mathbf{1}_3 - \mathbf{W})(\mathbf{1}_3 + \mathbf{W})_{,\alpha}\Big]_{\mathrm{lin}} = \Big[\mathbf{W}_{,\alpha} - \mathbf{W}\mathbf{W}_{,\alpha}\Big]_{\mathrm{lin}} \\ &=\; \Big[\mathbf{w}_{,\alpha} \times \mathbf{1}_3 - (\mathbf{w} \times \mathbf{1}_3)(\mathbf{w}_{,\alpha} \times \mathbf{1}_3)\Big]_{\mathrm{lin}} \\ &=\; \Big[\mathbf{w}_{,\alpha} \times \mathbf{1}_3 - \mathbf{w} \times (\mathbf{w}_{,\alpha} \times \mathbf{1}_3)\Big]_{\mathrm{lin}} \;=\; \mathbf{w}_{,\alpha} \times \mathbf{1}_3\,, \end{aligned} \tag{16}$$

so the linearization of $\operatorname{axl}(\mathbf{Q}_e^T\mathbf{Q}_{e,\alpha})$ is the vector $\mathbf{w}_{,\alpha}$. Then, from (6) we obtain the following expression of the *infinitesimal bending-curvature tensor*

$$\mathbf{k} \,=\, \mathbf{w}_{,\alpha}\otimes\mathbf{a}^{\alpha} \,=\, \mathrm{Grad}_s\mathbf{w}\,. \tag{17}$$

Remark 1 *The expressions of the infinitesimal strain measures (15) and (17) are in accordance with the strain and bending tensors presented in [12, 20]. The apparent difference between these expressions is due to the fact that the authors in [12, 20] adopt another definition for the surface gradient (namely, they consider the transpose of* Grad$_s$ *defined above).*

In what follows, we deduce the linearized equations of equilibrium from the virtual power statement: Equilibrium states are assumed to satisfy the relation

$$\dot{\mathcal{E}} = \mathcal{P}\,, \qquad \text{where} \qquad \mathcal{E} = \int_{\omega_\xi}\mathcal{W}(\mathbf{e},\mathbf{k})\,\mathrm{d}a\,, \tag{18}$$

where $\mathcal{W}(\mathbf{e},\mathbf{k})$ is the areal strain energy density (assumed to be quadratic), $\mathcal{E}$ is the total strain energy and $\mathcal{P}$ is the virtual power of external loading given by (26). Here, superposed dots represent variational derivatives. These are induced by the derivatives with respect to the parameter ε, of the one-parameter displacement and rotation fields $\mathbf{u}(x_\alpha;\varepsilon)$ and $\mathbf{w}(x_\alpha;\varepsilon)$ (evaluated at $\varepsilon=0$), where $\mathbf{u}(x_\alpha)=\mathbf{u}(x_\alpha;0)$ and $\mathbf{w}(x_\alpha)=\mathbf{w}(x_\alpha;0)$ are equilibrium fields.

For linear elastic shells, the surface stress tensor $\mathbf{N}$ and the surface couple stress tensor $\mathbf{M}$ are given by

$$\mathbf{N}=\frac{\partial\mathcal{W}}{\partial\mathbf{e}}\,, \qquad \mathbf{M}=\frac{\partial\mathcal{W}}{\partial\mathbf{k}}\,, \qquad \text{so} \qquad \dot{\mathcal{W}}(\mathbf{e},\mathbf{k})=\mathbf{N}:\dot{\mathbf{e}}+\mathbf{M}:\dot{\mathbf{k}}\,. \tag{19}$$

In view of relations (15) and (17) we have

$$\dot{\mathbf{e}} \,=\, \mathrm{Grad}_s\dot{\mathbf{u}}-\dot{\mathbf{w}}\times\mathbf{a} \qquad \text{and} \qquad \dot{\mathbf{k}} \,=\, \mathrm{Grad}_s\dot{\mathbf{w}}\,. \tag{20}$$

Then, from (19) and (20) we have

$$\dot{\mathcal{W}} \,=\, \mathbf{N}:(\mathrm{Grad}_s\dot{\mathbf{u}}-\dot{\mathbf{w}}\times\mathbf{a})+\mathbf{M}:\mathrm{Grad}_s\dot{\mathbf{w}}\,. \tag{21}$$

Further, we employ relations of the type

$$\mathbf{A}:\mathrm{Grad}_s\mathbf{v} \,=\, \mathrm{Div}_s(\mathbf{A}^T\mathbf{v})-\mathbf{v}\cdot\mathrm{Div}_s\mathbf{A}\,, \tag{22}$$

which holds for any second order tensor field $\mathbf{A}$ and vector field $\mathbf{v}$. Indeed, to prove the last relation we write

$$\begin{aligned}\mathbf{A} : \mathrm{Grad}_s\mathbf{v} + \mathbf{v}\cdot\mathrm{Div}_s\mathbf{A} \;&=\; \mathbf{A} : (\mathbf{v}_{,\alpha}\otimes\mathbf{a}^{\alpha}) + \mathbf{v}\cdot(\mathbf{A}_{,\alpha}\mathbf{a}^{\alpha})\\ &=\; \mathrm{tr}[\mathbf{A}^T(\mathbf{v}_{,\alpha}\otimes\mathbf{a}^{\alpha})] + (\mathbf{A}^T_{,\alpha}\mathbf{v})\cdot\mathbf{a}^{\alpha} \;=\; (\mathbf{A}^T\mathbf{v}_{,\alpha})\cdot\mathbf{a}^{\alpha} + (\mathbf{A}^T_{,\alpha}\mathbf{v})\cdot\mathbf{a}^{\alpha}\\ &=\; (\mathbf{A}^T\mathbf{v})_{,\alpha}\cdot\mathbf{a}^{\alpha} \;=\; \mathrm{Div}_s(\mathbf{A}^T\mathbf{v}),\end{aligned}$$

so equation (22) holds true. For the term $\mathbf{N} : (\dot{\mathbf{w}}\times\mathbf{a})$ which appears in (21) we make the following transformations

$$\begin{aligned}\mathbf{N} : (\dot{\mathbf{w}}\times\mathbf{a}) \;&=\; \mathbf{N} : [\dot{\mathbf{w}}\times(\mathbf{a}+\mathbf{n}_0\otimes\mathbf{n}_0)] \;=\; \mathbf{N} : (\dot{\mathbf{w}}\times\mathbf{1}_3)\\ &=\; \mathrm{skew}(\mathbf{N}) : (\dot{\mathbf{w}}\times\mathbf{1}_3) \;=\; 2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N})\cdot\mathrm{axl}(\dot{\mathbf{w}}\times\mathbf{1}_3)\\ &=\; 2\,\dot{\mathbf{w}}\cdot\mathrm{axl}(\mathrm{skew}\,\mathbf{N}).\end{aligned}\tag{23}$$

Here we have used that $\mathbf{N}\mathbf{n}_0 = \mathbf{0}$ (cf. (43)) and the relation

$$\mathbf{S} : \mathbf{T} = 2\,\mathrm{axl}(\mathbf{S})\cdot\mathrm{axl}(\mathbf{T}),$$

which holds for any skew-symmetric tensors $\mathbf{S},\mathbf{T}$. Using the relations (23) and (22) into (21), we get

$$\dot{\mathcal{W}} \;=\; \mathrm{Div}_s(\mathbf{N}^T\dot{\mathbf{u}}) - \dot{\mathbf{u}}\cdot\mathrm{Div}_s\mathbf{N} - 2\dot{\mathbf{w}}\cdot\mathrm{axl}(\mathrm{skew}\,\mathbf{N}) + \mathrm{Div}_s(\mathbf{M}^T\dot{\mathbf{w}}) - \dot{\mathbf{w}}\cdot\mathrm{Div}_s\mathbf{M}.\tag{24}$$

If we integrate this relation over the surface ω_ξ and use the divergence theorem (see, e.g., [21, Ch. 2]), we obtain

$$\begin{aligned}\int_{\omega_\xi}\dot{\mathcal{W}}\,\mathrm{d}a \;=\; &\int_{\partial\omega_\xi}(\dot{\mathbf{u}}\cdot\mathbf{N}\nu + \dot{\mathbf{w}}\cdot\mathbf{M}\nu)\,\mathrm{d}l\\ &-\int_{\omega_\xi}\Big(\dot{\mathbf{u}}\cdot\mathrm{Div}_s\mathbf{N} + \dot{\mathbf{w}}\cdot[\mathrm{Div}_s\mathbf{M} + 2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N})]\Big)\mathrm{d}a\,.\end{aligned}\tag{25}$$

According to the virtual power statement (18), we have to equate this with the virtual power of external loads $\mathcal{P}$, which is given by

$$\mathcal{P} \;=\int_{\omega_\xi}(\mathbf{f}\cdot\dot{\mathbf{u}} + \mathbf{l}\cdot\dot{\mathbf{w}})\,\mathrm{d}a + \int_{\partial\omega_\xi}(\mathbf{t}^*\cdot\dot{\mathbf{u}} + \mathbf{c}^*\cdot\dot{\mathbf{w}})\,\mathrm{d}l\,.\tag{26}$$

Here, $\mathbf{f}$ and $\mathbf{l}$ are densities of force and couple acting in the surface, while $\mathbf{t}^*$ and $\mathbf{c}^*$ are densities of force and couple acting on the boundary curve. Thus, we obtain the relation

$$\begin{aligned}&\int_{\partial\omega_\xi}(\dot{\mathbf{u}}\cdot\mathbf{N}\nu + \dot{\mathbf{w}}\cdot\mathbf{M}\nu)\,\mathrm{d}l - \int_{\omega_\xi}\Big(\dot{\mathbf{u}}\cdot\mathrm{Div}_s\mathbf{N} + \dot{\mathbf{w}}\cdot[\mathrm{Div}_s\mathbf{M}\\ &\quad + 2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N})]\Big)\mathrm{d}a \;=\int_{\omega_\xi}(\mathbf{f}\cdot\dot{\mathbf{u}} + \mathbf{l}\cdot\dot{\mathbf{w}})\,\mathrm{d}a + \int_{\partial\omega_\xi}(\mathbf{t}^*\cdot\dot{\mathbf{u}} + \mathbf{c}^*\cdot\dot{\mathbf{w}})\,\mathrm{d}l,\end{aligned}\tag{27}$$

which must hold for any variations $\dot{\mathbf{u}}$, $\dot{\mathbf{w}}$. Applying the fundamental lemma, we deduce the local form of the equilibrium equations for linear shells

$$\mathrm{Div}_s\mathbf{N}+\mathbf{f}=\mathbf{0} \qquad \text{and} \qquad \mathrm{Div}_s\mathbf{M}+2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N})+\mathbf{l}=\mathbf{0}\,, \tag{28}$$

together with the natural boundary conditions

$$\begin{aligned} &\mathbf{u}=\mathbf{u}^*, \qquad && \mathbf{w}=\mathbf{w}^* \qquad && \text{along } \partial\omega_d\,,\\ &\mathbf{N}\nu=\mathbf{t}^*, \qquad && \mathbf{M}\nu=\mathbf{c}^* \qquad && \text{along } \partial\omega_f\,. \end{aligned} \tag{29}$$

Here, the portion $\partial\omega_f$ is a subset of $\partial\omega_\xi$ where displacement and rotation are not assigned. We assume displacement and rotation to be assigned on the portion $\partial\omega_d$ (hence, we have $\dot{\mathbf{u}}=\mathbf{0}$, $\dot{\mathbf{w}}=\mathbf{0}$ on $\partial\omega_d$).

Remark 2 *Notice that the term* $2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N})$ *from the equilibrium equations can be written in an alternative form using the vector invariant (also called the Gibbsian cross) of the tensor* $\mathbf{N}$. *For any second order tensor* $\mathbf{T}=\sum_{i=1}^3\mathbf{x}_{(i)}\otimes\mathbf{y}_{(i)}$ *, the vector invariant or Gibbsian cross is defined by*

$$\mathbf{T}_\times=\Big(\sum_{i=1}^3\mathbf{x}_{(i)}\otimes\mathbf{y}_{(i)}\Big)_\times=\sum_{i=1}^3\mathbf{x}_{(i)}\times\mathbf{y}_{(i)}\;. \tag{30}$$

Then, the following relation holds

$$\mathbf{T}_\times=-2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{T})\,. \tag{31}$$

In particular, for rank-one tensors of the form $\mathbf{T}=\mathbf{x}\otimes\mathbf{y}$ *this equation reduce to the well-known formula*

$$\mathbf{x}\times\mathbf{y}\;=\;\mathrm{axl}(\mathbf{y}\otimes\mathbf{x}-\mathbf{x}\otimes\mathbf{y})\,.$$

In order to prove the relation (31), we consider a fixed orthonormal vector basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ *and decompose any second order tensor as* $\mathbf{T}=T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ *. Then, using the formula* $\mathrm{axl}(\mathbf{S})=-\frac12\,\epsilon_{ijk}S_{ij}\mathbf{e}_k$ *(where* $\mathbf{S}=S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ *is a skew-symmetric tensor and* ϵ_{ijk} *is the permutation tensor), we can write*

$$\begin{aligned} -2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{T}) \;&=\; \mathrm{axl}[(T_{ji}-T_{ij})\mathbf{e}_i\otimes\mathbf{e}_j] \;=\; -\tfrac12\,\epsilon_{ijk}(T_{ji}-T_{ij})\mathbf{e}_k\\ &=\; T_{ij}\epsilon_{ijk}\mathbf{e}_k \;=\; T_{ij}\mathbf{e}_i\times\mathbf{e}_j \;=\; (T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j)_\times=\mathbf{T}_\times\,, \end{aligned}$$

so the relation (31) is proved. Hence, we can put the equilibrium equation $(28)_2$ *in the form*

$$\mathrm{Div}_s\mathbf{M}-\mathbf{N}_\times+\mathbf{l}=\mathbf{0}\,. \tag{32}$$

This form is in accordance with the equilibrium equation presented in [8, 22, 23, 12], whereas the sign change is due to the transpose.

Let us present next the geometrical equations (15) and (17) written with the help of tensor components. Denote the components of the displacement vector $\mathbf{u}$, rotation vector $\mathbf{w}$ and strain measures $\mathbf{e}$, $\mathbf{k}$ as follows

$$\mathbf{u} = u_i\,\mathbf{a}^i, \quad \mathbf{w} = w_i\,\mathbf{a}^i, \quad \mathbf{e} = e_{i\beta}\,\mathbf{a}^i \otimes \mathbf{a}^\beta, \quad \mathbf{k} = k_{i\beta}\,\mathbf{a}^i \otimes \mathbf{a}^\beta. \tag{33}$$

Using relations of the type (for any vector $\mathbf{v} = v_i\,\mathbf{a}^i$)

$$\begin{gathered} \mathbf{v} = (v_{\alpha|\beta} - v_3 b_{\alpha\beta})\mathbf{a}^\alpha + (v_{3,\beta} + v_\alpha b^\alpha_\beta)\mathbf{n}_0 \\ \text{with} \qquad v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta} v_\lambda\,, \end{gathered} \tag{34}$$

we obtain from the geometrical equations (15) and (17) the alternative forms

$$\begin{aligned} &\mathbf{e} = (u_{\alpha|\beta} - u_3 b_{\alpha\beta} - w_3\varepsilon_{\alpha\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta + (u_{3,\beta} + u_\alpha b^\alpha_\beta + w_\alpha a^{\alpha\gamma}\varepsilon_{\gamma\beta})\mathbf{n}_0 \otimes \mathbf{a}^\beta, \\ &\mathbf{k} = (w_{\alpha|\beta} - w_3 b_{\alpha\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta + (w_{3,\beta} + w_\alpha b^\alpha_\beta)\mathbf{n}_0 \otimes \mathbf{a}^\beta. \end{aligned} \tag{35}$$

Here, $\varepsilon_{\alpha\beta}$ is the two-dimensional permutation tensor given by

$$\varepsilon_{\alpha\beta} = \sqrt{a}\;\epsilon_{\alpha\beta}\,, \qquad \varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}}\,\epsilon_{\alpha\beta}\,, \tag{36}$$

and $\epsilon_{\alpha\beta}$ is the two-dimensional alternator ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$).

Thus, the components of the strain tensor $\mathbf{e}$ and bending-curvature tensor $\mathbf{k}$ are given by the following geometrical equations

$$\begin{aligned} e_{\alpha\beta} &= u_{\alpha|\beta} - u_3 b_{\alpha\beta} - w_3\varepsilon_{\alpha\beta}\,, \qquad & e_{3\beta} &= u_{3,\beta} + u_\alpha b^\alpha_\beta + w_\alpha a^{\alpha\gamma}\varepsilon_{\gamma\beta}\,, \\ k_{\alpha\beta} &= w_{\alpha|\beta} - w_3 b_{\alpha\beta}\,, & k_{3\beta} &= w_{3,\beta} + w_\alpha b^\alpha_\beta\,. \end{aligned} \tag{37}$$

In the linear theory, the areal strain energy density $\mathcal{W}$ is assumed to be a quadratic function of the strain measures $\mathbf{e}$ and $\mathbf{k}$. We consider the general (quadratic) form

$$\mathcal{W}(\mathbf{e},\mathbf{k}) = \frac{1}{2}\,\mathbf{e} : \underline{\mathbf{A}} : \mathbf{e} + \mathbf{e} : \underline{\mathbf{C}} : \mathbf{k} + \frac{1}{2}\,\mathbf{k} : \underline{\mathbf{B}} : \mathbf{k}\,, \tag{38}$$

where $\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{C}}$ are the fourth order tensors of the elastic moduli, which in the case of shells can depend on the curvature of the reference. In view of $(33)_{3,4}$, we see that these tensors have the following structure

$$\begin{gathered} \underline{\mathbf{A}} = A^{i\alpha j\beta}\mathbf{a}_i \otimes \mathbf{a}_\alpha \otimes \mathbf{a}_j \otimes \mathbf{a}_\beta\,, \qquad \underline{\mathbf{B}} = B^{i\alpha j\beta}\mathbf{a}_i \otimes \mathbf{a}_\alpha \otimes \mathbf{a}_j \otimes \mathbf{a}_\beta \\ \underline{\mathbf{C}} = C^{i\alpha j\beta}\mathbf{a}_i \otimes \mathbf{a}_\alpha \otimes \mathbf{a}_j \otimes \mathbf{a}_\beta\,, \end{gathered} \tag{39}$$

and satisfy the major symmetries

$$A^{i\alpha j\beta} = A^{j\beta i\alpha}\,, \qquad B^{i\alpha j\beta} = B^{j\beta i\alpha}\,. \tag{40}$$

Then, the strain energy density (38) can be written as

$$\mathcal{W}(\mathbf{e},\mathbf{k}) = \frac{1}{2}\, A^{i\alpha j\beta} e_{i\alpha} e_{j\beta} + C^{i\alpha j\beta} e_{i\alpha} k_{j\beta} + \frac{1}{2}\, B^{i\alpha j\beta} k_{i\alpha} k_{j\beta}\;. \tag{41}$$

Taking into account $(19)_{1,2}$ and (38), we get the constitutive equations in the linear theory

$$\mathbf{N} = \frac{\partial\,\mathcal{W}(\mathbf{e},\mathbf{k})}{\partial \mathbf{e}} = \underline{\mathbf{A}} : \mathbf{e} + \underline{\mathbf{C}} : \mathbf{k}, \qquad \mathbf{M} = \frac{\partial\,\mathcal{W}(\mathbf{e},\mathbf{k})}{\partial \mathbf{k}} = \mathbf{e} : \underline{\mathbf{C}} + \underline{\mathbf{B}} : \mathbf{k}. \tag{42}$$

These relations can be written with the help of tensor components in the form

$$\begin{aligned} \mathbf{N} = N^{i\alpha}\mathbf{a}_i \otimes \mathbf{a}_\alpha \;&=\; (A^{i\alpha j\beta} e_{j\beta} + C^{i\alpha j\beta} k_{j\beta})\mathbf{a}_i \otimes \mathbf{a}_\alpha\;, \\ \mathbf{M} = M^{i\alpha}\mathbf{a}_i \otimes \mathbf{a}_\alpha \;&=\; (C^{k\gamma i\alpha} e_{k\gamma} + B^{i\alpha j\beta} k_{j\beta})\mathbf{a}_i \otimes \mathbf{a}_\alpha\;. \end{aligned} \tag{43}$$

In this paper, we assume that the strain energy density $\mathcal{W}(\mathbf{e},\mathbf{k})$ is coercive, in the sense that there exists a positive constant $C_0 > 0$ such that

$$\mathcal{W}(\mathbf{e},\mathbf{k}) \;\geq\; C_0(\,\|\mathbf{e}\|^2 + \|\mathbf{k}\|^2\,). \tag{44}$$

To avoid constants and norms carrying units, we suppose that all quantities and equations have been put in dimensionless form. The norm of any second order tensor $\mathbf{X}$ is given by $\|\mathbf{X}\|^2 = \mathrm{tr}(\mathbf{X}^T\mathbf{X})$.

Remark 3 *We mention that the equilibrium equations (28) can also be obtained by linearization of the nonlinear equations of equilibrium (7). To show this, we have to prove that the tensor* $(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T)$ *reduces upon linearization to* $2\,\mathrm{skew}\,\mathbf{N}$ *. In view of (11), we have* $\mathbf{F} = \mathrm{Grad}_s\mathbf{u} + \mathbf{a}$ *and using* $(42)_1$ *we get*

$$\begin{aligned} \mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T \;&=\; \mathbf{N}[(\mathrm{Grad}_s\mathbf{u})^T + \mathbf{a}] - (\mathrm{Grad}_s\mathbf{u} + \mathbf{a})\mathbf{N}^T \\ &=\; \mathbf{N}\mathbf{a} - \mathbf{a}\mathbf{N}^T + O(\epsilon^2) \;=\; (\mathbf{N} - \mathbf{N}^T) + O(\epsilon^2) \;=\; 2\,\mathrm{skew}\,\mathbf{N} + O(\epsilon^2). \end{aligned}$$

Hence, it follows that

$$\Big[\mathrm{axl}(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T)\Big]_{\mathrm{lin}} \;=\; 2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N})\,. \tag{45}$$

Finally, let us write the equilibrium equations (28) using tensor components. For any tensor of the form $\mathbf{T} = T^{i\alpha}\mathbf{a}_i \otimes \mathbf{a}_\alpha$ we can express the surface divergence as follows (see, e.g., [21, Ch. 2])

$$\begin{aligned} &\mathrm{Div}_s\mathbf{T} = (T^{\alpha\beta}{}_{|\beta} - b^\alpha_\beta T^{3\beta})\mathbf{a}_\alpha + (T^{3\beta}{}_{|\beta} + b_{\alpha\beta}T^{\alpha\beta})\mathbf{n}_0\,, \qquad \text{where} \\ &T^{\alpha\beta}{}_{|\gamma} = T^{\alpha\beta}{}_{,\gamma} + \Gamma^\alpha_{\gamma\lambda}T^{\lambda\beta} + \Gamma^\beta_{\gamma\lambda}T^{\alpha\lambda}\,, \qquad T^{3\beta}{}_{|\gamma} = T^{3\beta}{}_{,\gamma} + \Gamma^\beta_{\gamma\lambda}T^{3\lambda}\,. \end{aligned} \tag{46}$$

On the other hand, from the relation (31) we deduce

$$\begin{aligned} 2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N}) &= -\mathbf{N}_\times = -(N^{i\alpha}\mathbf{a}_i \otimes \mathbf{a}_\alpha)_\times = -N^{i\alpha}\mathbf{a}_i \times \mathbf{a}_\alpha \\ &= N^{\beta\alpha}\mathbf{a}_\alpha \times \mathbf{a}_\beta + N^{3\alpha}\mathbf{a}_\alpha \times \mathbf{n}_0 = \varepsilon_{\alpha\beta}N^{\beta\alpha}\mathbf{n}_0 + \varepsilon_{\beta\alpha}N^{3\alpha}\mathbf{a}^\beta\,, \end{aligned}$$

so we have

$$2\,\mathrm{axl}(\mathrm{skew}\,\mathbf{N}) = a^{\alpha\beta}\varepsilon_{\beta\gamma}N^{3\gamma}\mathbf{a}_\alpha + \varepsilon_{\alpha\beta}N^{\beta\alpha}\mathbf{n}_0\,. \tag{47}$$

Substituting $(46)_1$ and (47) into (28), we obtain the equilibrium equations written with tensor components:

$$\begin{aligned} &N^{\alpha\beta}{}_{|\beta} - b^\alpha_\beta N^{3\beta} + f^\alpha = 0, \qquad N^{3\beta}{}_{|\beta} + b_{\alpha\beta}N^{\alpha\beta} + f^3 = 0, \\ &M^{\alpha\beta}{}_{|\beta} - b^\alpha_\beta M^{3\beta} + a^{\alpha\beta}\varepsilon_{\beta\gamma}N^{3\gamma} + l^\alpha = 0, \\ &M^{3\beta}{}_{|\beta} + b_{\alpha\beta}M^{\alpha\beta} - \varepsilon_{\alpha\beta}N^{\alpha\beta} + l^3 = 0. \end{aligned} \tag{48}$$

Notice that we can replace

$$a^{\alpha\beta}\varepsilon_{\beta\gamma} = \varepsilon^{\alpha\beta}a_{\beta\gamma} \tag{49}$$

in the geometrical relations (37) and the equilibrium equations (48).

4 Existence and uniqueness of weak solutions

Let $(\mathbf{u}, \mathbf{w})$ be a classical solution of the boundary-value problem for linearly elastic 6-parameter shells, which consists in the equilibrium equations (28), the boundary conditions (29), the geometrical relations (15), (17), and the constitutive equations (42).

In order to formulate the variational problem and to define the weak solution, let us prove first the following relation

$$\begin{aligned} &\int_{\omega_\xi} \Big(\mathbf{N}(\mathbf{u},\mathbf{w}) : \mathbf{e}(\tilde{\mathbf{u}},\tilde{\mathbf{w}}) + \mathbf{M}(\mathbf{u},\mathbf{w}) : \mathrm{Grad}_s\tilde{\mathbf{w}}\Big)\,\mathrm{d}a \\ &\quad = \int_{\omega_\xi} (\mathbf{f}\cdot\tilde{\mathbf{u}} + \mathbf{l}\cdot\tilde{\mathbf{w}})\,\mathrm{d}a + \int_{\partial\omega_\xi} \Big(\mathbf{N}(\mathbf{u},\mathbf{w})\nu\cdot\tilde{\mathbf{u}} + \mathbf{M}(\mathbf{u},\mathbf{w})\nu\cdot\tilde{\mathbf{w}}\Big)\,\mathrm{d}l, \end{aligned} \tag{50}$$

where $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ is an arbitrary continuously differentiable displacement and rotation field. The relation (50) expresses the principle of virtual work for linear 6-parameter shells. To prove the equation (50), we use successively the geometrical relation (15), the formulas (22), (23), the equilibrium equations (28), and the divergence theorem to write

$$\begin{aligned}
&\int_{\omega_\xi} \Big(\mathbf{N}(\mathbf{u},\mathbf{w}) : (\mathrm{Grad}_s\tilde{\mathbf{u}} - \tilde{\mathbf{w}} \times \mathbf{a}) + \mathbf{M}(\mathbf{u},\mathbf{w}) : \mathrm{Grad}_s\tilde{\mathbf{w}}\Big)\,\mathrm{d}a \\
&\quad = \int_{\omega_\xi} \Big[\mathrm{Div}_s\Big(\mathbf{N}^T(\mathbf{u},\mathbf{w})\tilde{\mathbf{u}}\Big) - \tilde{\mathbf{u}} \cdot \mathrm{Div}_s\mathbf{N}(\mathbf{u},\mathbf{w}) - 2\tilde{\mathbf{w}} \cdot \mathrm{axl}\Big(\mathrm{skew}\,\mathbf{N}(\mathbf{u},\mathbf{w})\Big) \\
&\qquad\qquad + \mathrm{Div}_s\Big(\mathbf{M}^T(\mathbf{u},\mathbf{w})\tilde{\mathbf{w}}\Big) - \tilde{\mathbf{w}} \cdot \mathrm{Div}_s\mathbf{M}(\mathbf{u},\mathbf{w})\Big]\,\mathrm{d}a \\
&\quad = \int_{\omega_\xi} (\mathbf{f} \cdot \tilde{\mathbf{u}} + \mathbf{l} \cdot \tilde{\mathbf{w}})\,\mathrm{d}a + \int_{\partial\omega_\xi} \Big(\mathbf{N}^T(\mathbf{u},\mathbf{w})\tilde{\mathbf{u}} \cdot \nu + \mathbf{M}^T(\mathbf{u},\mathbf{w})\tilde{\mathbf{w}} \cdot \nu\Big)\,\mathrm{d}l,
\end{aligned}$$

so the relation (50) holds true. Next, using the constitutive equations (42) we can put the relation (50) in the alternative form

$$\begin{aligned}
&\int_{\omega_\xi} \Big(\mathbf{e}(\mathbf{u},\mathbf{w}) : \underline{\mathbf{A}} : \mathbf{e}(\tilde{\mathbf{u}},\tilde{\mathbf{w}}) + \mathbf{e}(\tilde{\mathbf{u}},\tilde{\mathbf{w}}) : \underline{\mathbf{C}} : \mathrm{Grad}_s\mathbf{w} \\
&\qquad + \mathbf{e}(\mathbf{u},\mathbf{w}) : \underline{\mathbf{C}} : \mathrm{Grad}_s\tilde{\mathbf{w}} + \mathrm{Grad}_s\mathbf{w} : \underline{\mathbf{B}} : \mathrm{Grad}_s\tilde{\mathbf{w}}\Big)\,\mathrm{d}a \qquad (51) \\
&\quad = \int_{\omega_\xi} (\mathbf{f} \cdot \tilde{\mathbf{u}} + \mathbf{l} \cdot \tilde{\mathbf{w}})\,\mathrm{d}a + \int_{\partial\omega_\xi} \Big(\mathbf{N}(\mathbf{u},\mathbf{w})\nu \cdot \tilde{\mathbf{u}} + \mathbf{M}(\mathbf{u},\mathbf{w})\nu \cdot \tilde{\mathbf{w}}\Big)\,\mathrm{d}l.
\end{aligned}$$

For the sake of simplicity, we consider the equilibrium problem with homogeneous boundary conditions, i.e. the solution must satisfy the boundary conditions

$$\begin{aligned}
\mathbf{u} = \mathbf{0}, &\qquad \mathbf{w} = \mathbf{0} \qquad \text{for } (x_1, x_2) \in \partial\omega_u\,, \\
\mathbf{N}\nu = \mathbf{0}, &\qquad \mathbf{M}\nu = \mathbf{0} \qquad \text{for } (x_1, x_2) \in \partial\omega_t\,,
\end{aligned} \tag{52}$$

where $\partial\omega_u \cup \partial\omega_t = \partial\omega$ is a disjoint partition (with $\mathrm{length}(\partial\omega_u) > 0$) of the boundary curve of the parameter domain $\omega \subset \mathbb{R}^2$.

Let us introduce the functional framework for the weak formulation of the problem. Consider the Banach spaces $(L^2(\omega,\mathbb{R}^3) \times L^2(\omega,\mathbb{R}^3), \|\cdot\|_{L^2})$ and $(H^1(\omega,\mathbb{R}^3) \times H^1(\omega,\mathbb{R}^3), \|\cdot\|_{H^1})$ equiped with the usual norms

$$\begin{aligned}
\|(\mathbf{u},\mathbf{w})\|^2_{L^2} &= \int_\omega (\mathbf{u} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{w})\,\mathrm{d}x_1\mathrm{d}x_2\,, \\
\|(\mathbf{u},\mathbf{w})\|^2_{H^1} &= \int_\omega (\mathbf{u} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{w} + \mathbf{u}_{,\alpha} \cdot \mathbf{u}_{,\alpha} + \mathbf{w}_{,\alpha} \cdot \mathbf{w}_{,\alpha})\,\mathrm{d}x_1\mathrm{d}x_2\,,
\end{aligned} \tag{53}$$

and define the admissible set $V(\omega)$ by

$$V(\omega)=\Big\{(\mathbf{u},\mathbf{w})\in H^1(\omega,\mathrm{I\!R}^3)\times H^1(\omega,\mathrm{I\!R}^3)\;\Big|\;\mathbf{u}=\mathbf{0},\mathbf{w}=\mathbf{0}\text{ on }\partial\omega_u\Big\}. \quad (54)$$

In view of (51) and (52) we deduce that any solution $(\mathbf{u},\mathbf{w})$ satisfies

$$\begin{aligned}\int_\omega\Big(&\mathbf{e}(\mathbf{u},\mathbf{w}):\underline{\mathbf{A}}:\mathbf{e}(\tilde{\mathbf{u}},\tilde{\mathbf{w}})+\mathbf{e}(\tilde{\mathbf{u}},\tilde{\mathbf{w}}):\underline{\mathbf{C}}:\mathrm{Grad}_s\mathbf{w}\\&+\mathbf{e}(\mathbf{u},\mathbf{w}):\underline{\mathbf{C}}:\mathrm{Grad}_s\tilde{\mathbf{w}}+\mathrm{Grad}_s\mathbf{w}:\underline{\mathbf{B}}:\mathrm{Grad}_s\tilde{\mathbf{w}}\Big)\sqrt{a}\,\mathrm{d}x_1\mathrm{d}x_2 \quad (55)\\&=\int_\omega(\mathbf{f}\cdot\tilde{\mathbf{u}}+\mathbf{l}\cdot\tilde{\mathbf{w}})\sqrt{a}\,\mathrm{d}x_1\mathrm{d}x_2\,,\end{aligned}$$

for any admissible field $(\tilde{\mathbf{u}},\tilde{\mathbf{w}})\in V(\omega)$. We assume that the body forces are such that

$$\mathbf{f},\mathbf{l}\in L^2(\omega,\mathrm{I\!R}^3). \quad (56)$$

Suggested by relation (55) we introduce the bilinear form $B:V(\omega)\times V(\omega)\to\mathrm{I\!R}$ and the linear functional $F:V(\omega)\to\mathrm{I\!R}$ given by

$$\begin{aligned}B((\mathbf{u},\mathbf{w}),(\tilde{\mathbf{u}},\tilde{\mathbf{w}}))&=\int_\omega\Big(\mathbf{e}(\mathbf{u},\mathbf{w}):\underline{\mathbf{A}}:\mathbf{e}(\tilde{\mathbf{u}},\tilde{\mathbf{w}})+\mathbf{e}(\tilde{\mathbf{u}},\tilde{\mathbf{w}}):\underline{\mathbf{C}}:\mathrm{Grad}_s\mathbf{w}\\&\quad+\mathbf{e}(\mathbf{u},\mathbf{w}):\underline{\mathbf{C}}:\mathrm{Grad}_s\tilde{\mathbf{w}}+\mathrm{Grad}_s\mathbf{w}:\underline{\mathbf{B}}:\mathrm{Grad}_s\tilde{\mathbf{w}}\Big)\sqrt{a}\,\mathrm{d}x_1\mathrm{d}x_2\,,\\F(\mathbf{u},\mathbf{w})&=\int_\omega(\mathbf{f}\cdot\mathbf{u}+\mathbf{l}\cdot\mathbf{w})\sqrt{a}\,\mathrm{d}x_1\mathrm{d}x_2\,.\end{aligned} \quad (57)$$

Then, the displacement and rotation field $(\mathbf{u},\mathbf{w})\in V(\omega)$ is called a *weak solution* of the equilibrium boundary-value problem with boundary conditions (52), if $(\mathbf{u},\mathbf{w})$ satisfies

$$B((\mathbf{u},\mathbf{w}),(\tilde{\mathbf{u}},\tilde{\mathbf{w}}))=F(\tilde{\mathbf{u}},\tilde{\mathbf{w}})\qquad\text{for any}\quad(\tilde{\mathbf{u}},\tilde{\mathbf{w}})\in V(\omega). \quad (58)$$

Let us state the general existence result for this variational problem.

Theorem 1 *Assume that the reference midsurface of the shell satisfies the regularity conditions*

$$\mathbf{y}_0\in H^1(\omega,\mathrm{I\!R}^3),\qquad\mathbf{a}_\alpha=\mathbf{y}_{0,\alpha}\in L^\infty(\omega,\mathrm{I\!R}^3) \quad (59)$$

and there exists a positive constant a_0 such that

$$\det\left(a_{\alpha\beta}(x_1,x_2)\right)_{2\times2}\geq a_0>0\qquad\textit{for any}\quad(x_1,x_2)\in\omega. \quad (60)$$

Further, assume that the body forces $\mathbf{f}, \mathbf{l}$ *fulfill the conditions (56), and the strain energy density* $\mathcal{W}(\mathbf{e}, \mathbf{k})$ *is coercive, i.e. the inequality (44) holds.*

Then, the equilibrium boundary-value problem for linearly elastic 6-parameter shells admits an unique weak solution $(\mathbf{u}, \mathbf{w}) \in V(\omega)$*. This solution can be characterized as the minimizer on the space* $V(\omega)$ *of the functional*

$$J(\mathbf{u}, \mathbf{w}) = \int_\omega \mathcal{W}(\mathbf{e}, \mathbf{k}) \sqrt{a}\, \mathrm{d}x_1 \mathrm{d}x_2 - \int_\omega (\mathbf{f} \cdot \mathbf{u} + \mathbf{l} \cdot \mathbf{w}) \sqrt{a}\, \mathrm{d}x_1 \mathrm{d}x_2 \,, \tag{61}$$

where the tensors $\mathbf{e}$ *and* $\mathbf{k}$ *are expressed in terms of* $(\mathbf{u}, \mathbf{w})$ *by the geometrical relations (15), (17).*

Proof. Taking into account the definitions (57) and the hypotheses, we see that the bilinear form $B(\cdot, \cdot)$ is symmetric and continuous, while $F(\cdot)$ is continuous. With a view toward applying the Lax-Milgram lemma for the problem (58) we need to prove that $B(\cdot, \cdot)$ is also coercive on $V(\omega)$, i.e. there exists a constant $C_1 > 0$ such that

$$B((\mathbf{u}, \mathbf{w}), (\mathbf{u}, \mathbf{w})) \,\geq\, C_1 \|(\mathbf{u}, \mathbf{w})\|^2_{H^1} \qquad \text{for any} \quad (\mathbf{u}, \mathbf{w}) \in V(\omega). \tag{62}$$

Using the relations (38), (44) and (60), we deduce that

$$\begin{aligned} B((\mathbf{u}, \mathbf{w}), (\mathbf{u}, \mathbf{w})) \;&=\; \int_\omega 2\, \mathcal{W}(\mathbf{e}, \mathbf{k}) \sqrt{a}\, \mathrm{d}x_1 \mathrm{d}x_2 \\ &\geq\; C_2 \int_\omega (\, \|\mathbf{e}\|^2 + \|\mathbf{k}\|^2\,)\, \mathrm{d}x_1 \mathrm{d}x_2 \,, \end{aligned} \tag{63}$$

where $C_2 > 0$ is some constant. Then, in order to prove (62) we still have to show that the following inequality of Korn-type holds

$$\begin{aligned} &\int_\omega (\, \|\mathbf{e}\|^2 + \|\mathbf{k}\|^2\,)\, \mathrm{d}x_1 \mathrm{d}x_2 \\ &\qquad \geq C_3 \int_\omega (\mathbf{u} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{w} + \mathbf{u}_{,\alpha} \cdot \mathbf{u}_{,\alpha} + \mathbf{w}_{,\alpha} \cdot \mathbf{w}_{,\alpha})\, \mathrm{d}x_1 \mathrm{d}x_2 \,, \end{aligned} \tag{64}$$

for some positive constant C_3. To prove this inequality, let us estimate the norms $\|\mathbf{e}\|$ and $\|\mathbf{k}\|$ appearing in the left-hand side. By virtue of the hypotheses (59) and (60), it follows that the matrix $(a_{\alpha\beta})_{2\times2}$ and its inverse $(a^{\alpha\beta})_{2\times2} = (a_{\alpha\beta})^{-1}_{2\times2}$ satisfy the relations

$$(a_{\alpha\beta}) \in L^\infty(\omega, \mathbb{R}^{2\times2}) \quad \text{and} \quad (a^{\alpha\beta}) \in L^\infty(\omega, \mathbb{R}^{2\times2}).$$

Then, the smallest eigenvalue (over ω) of the positive definite symmetric matrix $(a^{\alpha\beta}(x_1, x_2))_{2\times2}$ is greater than a positive constant $\lambda_0 > 0$. Hence,

$$a^{\alpha\beta}(x_1, x_2)\, v_\alpha v_\beta \,\geq\, \lambda_0\, v_\gamma\, v_\gamma \,, \quad \forall\, (x_1, x_2) \in \omega, \;\; \forall\, v_1, v_2 \in \mathbb{R}. \tag{65}$$

By a straightforward argument we can extend this inequality for the case of two arbitrary constant vectors $\mathbf{v}_1$, $\mathbf{v}_2$, i.e. it holds

$$a^{\alpha\beta}(x_1,x_2)\,\mathbf{v}_\alpha\cdot\mathbf{v}_\beta \,\geq\, \lambda_0\,\mathbf{v}_\gamma\cdot\mathbf{v}_\gamma\,, \quad \forall\,(x_1,x_2)\in\omega,\;\; \forall\,\mathbf{v}_1,\mathbf{v}_2\in\mathbb{R}^3. \tag{66}$$

Now, using the relation (66) we can write

$$\begin{aligned}\|\mathbf{e}\|^2 \;&=\; \mathrm{tr}\Big[\Big((\mathbf{u}_{,\alpha}-\mathbf{w}\times\mathbf{a}_\alpha)\otimes\mathbf{a}^\alpha\Big)\Big(\mathbf{a}^\beta\otimes(\mathbf{u}_{,\beta}-\mathbf{w}\times\mathbf{a}_\beta)\Big)\Big]\\ &=\; a^{\alpha\beta}(\mathbf{u}_{,\alpha}-\mathbf{w}\times\mathbf{a}_\alpha)\cdot(\mathbf{u}_{,\beta}-\mathbf{w}\times\mathbf{a}_\beta)\\ &\geq\; \lambda_0\,(\mathbf{u}_{,\gamma}-\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{u}_{,\gamma}-\mathbf{w}\times\mathbf{a}_\gamma)\\ &=\; \lambda_0\,[\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}-2\,\mathbf{u}_{,\gamma}\cdot(\mathbf{w}\times\mathbf{a}_\gamma)+(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma)].\end{aligned} \tag{67}$$

For the scalar product $\mathbf{u}_{,\gamma}\cdot(\mathbf{w}\times\mathbf{a}_\gamma)$ we have the inequality

$$2\,\mathbf{u}_{,\gamma}\cdot(\mathbf{w}\times\mathbf{a}_\gamma)\,\leq\,\varepsilon\,\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}+\frac{1}{\varepsilon}(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma), \tag{68}$$

which holds for any $\varepsilon>0$. The choice of the constant ε will be made explicit later in (77). Then, from (67) and (68) we obtain

$$\|\mathbf{e}\|^2\,\geq\,\lambda_0\,\Big[(1-\varepsilon)\,\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}+\Big(1-\frac{1}{\varepsilon}\Big)(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma)\Big]. \tag{69}$$

In the same way, we can estimate

$$\|\mathbf{k}\|^2\,=\,\mathrm{tr}\Big[\Big(\mathbf{w}_{,\alpha}\otimes\mathbf{a}^\alpha\Big)\Big(\mathbf{a}^\beta\otimes\mathbf{w}_{,\beta}\Big)\Big]\,=\,a^{\alpha\beta}\,\mathbf{w}_{,\alpha}\cdot\mathbf{w}_{,\beta}\,\geq\,\lambda_0\,\mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma}\,. \tag{70}$$

Since $\mathbf{w}\in H^1(\omega,\mathbb{R}^3)$ and $\mathbf{w}=\mathbf{0}$ on the boundary portion $\partial\omega_u$, we deduce from the Poincaré inequality that

$$\int_\omega \mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma}\,\mathrm{d}x_1\mathrm{d}x_2\,\geq\,c_p\int_\omega \mathbf{w}\cdot\mathbf{w}\,\mathrm{d}x_1\mathrm{d}x_2\,=\,c_p\,\|\mathbf{w}\|^2_{L^2}\,, \tag{71}$$

for some positive constant $c_p>0$. On the other side, using the Lagrange identity for vector products, we can write

$$(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma)\,=\,(\mathbf{w}\cdot\mathbf{w})(\mathbf{a}_\gamma\cdot\mathbf{a}_\gamma)-(\mathbf{a}_\gamma\cdot\mathbf{w})(\mathbf{a}_\gamma\cdot\mathbf{w})\,\leq\,(\mathbf{a}_\gamma\cdot\mathbf{a}_\gamma)\,\|\mathbf{w}\|^2. \tag{72}$$

In view of the assumption $\mathbf{a}_\alpha\in L^\infty(\omega,\mathbb{R}^3)$ we infer the existence of a positive constant M such that

$$\mathbf{a}_\gamma\cdot\mathbf{a}_\gamma\,\leq\,M\,,\quad \text{for any }\;(x_1,x_2)\in\omega. \tag{73}$$

From the inequalities (72) and (73) we get

$$(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma)\;\leq\; M\,\|\mathbf{w}\|^2\,. \tag{74}$$

Then, making use of (71) and (74) we derive that

$$\int_\omega \mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma}\,\mathrm{d}x_1\mathrm{d}x_2\;\geq\;\frac{c_p}{M}\int_\omega(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma)\,\mathrm{d}x_1\mathrm{d}x_2\,. \tag{75}$$

By virtue of the inequalities (69), (70) and (75), we can estimate the sum $\|\mathbf{e}\|^2+\|\mathbf{k}\|^2$ as follows

$$\begin{aligned}\int_\omega(\,\|\mathbf{e}\|^2+\|\mathbf{k}\|^2\,)\,\mathrm{d}x_1\mathrm{d}x_2\;\geq\;&\int_\omega\lambda_0\Big[(1-\varepsilon)\,\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}\\&+\Big(1-\frac{1}{\varepsilon}\Big)(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma)+\Big(\frac{1}{2}+\frac{1}{2}\Big)\mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma}\Big]\mathrm{d}x_1\mathrm{d}x_2\\\geq\;\lambda_0\int_\omega&\Big[(1-\varepsilon)\,\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}+\Big(1-\frac{1}{\varepsilon}+\frac{c_p}{2M}\Big)(\mathbf{w}\times\mathbf{a}_\gamma)\cdot(\mathbf{w}\times\mathbf{a}_\gamma)\\&+\frac{1}{2}\mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma}\Big]\,\mathrm{d}x_1\mathrm{d}x_2\,.\end{aligned} \tag{76}$$

Now, we can choose the scalar $\varepsilon>0$ such that all the coefficients in (76) be positive, i.e.

$$1-\varepsilon>0\qquad\text{and}\qquad 1-\frac{1}{\varepsilon}+\frac{c_p}{2M}>0,$$

which means

$$\Big(1+\frac{c_p}{2M}\Big)^{-1}\;<\;\varepsilon\;<\;1. \tag{77}$$

Hence, from (76) we deduce that there exists a constant $c_1>0$ such that

$$\int_\omega(\,\|\mathbf{e}\|^2+\|\mathbf{k}\|^2\,)\,\mathrm{d}x_1\mathrm{d}x_2\;\geq\;c_1\int_\omega(\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}+\mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma})\,\mathrm{d}x_1\mathrm{d}x_2\,. \tag{78}$$

Finally, on the basis of the Poincaré inequality, we have

$$\begin{aligned}&\int_\omega(\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}+\mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma})\,\mathrm{d}x_1\mathrm{d}x_2\\&\qquad\geq\;c_2\int_\omega(\mathbf{u}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{w}+\mathbf{u}_{,\gamma}\cdot\mathbf{u}_{,\gamma}+\mathbf{w}_{,\gamma}\cdot\mathbf{w}_{,\gamma})\,\mathrm{d}x_1\mathrm{d}x_2\,,\end{aligned} \tag{79}$$

for some positive constant c_2 . Using the inequalities (78) and (79) we derive that

$$\int_\omega(\,\|\mathbf{e}\|^2+\|\mathbf{k}\|^2\,)\,\mathrm{d}x_1\mathrm{d}x_2\;\geq\;C_3\|(\mathbf{u},\mathbf{w})\|^2_{H^1}\,, \tag{80}$$

for some positive constant C_3, which means that the Korn-type inequality (64) holds true.

From the relations (63) and (80) we obtain that the bilinear form $B(\cdot,\cdot)$ is also coercive on $V(\omega)$, i.e. the inequality (62) holds. Thus, all the hypotheses of the Lax-Milgram lemma are fulfilled (see, e.g. [24, Corollary V.8]). Applying the Lax-Milgram lemma for the variational problem (58) we deduce the existence and uniqueness of the weak solution $(\mathbf{u},\mathbf{w}) \in V(\omega)$.

Let us justify that the solution $(\mathbf{u},\mathbf{w})$ minimizes the functional (61) on $V(\omega)$. In view of (63), the functional (61) can be rewriten in the form

$$J(\mathbf{u},\mathbf{w}) = \frac{1}{2}\, B((\mathbf{u},\mathbf{w}),(\mathbf{u},\mathbf{w})) - F(\mathbf{u},\mathbf{w}). \tag{81}$$

Then, we obtain the relation

$$J(\tilde{\mathbf{u}},\tilde{\mathbf{w}}) = \frac{1}{2}\, B((\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{w}}-\mathbf{w}),(\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{w}}-\mathbf{w})) + J(\mathbf{u},\mathbf{w}) \geq J(\mathbf{u},\mathbf{w}), \tag{82}$$

which holds for any displacement and rotation field $(\tilde{\mathbf{u}},\tilde{\mathbf{w}}) \in V(\omega)$. Indeed, using the equation (58) we deduce immediately

$$\begin{aligned}\tfrac{1}{2}\, B((\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{w}}-\mathbf{w}),(\tilde{\mathbf{u}}-\mathbf{u},\tilde{\mathbf{w}}-\mathbf{w})) + J(\mathbf{u},\mathbf{w}) = \tfrac{1}{2}\, B((\tilde{\mathbf{u}},\tilde{\mathbf{w}}),(\tilde{\mathbf{u}},\tilde{\mathbf{w}})) \\ -B((\mathbf{u},\mathbf{w}),(\tilde{\mathbf{u}},\tilde{\mathbf{w}})) + \tfrac{1}{2}B((\mathbf{u},\mathbf{w}),(\mathbf{u},\mathbf{w})) + \tfrac{1}{2}B((\mathbf{u},\mathbf{w}),(\mathbf{u},\mathbf{w})) - F(\mathbf{u},\mathbf{w}) \\ = \tfrac{1}{2}\, B((\tilde{\mathbf{u}},\tilde{\mathbf{w}}),(\tilde{\mathbf{u}},\tilde{\mathbf{w}})) - F(\tilde{\mathbf{u}},\tilde{\mathbf{w}}) = J(\tilde{\mathbf{u}},\tilde{\mathbf{w}}).\end{aligned}$$

Hence, the inequality (82) holds, which shows that the weak solution $(\mathbf{u},\mathbf{w})$ is the unique minimizer of the functional over $V(\omega)$. The proof is complete.

In the next section, we apply this general existence theorem to the special case of isotropic Cosserat 6-parameter shells.

5 Applications to isotropic linear Cosserat shells

In order to use the 6-parameter shell model in practice, one needs to know the specific expression of the strain energy density $\mathcal{W}(\mathbf{e},\mathbf{k})$ of the form (38).

In this section, we present three different specific isotropic shell models which are available in the literature and investigate whether the existence result stated by Theorem 1 is applicable for these models.

5.1 Classical linear model for 6-parameter shells

In [10, 12, 20] the following quadratic strain energy density for 6-parameter shells made of a Cauchy continuum is considered

$$2\widehat{\mathcal{W}}(\mathbf{e},\mathbf{k}) = \begin{aligned}[t] &\alpha_1[\mathrm{tr}(\mathbf{ae})]^2 + \alpha_2\mathrm{tr}[(\mathbf{ae})^2] + \alpha_3\|\mathbf{ae}\|^2 + \alpha_4\|\mathbf{n}_0\mathbf{e}\|^2 \\ &+\beta_1[\mathrm{tr}(\mathbf{ak})]^2 + \beta_2\mathrm{tr}[(\mathbf{ak})^2] + \beta_3\|\mathbf{ak}\|^2 + \beta_4\|\mathbf{n}_0\mathbf{k}\|^2, \end{aligned} \tag{83}$$

where $\mathbf{ae} = e_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta$, $\mathbf{n}_0\mathbf{e} = e_{3\beta}\,\mathbf{a}^\beta$, $\mathbf{ak} = k_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta$, and $\mathbf{n}_0\mathbf{k} = k_{3\beta}\,\mathbf{a}^\beta$. In the above relation, α_k and β_k $(k = 1, 2, 3, 4)$ are constant constitutive coefficients (elastic moduli). In view of the relation $\mathrm{tr}(\mathbf{X}^2) = \|\mathrm{sym}(\mathbf{X})\|^2 - \|\mathrm{skew}(\mathbf{X})\|^2$, we can put (83) in the equivalent form

$$\begin{aligned} 2\widehat{\mathcal{W}}(\mathbf{e},\mathbf{k}) = {}& (\alpha_2+\alpha_3)\|\mathrm{sym}(\mathbf{ae})\|^2 + (\alpha_3-\alpha_2)\|\mathrm{skew}(\mathbf{ae})\|^2 \\ &+\alpha_1[\mathrm{tr}(\mathbf{ae})]^2 + \alpha_4\|\mathbf{n}_0\mathbf{e}\|^2 + (\beta_2+\beta_3)\|\mathrm{sym}(\mathbf{ae})\|^2 \\ &+(\beta_3-\beta_2)\|\mathrm{skew}(\mathbf{ae})\|^2 + \beta_1[\mathrm{tr}(\mathbf{ae})]^2 + \beta_4\|\mathbf{n}_0\mathbf{e}\|^2. \end{aligned} \tag{84}$$

It was shown that this strain energy function satisfies the coercivity condition (44) provided that the elastic moduli verify the inequalities

$$\begin{array}{llll} 2\alpha_1+\alpha_2+\alpha_3>0, & \alpha_2+\alpha_3>0, & \alpha_3-\alpha_2>0, & \alpha_4>0, \\ 2\beta_1+\beta_2+\beta_3>0, & \beta_2+\beta_3>0, & \beta_3-\beta_2>0, & \beta_4>0. \end{array} \tag{85}$$

Under these conditions, we can apply the Theorem 1 and obtain the following result concerning existence and uniqueness of weak solutions.

Corollary 1 *Assume that the hypotheses (56), (59), (60) of Theorem 1 are satisfied. Consider that the strain energy density is a quadratic function of the form (83) (or equivalently (84)) and the elastic moduli satisfy the inequalities (85).*

Then, the variational problem (58) associated to the equilibrium of linearly elastic 6-parameter shells admits an unique solution $(\mathbf{u},\mathbf{w})$ *in the admissible set* $V(\omega)$. *This solution* $(\mathbf{u},\mathbf{w})$ *is the minimizer on the space* $V(\omega)$ *of the functional*

$$\widehat{J}(\mathbf{u},\mathbf{w}) = \int_\omega \widehat{\mathcal{W}}(\mathbf{e},\mathbf{k})\,\sqrt{a}\,\mathrm{d}x_1\mathrm{d}x_2 - \int_\omega(\mathbf{f}\cdot\mathbf{u}+\mathbf{l}\cdot\mathbf{w})\,\sqrt{a}\,\mathrm{d}x_1\mathrm{d}x_2\,. \tag{86}$$

Remark 4 *Using a different functional framework and another method, Eremeyev and Lebedev have proved previously in [12] the existence and uniqueness of weak solutions in the energy space associated to linear 6-parameter shells.*

5.2 Linear Cosserat shell model of order h^3

Starting from an isotropic three-dimensional Cosserat parent model and using a method suggested by the classical shell theory [21], we have derived in [19] a 6-parameter Cosserat shell model of order $O(h^3)$.

In order to present the linearized version of this Cosserat shell model we denote by $\mathbf{b}$ the second fundamental tensor given by

$$\mathbf{b} = -\mathrm{Grad}_s\, \mathbf{n}_0 = -\mathbf{n}_{0,\alpha} \otimes \mathbf{a}^\alpha = b_{\alpha\beta}\, \mathbf{a}^\alpha \otimes \mathbf{a}^\beta = b^\alpha_\beta\, \mathbf{a}_\alpha \otimes \mathbf{a}^\beta. \tag{87}$$

We also introduce the so-called alternator tensor $\mathbf{c}$ of the midsurface ω_ξ by the relation

$$\mathbf{c} = \varepsilon^{\alpha\beta}\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta = \varepsilon_{\alpha\beta}\, \mathbf{a}^\alpha \otimes \mathbf{a}^\beta\,, \tag{88}$$

where $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are defined in (36). Let $H = \frac{1}{2}\mathrm{tr}\,\mathbf{b} = \frac{1}{2}b^\alpha_\alpha$ be the mean curvature and $K = \det \mathbf{b} = \det(b^\alpha_\beta)_{2\times 2}$ be the Gauß curvature of the surface ω_ξ. In view of the relation $\mathbf{b}(2H\mathbf{a} - \mathbf{b}) = K\mathbf{a}$, we introduce the tensor

$$\mathbf{b}^* = 2H\mathbf{a} - \mathbf{b}, \tag{89}$$

which can be regarded as the cofactor of $\mathbf{b}$ in the tangent plane. We also denote by κ_1, κ_2 the principal curvatures of the reference midsurface and we assume as usual that $|\kappa_\alpha h| < 1$ $(\alpha = 1, 2)$, where h is the thickness of the shell. Let κ be the maximum of the absolute value of principal curvatures $|\kappa_\alpha|$ on the midsurface ω_ξ, i.e. $\kappa = \max\limits_{\omega_\xi}\{|\kappa_1|\,, |\kappa_2|\}$.

The explicit form of the areal strain energy density of order h^3 has been obtained in [19] as a quadratic function of the strain measures, in which the coefficients are expressed in terms of the three-dimensional material constants and depend also on the curvature of the reference midsurface. This explicit form for linear shells is (cf. [19, f. (68)])

$$\begin{aligned} \mathcal{W}^{(3)}(\mathbf{e}, \mathbf{k}) = {} & \Big(h - K\,\frac{h^3}{12}\Big)\Big[W_{\mathrm{Coss}}(\mathbf{e}) + W_{\mathrm{curv}}(\mathbf{k})\Big] \\ & + \frac{h^3}{12}\Big[W_{\mathrm{Coss}}(\mathbf{e}\mathbf{b} + \mathbf{c}\mathbf{k}) - 2W_{\mathrm{Coss}}(\mathbf{e}, \mathbf{c}\mathbf{k}\mathbf{b}^*) + W_{\mathrm{curv}}(\mathbf{k}\mathbf{b})\Big]. \end{aligned} \tag{90}$$

Here, the bilinear form $W_{\mathrm{Coss}}(\cdot\,,\cdot)$ and the quadratic form $W_{\mathrm{Coss}}(\cdot)$ are defined for any tensors $\mathbf{X} = X_{i\alpha}\mathbf{a}^i \otimes \mathbf{a}^\alpha$, $\mathbf{Y} = Y_{i\alpha}\mathbf{a}^i \otimes \mathbf{a}^\alpha$ by

$$\begin{aligned} W_{\mathrm{Coss}}(\mathbf{X}, \mathbf{Y}) \;=\; & \mu\, \mathrm{sym}(\mathbf{a}\mathbf{X}) : \mathrm{sym}(\mathbf{a}\mathbf{Y}) \;+\; \mu_c\, \mathrm{skew}(\mathbf{a}\mathbf{X}) : \mathrm{skew}(\mathbf{a}\mathbf{Y}) \\ & + \frac{\lambda\,\mu}{\lambda + 2\mu}\, \mathrm{tr}(\mathbf{a}\mathbf{X})\, \mathrm{tr}(\mathbf{a}\mathbf{Y}) + \frac{2\mu\,\mu_c}{\mu + \mu_c}\, (\mathbf{n}_0\mathbf{X}) \cdot (\mathbf{n}_0\mathbf{Y}), \end{aligned} \tag{91}$$

and, respectively,

$$W_{\text{Coss}}(\mathbf{X}) = \mu\,\|\text{sym}(\mathbf{aX})\|^2 + \mu_c\|\text{skew}(\mathbf{aX})\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}[\text{tr}(\mathbf{aX})]^2 + \frac{2\mu\,\mu_c}{\mu+\mu_c}\|\mathbf{n}_0\mathbf{X}\|^2, \tag{92}$$

where λ, μ and μ_c represent the Lamé constants and the Cosserat couple modulus of the three-dimensional isotropic Cosserat material. Also the quadratic form $W_{\text{curv}}(\cdot)$ is defined by

$$W_{\text{curv}}(\mathbf{X}) = \mu\,L_c^2\left[\,b_1\,\|\,\text{sym}\,\mathbf{X}\|^2 + b_2\,\|\,\text{skew}\,\mathbf{X}\|^2 + (b_3 - \frac{b_1}{3})(\text{tr}\,\mathbf{X})^2\,\right], \tag{93}$$

where the coefficients b_1, b_2, b_3 are dimensionless constitutive coefficients and the parameter $L_c > 0$ introduces an internal length (characteristic for the Cosserat material, see details in [13, 14]).

For this linear Cosserat shell model, we state the following result concerning existence and uniqueness of weak solutions.

Corollary 2 *Assume that the reference midsurface satisfies the regularity* $\mathbf{y}_0 \in H^2(\omega,\mathbb{R}^3)$*, together with the conditions (59)*$_2$ *and (60), while the body forces verify the requirements (56). Further, assume that the constitutive coefficients fulfill the inequalities*

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \qquad \mu_c > 0, \qquad b_i > 0, \tag{94}$$

and the product κh *satisfies the following condition*

$$\kappa h < \min\left\{\frac{1}{2},\,\left(\frac{47}{32}\cdot\frac{\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{\lambda+\mu}\right)^{1/2},\left(\frac{47}{8}\cdot\frac{\min\{\mu,\mu_c\}}{\mu+\mu_c}\right)^{1/2}\right\}. \tag{95}$$

Then, the equilibrium boundary-value problem for linear Cosserat 6-parameter shells (with strain energy density $\mathcal{W}^{(3)}$ *given by (90)) admits an unique weak solution* $(\mathbf{u},\mathbf{w}) \in V(\omega)$*. The weak solution* $(\mathbf{u},\mathbf{w})$ *is the minimizer on the space* $V(\omega)$ *of the functional*

$$J^{(3)}(\mathbf{u},\mathbf{w}) = \int_\omega \mathcal{W}^{(3)}(\mathbf{e},\mathbf{k})\,\sqrt{a}\,\text{d}x_1\text{d}x_2 - \int_\omega (\mathbf{f}\cdot\mathbf{u} + \mathbf{l}\cdot\mathbf{w})\,\sqrt{a}\,\text{d}x_1\text{d}x_2\,. \tag{96}$$

Proof. In view of $\mathbf{y}_0 \in H^2(\omega,\mathbb{R}^3)$ and the definition (87), we see that the tensor $\mathbf{b}$ is of class $L^2(\omega)$. In a recent paper, we have proved that the strain energy density $\mathcal{W}^{(3)}$ is coercive provided that the conditions (94) and (95) are satisfied (see [25, Th. 1]). Hence, it holds

$$\mathcal{W}^{(3)}(\mathbf{e},\mathbf{k}) \;\geq\; C_0(\,\|\mathbf{e}\|^2+\|\mathbf{k}\|^2\,) \tag{97}$$

and from Theorem 1 we deduce the existence and uniqueness of the weak solution, as well as the minimization property.

Remark 5 *The statement of Corollary 2 remains valid if we replace the condition (95) with the alternative condition*

$$\kappa h < \min\left\{ \sqrt{\frac{12\,\min\{\lambda+2\mu\,,3\lambda+2\mu\}}{8(\lambda+\mu)+\min\{\lambda+2\mu\,,3\lambda+2\mu\}}}\,,\, \sqrt{\frac{12\,\min\{\mu,\mu_c\}}{2(\mu+\mu_c)+\min\{\mu,\mu_c\}}} \right\}. \tag{98}$$

Indeed, according to the result in [25, Th. 1] the conditions (94) and (98) insure the coercivity of the strain energy density $\mathcal{W}^{(3)}$*, so we can apply again the general Theorem 1 established in the previous section to prove the existence and uniqueness of weak solutions.*

Remark 6 *The existence of weak solutions for a related linear Cosserat shell model of order* $O(h^3)$ *has been established previously in [17]. Notice that Corollary 2 is an improvement of the existence theorem presented in [17], in the sense that the hypotheses on the coefficients (94) are less restrictive and the conditions (95) have been optimized.*

5.3 Higher order Cosserat shell model

In [18] we have derived a refined Cosserat shell model of order $O(h^5)$ by a dimensional descent from the three-dimensional nonlinear Cosserat elasticity. This 6-parameter model is able to describe isotropic shells made of Cosserat materials.

In what follows we consider the corresponding linearized 6-parameter shell model of order $O(h^5)$ and establish the existence of weak solutions. The quadratic strain energy density derived in [18, f. (119)] has the form

$$\begin{aligned}\mathcal{W}^{(5)}(\mathbf{e},\mathbf{k}) =\; & \Big(h-K\frac{h^3}{12}\Big)\Big[W_{\mathrm{Coss}}(\mathbf{e})+W_{\mathrm{curv}}(\mathbf{k})\Big]-\frac{h^3}{6}W_{\mathrm{Coss}}(\mathbf{e},\mathbf{ckb}^*)\\ & +\Big(\frac{h^3}{12}-K\,\frac{h^5}{80}\Big)\Big[W_{\mathrm{Coss}}(\mathbf{eb}+\mathbf{ck})+W_{\mathrm{curv}}(\mathbf{kb})\Big]\\ & +\frac{h^5}{80}\Big[W_{\mathrm{Coss}}((\mathbf{eb}+\mathbf{ck})\mathbf{b})+W_{\mathrm{curv}}(\mathbf{kb}^2)\Big].\end{aligned} \tag{99}$$

Under certain conditions on the constitutive coefficients, the coercivity of this strain energy function has been proved in [25, Theorem 3]. Then, we can formulate the following existence and uniqueness result.

Corollary 3 *Assume that the reference midsurface fulfill the regularity* $\mathbf{y}_0 \in H^2(\omega, \mathbb{R}^3)$*, together with the conditions* $(59)_2$ *and (60), while the body forces verify the requirements (56). Further, assume that* $\kappa h < \frac{1}{2}$ *and the constitutive coefficients satisfy the inequalities (94).*

Then, the equilibrium boundary-value problem for linear Cosserat 6-parameter shells (with higher-order strain energy density $\mathcal{W}^{(5)}$ *given by (99)) admits an unique weak solution* $(\mathbf{u}, \mathbf{w})$ *in the admissible space* $V(\omega)$*. This weak solution is the minimizer on the space* $V(\omega)$ *of the functional*

$$J^{(5)}(\mathbf{u}, \mathbf{w}) = \int_\omega \mathcal{W}^{(5)}(\mathbf{e}, \mathbf{k}) \sqrt{a}\, \mathrm{d}x_1 \mathrm{d}x_2 - \int_\omega (\mathbf{f} \cdot \mathbf{u} + \mathbf{l} \cdot \mathbf{w}) \sqrt{a}\, \mathrm{d}x_1 \mathrm{d}x_2 \,. \quad (100)$$

Proof. Since the constitutive coefficients verify the conditions (94) and $\kappa h < \frac{1}{2}$ we can use the Theorem 3 in [25], which affirms that the areal strain energy density $\mathcal{W}^{(5)}$ is coercive. Therefore, all the hypotheses of Theorem 1 are satisfied. According to the statement of Theorem 1, the variational problem (58) admits an unique solution $(\mathbf{u}, \mathbf{w}) \in V(\omega)$, which is also the minimizer of the functional (100). This completes the proof.

Remark 7 *The first Cosserat 6-parameter shell model of order* $O(h^5)$ *has been established in [13, 14]. For this related Cosserat shell model, the existence of weak solutions in the linearized theory has been proved similarly in [16, 17]. A detailed comparison between the two 6-parameter Cosserat shell models of order* $O(h^5)$ *has been presented in [18, Sect. 5.3].*

In conclusion, we have shown in this section that the general existence result Theorem 1 established in Section 4 is applicable for the mathematical study of various 6-parameter models existing in the literature on linear elastic shells.

References

[1] P.G. Ciarlet. *Mathematical Elasticity, Vol. III: Theory of Shells.* North-Holland, Amsterdam, 2000.

[2] P.G. Ciarlet. *Introduction to Linear Shell Theory.* Gauthier-Villars, Paris, 1998.

[3] J. Sprekels, D. Tiba. An analytic approach to a generalized Naghdi shell model. *Adv. Math. Sci. Appl.* 12:175-190, 2002.

[4] M. Bîrsan. Inequalities of Korn's type and existence results in the theory of Cosserat elastic shells. *J. Elasticity* 90:227-239, 2008.

[5] M. Bîrsan, H. Altenbach. A mathematical study of the linear theory for orthotropic elastic simple shells. *Math. Methods Appl. Sci.* 33:1399-1413, 2010.

[6] M. Bîrsan, H. Altenbach. On the dynamical theory of thermoelastic simple shells. *Z. Angew. Math. Mech.* 91:443-457, 2011.

[7] A. Libai, J. Simmonds. *The Nonlinear Theory of Elastic Shells.* Cambridge University Press, Cambridge, 1998.

[8] J. Chróścielewski, J. Makowski, W. Pietraszkiewicz. *Statics and Dynamics of Multifold Shells: Nonlinear Theory and Finite Element Method* (in Polish). Wydawnictwo IPPT PAN, Warsaw, 2004.

[9] V. Eremeyev, W. Pietraszkiewicz. The nonlinear theory of elastic shells with phase transitions. *J. Elasticity* 74:67-86, 2004.

[10] V. Eremeyev, W. Pietraszkiewicz. Local symmetry group in the general theory of elastic shells. *J. Elasticity* 85:125-152, 2006.

[11] M. Bîrsan, P. Neff. Existence of minimizers in the geometrically nonlinear 6-parameter resultant shell theory with drilling rotations. *Math. Mech. Solids* 19:376-397, 2014.

[12] V. Eremeyev, L. Lebedev. Existence theorems in the linear theory of micropolar shells. *Z. Angew. Math. Mech.* 91:468-476, 2011.

[13] M. Bîrsan, I.D. Ghiba, R. Martin, P. Neff. Refined dimensional reduction for isotropic elastic Cosserat shells with initial curvature. *Math. Mech. Solids* 24:4000-4019. 2019.

[14] I.D. Ghiba, M. Bîrsan, P. Lewintan, P. Neff. The isotropic Cosserat shell model including terms up to $O(h^5)$. Part I: Derivation in matrix notation. *J. Elasticity* 142:201-262, 2020.

[15] I.D. Ghiba, M. Bîrsan, P. Lewintan, P. Neff. The isotropic Cosserat shell model including terms up to $O(h^5)$. Part II: Existence of Minimizers. *J. Elasticity* 142:263-290, 2020.

[16] M. Bîrsan, I.D. Ghiba, P. Neff. Existence results for the higher order linear Cosserat shell model. *PAMM – Proc. Appl. Math. Mech.*, 22:1, DOI: 10.1002/pamm.202200030, 2022.

[17] I.D. Ghiba, M. Bîrsan, P. Neff. A linear isotropic Cosserat shell model including terms up to $O(h^5)$. Existence and uniqueness. *J. Elasticity*, DOI: 10.1007/s10659-022-09981-6, 2023.

[18] M. Bîrsan. Alternative derivation of the higher-order constitutive model for six-parameter elastic shells. *Z. Angew. Math. Phys.* 72: art. 50, 2021.

[19] M. Bîrsan. Derivation of a refined six-parameter shell model: descent from the three-dimensional Cosserat elasticity using a method of classical shell theory. *Math. Mech. Solids* 25:1318-1339, 2020.

[20] H. Altenbach, V. Eremeyev. Cosserat-type shells. In: H. Altenbach, V. Eremeyev (eds.) *Generalized Continua – from the Theory to Engineering Applications*, CISM Courses and Lectures, Vol. 541, pp. 131–178. Springer, Wien, 2013.

[21] D.J. Steigmann, M. Bîrsan, M. Shirani. *Lecture Notes on the Theory of Plates and Shells.* Ser. Solid Mechanics and Its Applications, Vol. 274, Springer, 2023.

[22] V. Eremeyev, L. Zubov. *Mechanics of Elastic Shells* (in Russian). Nauka, Moscow, 2008.

[23] L.P. Lebedev, M.J. Cloud, V. Eremeyev. *Tensor Analysis with Applications in Mechanics.* World Scientific, New Jersey, 2010.

[24] H. Brezis. *Analyse fonctionelle. Théorie et applications* (in French). Masson, Paris, 1983.

[25] M. Bîrsan, P. Neff. On the coercivity of strain energy functions in generalized models of 6-parameter shells. In: H. Altenbach et al. (eds.), *Sixty Shades of Generalized Continua.* Ser. Advanced Structured Materials, Vol. 170, pp. 63–90, Springer, 2023.