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ON THE STABILITY AND MEAN SQUARE STABILIZATION OF A CLASS OF LINEAR STOCHASTIC SYSTEMS CONTROLLED BY IMPULSES*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

This paper is devoted to the analysis of the mean-square stability of a class of linear time-varying impulsive Itô-type stochastic systems. We succeed to obtain necessary and sufficient stability conditions in the general time-varying case. This result is obtained thanks to the theory of positive operators on Hilbert spaces. The problem of state-feedback stabilization is treated as well.

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1 Introduction

In many applications, systems are subject to some abrupt changes at certain time moments and cannot be considered continuously. Such a phenomenon is called the impulsive effect. More formally, impulsive systems are a class of dynamic systems in which the state propagates according to continuous-time dynamics except for a countable set of times at which the state can change instantaneously. These systems are useful in representing a number of real world applications from engineering, environmental to mathematical finance applications [1, 4, 5, 9, 13, 16, 18, 19, 20, 22]. A privileged mathematical tool used in the analysis of such systems is the theory of impulsive differential equations. It has been extensively used in the investigation of dynamical impulsive systems in the past several years [2, 10, 11, 12, 15, 17]. For a recent overview on the subject, one can refer to the review article [21].

In this paper, we first address the problem of mean-square stability of a class of linear time-varying impulsive Itô-type stochastic systems. We succeed to obtain necessary and sufficient stability conditions in the general time-varying case (*i.e.* without any assumption on the time-variation nature of the system). This result is obtained thanks to the theory of positive operators on Hilbert spaces. Indeed, the stability conditions are formulated as a global existence condition for the solution of an adequately defined backward jump Lyapunov differential equation. These very general conditions allow us, when the time variation nature of the system is constrained, to obtain several auxiliary results (viewed as particular cases), namely: the periodic case and the time invariant case. Another auxiliary result covered by the theory developed in our paper is the case of impulsive linear time invariant systems with ranged dwell-time. By specializing the obtained stability conditions to this particular case, one can obtain necessary and sufficient stability conditions for such a class of systems. This has to be directly compared to the result obtained in [3] where only sufficient stability conditions have been proposed.
Finally, by using the obtained stability conditions, we address the problem of mean square stabilization (via state-feedback) of time-varying impulsive Itô-type stochastic systems. We succeed to obtain necessary and sufficient stabilization conditions under the linear matrix inequalities framework.

The rest of the paper is organized as follows: in Section 2 we give the problem setting. In Section 3, we describe a class of jump Lyapunov-type differential equations that play a key role for the stability analysis. In Section 4 we state our main results regarding the stability analysis problem. Section 5 is devoted to the state-feedback stabilization problem.

2 Model description and the problem setting

Let us consider the controlled system having the state-space representation described by:

$$dx(t) = A_0(t)x(t)dt + A_1(t)x(t)dw(t),\ kh < t \le (k+1)h \tag{1a}$$

$$x(kh^+) = \mathcal{A}_0(k)x(kh) + \mathcal{B}_0(k)u(k) + w_d(k)(\mathcal{A}_1(k)x(kh) + \mathcal{B}_1(k)u(k)) \tag{1b}$$

$k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $h > 0$ being a given constant, where $x(t) \in \mathbb{R}^n$ is the vector of the state parameters and $u(k) \in \mathbb{R}^m$ are the control parameters. In (1a) $\{w(t)\}_{t\ge 0}$ is a 1-dimensional standard Wiener process defined on the probability space $(\Omega, \mathfrak{F}, \mathcal{P})$ and $\{w_d(k)\}_{k\in\mathbb{Z}_+}$ is a sequence of independent random variables with zero mean and variance 1.

We assume that $\{w(t)\}_{t\ge 0}$ and $\{w_d(k)\}_{k\in\mathbb{Z}_+}$ are independent stochastic processes.

Foe each $t > 0$, $\mathcal{F}_t \subset \mathfrak{F}$ stands for the $\sigma-$algebra generated by the random variables $w(s)$, $0 \le s \le t$ and $w_d(k)$, $0 \le kh < t$. For $t = 0$, $\mathcal{F}_0$ is the $\sigma-$algebra consisting of $\emptyset$ and Ω. For each $t \ge 0$, we assume that $\mathcal{F}_t$ is augmented by all subsets $\mathfrak{A} \in \mathfrak{F}$ with $\mathcal{P}(\mathfrak{A}) = 0$. If $[a, b] \subset \mathbb{R}_+$ is an interval, we denote $\mathcal{L}^2_{\mathcal{F}}\{[a, b], \mathbb{R}^d\}$ the linear space of the measurable stochastic processes $v : [a, b] \times \Omega \to \mathbb{R}^d$ with the property:

(a) for each $t \in [a, b]$, $v(t)$ is $\mathcal{F}_t-$measurable;

(b) $\mathbb{E}\left[\int_a^b |v(t)|^2 dt\right] < \infty.$

Throughout the paper $\mathbb{E}[\cdot]$ denotes the mathematical expectation. Based on Theorem 1.6.1 from [7] one proves that $\mathcal{L}^2_{\mathcal{F}}\{[a, b], \mathbb{R}^d\}$ is a real Hilbert space with respect to the inner product

$$\langle v_1(\cdot), v_2(\cdot)\rangle = \mathbb{E}\left[\int_a^b v_1^\top(t)v_2(t)dt\right],$$

for all $v_1(\cdot), v_2(\cdot) \in \mathcal{L}^2_{\mathcal{F}}\{[a,b], \mathbb{R}^d\}$. One sees that the control parameters act to the system (1) only at the instances $t_k = kh$, $k \in \mathbb{Z}_+$. This is why, the system of type (1) will be called impulsive controlled linear stochastic system (ICLSS) and the time instances $t_k = kh$ will be called impulsive times.

If $t_0 \in \mathbb{R}_+$ we denote $k(t_0)$ the integer which is related by t_0 through the condition

$$k(t_0) = 1 + [\frac{t_0}{h}]. \tag{2}$$

Hence we have

$$(k(t_0) - 1)h \leq t_0 < k(t_0)h.$$

We denote $\mathcal{U}_{ad}(t_0)$ the set of the sequences of random vectors $\mathbf{u} = \{u(k)\}_{k \geq k(t_0)}$ with the properties:

(a) for each $k \geq k(t_0)$, $u(k) : \Omega \to \mathbb{R}^m$ are random vectors $\mathcal{F}_{kh}-$measurable;

(b) $\mathbb{E}\left[|u(k)|^2\right] < \infty$.

Regarding the solutions of the system (1) one may prove:

Proposition 1. *Assume that $A_j(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$, $j = 0, 1$, are bounded and continuous matrix valued functions. Let $t_0 \in \mathbb{R}_+$ and $x_0 : \Omega \to \mathbb{R}^n$ be a random vector $\mathcal{F}_{t_0}-$measurable such that $\mathbb{E}[|x_0|^2] < \infty$. Under these conditions for each $\mathbf{u} = \{u(k)\}_{k \geq k(t_0)}$ the ICLSS (1) has a unique solution $x(t) = x(t; t_0, x_0, \mathbf{u})$ which has the properties:*

(a) *$t \to x(t)$ is continuous with probability 1 in any $t \geq t_0$, $t \neq kh$ and it is left continuous with probability 1 in any $t = kh$, $k \geq k(t_0)$;*

(b) *$x(\cdot) \in \mathcal{L}^2_{\mathcal{F}}\{[t_0, T], \mathbb{R}^n\}$ for all $T > t_0$;*

(c) *$x(t_0) = x_0$ and $\lim\limits_{t \searrow kh} x(t) = x(kh^+)$ a.s. for all $k \geq k(t_0)$.*

Proof. One may apply Theorem 1.1 from Chapter 5 of [8] or Theorem 5.2.1 of [14] first on the interval $[t_0, k(t_0)h]$ taking x_0 as initial condition and next for each interval $[kh, (k+1)h]$ when $k \geq k(t_0)$ taking $x(kh^+)$ provided by (1b), as initial condition. From (1b) one sees that $x(kh^+)$ is measurable with respect to the $\sigma-$algebra $\mathcal{F}_{kh} \vee \sigma(w_d(k)) \subset \mathcal{F}_t$, for all $t > kh$. □

If in (1) we take $\mathcal{B}_j(k) = 0$, $k \in \mathbb{Z}_+$, $j = 0, 1$, we obtain the following jump stochastic linear differential equation (JSLDE):

$$dx(t) = A_0(t)x(t)dt + A_1(t)x(t)dw(t), \ kh < t \leq (k+1)h \tag{3a}$$

$$x(kh^+) = (\mathcal{A}_0(k) + w_d(k)\mathcal{A}_1(k))x(kh), \ \ k \in \mathbb{Z}_+. \tag{3b}$$

Since the solutions of the JSLDE (3) may be viewed as solutions of the ICLSS (1) when $u(k) = 0$, $k \in \mathbb{Z}_+$, it follows that the result stated in Proposition 1 may be applied in the case of the solution $x(\cdot; t_0, x_0)$ of the JSLDE (3). Now, we introduce the following definitions which help us to state the problems investigated in this work.

Definition 1. *We say that JSLDE (3) is exponentially stable in mean square (ESMS) if there exist two constants $\beta \geq 1$, $\alpha > 0$ with the property that its solutions $x(\cdot; t_0, x_0)$ are satisfying:*

$$\mathbb{E}[|x(t; t_0, x_0)|^2] \leq \beta e^{-\alpha(t-t_0)} |x_0|^2, \tag{4}$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$.

Definition 2. *We say that the ICLSS (1) is mean square stabilizable by a linear state feedback, if there exists a sequence of matrices $\{F(k)\}_{k \in \mathbb{Z}_+} \subset \mathbb{R}^{m \times n}$ with the property that the closed-loop JSLDE*

$$dx(t) = A_0(t)x(t)dt + A_1(t)x(t)dw(t), \; kh < t \leq (k+1)h \tag{5a}$$

$$x(kh^+) = [\mathcal{A}_0(k) + \mathcal{B}_0(k)F(k) + w_d(k)(\mathcal{A}_1(k) + \mathcal{B}_1(k)F(k))]x(kh), \quad k \in \mathbb{Z}_+ \tag{5b}$$

is ESMS.

Our aim is to provide necessary and sufficient conditions which guarantee the fact that a JSLDE of type (3) is ESMS. Then, using the criteria obtained for exponential stability in mean square of a JSLDE, we shall derive criteria for mean-square stabilizability by state feedback of a ICLSS of type (1). At the end of this section we display two important particular cases of an ICLSS (1) and of a JSLDE (3).

(i) The periodic case.

Definition 3. *We say that an ICLSS of type (1) is in the periodic case if the following conditions are simultaneously fulfilled:*

(a) *there exist a real number $\theta_c > 0$ with the property that $A_j(t+\theta_c) = A_j(t)$, for all $t \in \mathbb{R}_+$, $j = 0, 1$;*

(b) *there exists an integer number $\theta_d \geq 1$ such that $\mathcal{A}_j(k+\theta_d) = \mathcal{A}_j(k)$ and $\mathcal{B}_j(k+\theta_d) = \mathcal{B}_j(k)$, for all $k \in \mathbb{Z}_+$, $j = 0, 1$;*

(c) *$\frac{\theta_c}{h\theta_d}$ is a rational number.*

(ii) The time invariant case.

Definition 4. *We say that an ICLSS as (1) is time invariant if* $A_j(t) = A_j \in \mathbb{R}^{n\times n}$ *for all* $t \in \mathbb{R}_+$, $j = 0, 1$ *and* $\mathcal{A}_i(k) = \mathcal{A}_i \in \mathbb{R}^{n\times n}$, $\mathcal{B}_i(k) = \mathcal{B} \in \mathbb{R}^{n\times m}$ *for all* $k \in \mathbb{Z}_+$, $i = 0, 1$.

Remark 1. (a) *If the condition* (*c*) *from Definition 3 is satisfied then there exist two coprime natural numbers* $\tilde{p} \geq 1$ *and* $\tilde{q} \geq 1$ *such that*

$$\frac{\theta_c}{h\theta_d} = \frac{\tilde{p}}{\tilde{q}}. \tag{6}$$

In this case we can take

$$\tilde{\theta} \triangleq \tilde{q}\theta_c = \tilde{p}h\theta_d \tag{7a}$$

$$\tilde{\mathbf{k}} \triangleq \tilde{p}\theta_d. \tag{7b}$$

(b) *In the time invariant case we may take* $\theta_c = h$ *and* $\theta_d = 1$ *because a constant function can be viewed as a periodic function of an arbitrary period and a constant sequence can be viewed as a periodic sequence of period* $\theta_d = 1$. *In this case (7) yields*

$$\tilde{\theta} = h \tag{8a}$$

$$\tilde{\mathbf{k}} = 1. \tag{8b}$$

(c) *One may prove that if the ICLSS (1) is periodic in the sense of the Definition 3, then it is stabilizable in mean square if and only if there exist a sequence of feedback gains* $\{f(k)\}_{k\in\mathbb{Z}_+}$ *for which the closed-loop system (5) is ESMS and additionally* $F(k + \tilde{\mathbf{k}}) = F(k)$, *for all* $k \in \mathbb{Z}_+$, $\tilde{\mathbf{k}}$ *being defined in (7b). Particularly, if the ICLSS is time invariant, then it is stabilizable in mean square if and only if there exists a constant sequence of feedback gains for which the corresponding closed-loop system of type (5) is ESMS.*

3 A deterministic jump matrix linear differential equation associated to a jump stochastic linear differential equation

In this section, we associate a deterministic jump matrix linear differential equation (JMLDE) to a jump stochastic linear differential equation of type (3). Further, to the obtained JMLDE we associate a discrete-time linear equation DTLE with positive evolution on an ordered Hilbert space. Employing the criteria derived in Chapter 2 from [6] for the exponential stability of a DTLE with positive evolution we can derive useful criteria for exponential stability in mean square of a JSLDE of type (3).

Let $x(t) = x(t; t_0, x_0)$ be an arbitrary solution of (3) where x_0 is an $n-$dimensinal random vector $\mathcal{F}_{t_0}-$measurable with $\mathbb{E}[|x_0|^2] < \infty$. We set $Y(t) \stackrel{\Delta}{=} \mathbb{E}[x(t)x^\top(t)]$. Applying Itó formula on each interval $[kh, (k+1)h]$ we obtain that $t \to Y(t)$ solves the following deterministic jump matrix linear differential equation:

$$\dot{Y}(t) = \mathcal{L}(t)[Y(t)], \quad kh < t \leq (k+1)h \tag{9a}$$

$$Y(kh^+) = \mathbb{E}[x(kh^+)x^\top(kh^+)], \quad k \geq k(t_0), \tag{9b}$$

where $Y \to \mathcal{L}(t)[Y] : \mathcal{S}_n \to \mathcal{S}_n$ is the linear operator defined by

$$\mathcal{L}(t)[Y] = A_0(t)Y + YA_0^\top(t) + A_1(t)YA_1^\top(t). \tag{10}$$

Here and in the sequel $\mathcal{S}_n$ is the linear space of symmetric matrices of dimension $n \times n$.

To obtain an explicit formula of the right hand side of (9b) we remark that $\sigma(w_d(k))$ is independent of the $\sigma-$algebra $\mathcal{F}_{kh}$. Using (3b) we obtain that

$$\begin{aligned}\mathbb{E}[x(kh^+)x^\top(kh^+)] &= \mathbb{E}[\mathbb{E}[(\mathcal{A}_0(k) + w_d(k)\mathcal{A}_1(k))x(kh)x^\top(kh)\\ &\qquad \cdot (\mathcal{A}_0(k) + w_d(k)\mathcal{A}_1(k))^\top|\mathcal{F}_{kh}]]\\ &= \mathcal{A}_0(k)\mathbb{E}[x(kh)x^\top(kh)]\mathcal{A}_0^\top(k) + \mathcal{A}_1(k)\mathbb{E}[x(kh)x^\top(kh)]\mathcal{A}_1^\top(k)\\ &= \sum_{j=0}^{1}\mathcal{A}_j(k)Y(kh)\mathcal{A}_j^\top(k).\end{aligned} \tag{11}$$

From (9a) and (11) we may conclude that $Y(t) \stackrel{\Delta}{=} \mathbb{E}[x(t)x^\top(t)]$ solves the following problem with given initial values (IVP):

$$\dot{Y}(t) = \mathcal{L}(t)[Y(t)], \quad kh < t \leq (k+1)h \tag{12a}$$

$$Y(kh^+) = \sum_{j=0}^{1}\mathcal{A}_j(k)Y(kh)\mathcal{A}_j^\top(k), \quad k \geq k(t_0), \tag{12b}$$

$$Y(t_0) = \mathbb{E}[x_0x_0^\top]. \tag{12c}$$

Remark 2. *Using the JMLDE (12a), (12b) we may consider the following IVP on* $\mathcal{S}_n$:

$$\dot{Y}(t) = \mathcal{L}(t)[Y(t)], \quad kh < t \le (k+1)h \tag{13a}$$

$$Y(kh^+) = \sum_{j=0}^{1} \mathcal{A}_j(k) Y(kh) \mathcal{A}_j^\top(k), \quad k \ge k(t_0), \tag{13b}$$

$$Y(t_0) = H \tag{13c}$$

where $(t_0, H) \in \mathbb{R}_+ \times \mathcal{S}_n$ *are arbitrary and* $k(t_0) \in \mathbb{Z}_+$ *is associated to* t_0 *via (2).*

We recall that the vector space $\mathcal{S}_n$ equipped with the inner product

$$\langle X_1, X_2 \rangle \triangleq Tr[X_1 X_2], \tag{14}$$

for all $X_1, X_2 \in \mathcal{S}_n$ becomes a real Hilbert space. In (14), $Tr[\cdot]$ denotes the trace operator. On $\mathcal{S}_n$ one considers the ordering relation $\succeq$ induced by the convex cone

$$\mathcal{S}_n^+ = \{X \in \mathcal{S}_n \mid X \ge 0\}.$$

Here $X \ge 0$ means that X is a positive semidefinite matrix. By direct calculation one obtains that the adjoint $\mathcal{L}^*(t)$ with respect to the inner product (14) of the operator $\mathcal{L}(t)$ introduced via (10) is:

$$\mathcal{L}^*(t)[Z] = A_0^\top(t) Z + Z A_0(t) + A_1^\top(t) Z A_1(t), \tag{15}$$

for all $Z \in \mathcal{S}_n$.

Definition 5. *We say that the JMLDE (13) is exponentially stable if there exist the constants* $\beta \ge 1$, $\alpha > 0$ *with the property that its solutions* $Y(\cdot; t_0, H)$ *satisfy*

$$\|Y(t; t_0, H)\| \le \beta e^{-\alpha(t-t_0)} \|H\|, \tag{16}$$

for all $t \ge t_0 \ge 0$, $H \in \mathcal{S}_n$.

In (16), $\|\cdot\|$ is the Euclidian norm of a symmetric matrix. The next result highlights the equivalence between the property of ESMS of the JSLDE (3) and the exponential stability of the accompanying JMLDE (13).

Proposition 2. *If* $A_j(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $j = 0, 1$, *are bounded and continuous matrix valued functions and* $\{\mathcal{A}_j(k)\}_{k\in\mathbb{Z}_+} \subset \mathbb{R}^{n\times n}$, $j = 0, 1$, *are bounded matrix valued sequences, then the following are equivalent:*

(i) *the JSLDE (3) is ESMS;*

(ii) *the accompanying JMLDE (13) is exponentially stable.*

Proof. See Appendix A. □

Let $\mathbf{T}(t, t_0)$ be the linear evolution operator defined on $\mathcal{S}_n$ by the linear differential equation

$$\dot{X}(t) = \mathcal{L}(t)[X(t)]. \tag{17}$$

By definition,

$$\mathbf{T}(t, t_0)[X_0] = X(t; t_0, X_0),$$

for all $t, t_0 \in \mathbb{R}_+$, $X_0 \in \mathcal{S}_n$, $X(\cdot; t_0, X_0)$ being the solution of the differential equation (17) with the initial condition

$$X(t_0; t_0, X_0) = X_0.$$

Applying Theorem 2.6.1 from [7] in the special case of the linear differential equation (17) defined on the ordered Hilbert space $(\mathcal{S}_n, \mathcal{S}_n^+)$ we have:

Corollary 1. *The linear operator* $\mathbf{T}(t, t_0)$ *and its adjoint operator* $\mathbf{T}^*(t, t_0)$ *are positive operators on the ordered space* $(\mathcal{S}_n, \mathcal{S}_n^+)$, *that is*

$$\mathbf{T}(t, t_0)[\mathcal{S}_n^+] \subset \mathcal{S}_n^+$$

and

$$\mathbf{T}^*(t, t_0)[\mathcal{S}_n^+] \subset \mathcal{S}_n^+$$

for all $t \geq t_0$.

Let $Y(\cdot)$ be a solution of the JMLDE (13). From (13a) we deduce that

$$Y((k+1)h) = \mathbf{T}((k+1)h, kh)[Y(kh^+)].$$

Substituting $Y(kh^+)$ in the previous equality, we deduce that the matrix valued sequence $\{Z_k\}_{k \in \mathbb{Z}_+}$ defined by $Z_k := Y(kh)$ for all $k \in \mathbb{Z}_+$ (for which kh is in the domain of definition of the solution $Y(\cdot)$), solves the following discrete-time linear equation (DTLE):

$$Z_{k+1} = \mathcal{L}_d(k)[Z_k] \tag{18}$$

where $Z \to \mathcal{L}_d(k)[Z] : \mathcal{S}_n \to \mathcal{S}_n$ is described by

$$\mathcal{L}_d(k)[Z] = \sum_{j=0}^{1} \mathbf{T}((k+1)h, kh)[\mathcal{A}_j(k) Z \mathcal{A}_j^\top(k)], \tag{19}$$

for all $Z \in \mathcal{S}_n$. Often the DTLE (18)-(19) will be named the accompanying DTLE associated to the JMLDE (13).

Definition 6. *We say that the DTLE (18) is exponentially stable if there exist* $\gamma \geq 1$, $\delta \in (0,1)$ *with the property that the solutions* $\{Z_k\}_{k\geq k_0}$ *of (18) are satisfying*

$$\|Z_k\| \leq \gamma \delta^{k-k_1} \|Z_{k_1}\|, \tag{20}$$

for all $k \geq k_1 \geq k_0 \geq 0$.

The next result highlights the equivalence between the exponential stability of JMLDE (13) and its accompanying DTLE (18)-(19).

Proposition 3. *Under the assumption of Proposition 2 the following are equivalent:*

(i) *the JMLDE (13) is exponentially stable;*

(ii) *the accompanying DTLE (18)-(19) is exponentially stable.*

Proof. See Appendix B. □

Remark 3. **(a)** *Combining the results proved in Proposition 2 and Proposition 3 we deduce that we can obtain criteria for exponential stability in mean square for a JSLDE (3) based on the criteria for exponential stability of the accompanying DTLE (18)-(19).*

(b) *Employing the properties of the trace operator we obtain that the adjoint operator with respect to the inner product (14) of the linear operator* $\mathcal{L}_d(k)$ *defined in (19) is given by*

$$\mathcal{L}_d^*(k)[Z] = \sum_{i=0}^{1} \mathcal{A}_i^\top(k) \mathbf{T}^*((k+1)h, kh)[Z] \mathcal{A}_i(k), \tag{21}$$

for all $Z \in \mathcal{S}_n$. *From Corollary 1 we may infer that both* $\mathcal{L}_d(k)[\cdot]$ *as well as its adjoint* $\mathcal{L}_d^*(k)[\cdot]$ *are positive operators on the ordered Hilbert space* $(\mathcal{S}_n, \mathcal{S}_n^+)$. *Hence, criteria for exponential stability of the DTLE (18)-(19) can be derived from Theorem 2.4, Theorem 2.5 and Theorem 2.7 from [6].*

(c) *If the JSLDE (3) is periodic in the sense of Definition 3 (written for* $\mathcal{B}_i(k) = 0$, $i = 0, 1$, $k \in \mathbb{Z}_+$*) then the sequence* $\{\mathcal{L}_d(k)\}_{k\in\mathbb{Z}_+}$ *is a periodic sequence of period* $\tilde{\mathbf{k}}$, *where* $\tilde{\mathbf{k}}$ *is introduced via (7b). Indeed*

(19) and (7) give

$$\begin{aligned}\mathcal{L}_d(k+\tilde{\mathbf{k}})[Z] &= \sum_{i=0}^{1} \mathbf{T}((k+1)h+\tilde{q}\theta_c, kh+\tilde{q}\theta_c)[\mathcal{A}_i(k+\tilde{p}\theta_d)Z\mathcal{A}_i^\top(k+\tilde{p}\theta_d)] \\ &= \sum_{i=0}^{1} \mathbf{T}((k+1)h, kh)[\mathcal{A}_i(k)Z\mathcal{A}_i^\top(k)] \\ &= \mathcal{L}_d(k)[Z]\end{aligned}$$

for all $k \in \mathbb{Z}_+$, $Z \in \mathcal{S}_n$.

(d) *If the JSLDE (3) is time invariant in the sense of Definition 4 (written for* $\mathcal{B}_i(k) = 0$, $i = 0, 1$, $k \in \mathbb{Z}_+$*) then the sequence of linear operators* $\{\mathcal{L}_d(k)\}_{k\in\mathbb{Z}_+}$ *introduced via (19) is a constant sequence. In this case, (19) reduces to*

$$\mathcal{L}_d(k)[Z] = \sum_{i=0}^{1} e^{\mathcal{L}h}[\mathcal{A}_i Z \mathcal{A}_i^\top] \triangleq \mathcal{L}_d[Z] \tag{22}$$

for all $Z \in \mathcal{S}_n$.

4 Criteria for exponential stability in mean square of a jump stochastic linear differential equation

4.1 The general time-varying case

Proceeding according to Remark 3 (a) we obtain:

Theorem 1. *Under the assumptions of Proposition 2, the following are equivalent:*

(i) *the JSLDE (3) is ESMS;*

(ii) *for each bounded and piecewise continuous matrix valued function* $H(\cdot): \mathbb{R}_+ \to \mathcal{S}_n^+$ *and each matrix valued sequence* $\{Q(k)\}_{k\in\mathbb{Z}_+} \subset \mathcal{S}_n^+$ *with the property that*

$$0 \prec \mu_1 I_n \preceq Q(k) \preceq \mu_2 I_n \tag{23}$$

for all $k \in \mathbb{Z}_+$*, the non-homogeneous backward jump Lyapunov differential equation:*

$$-\dot{Y}(t) = \mathcal{L}^*(t)[Y(t)] + H(t), \quad kh \le t < (k+1)h \tag{24a}$$

$$Y(kh^-) = \sum_{i=0}^{1} \mathcal{A}_i^\top(k)Y(kh)\mathcal{A}_i(k) + Q(k), \quad k \in \mathbb{Z}_+, \tag{24b}$$

has a unique bounded solution $Y(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *with the property that there exist positive constants* ν_k *such that*

$$0 \prec \nu_1 I_n \preceq Y(t) \preceq \nu_2 I_n \tag{25}$$

for all $t \in \mathbb{R}_+$*;*

(iii) *there exists a bounded and piecewise continuous function* $\tilde{H}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *and a matrix valued sequence* $\{\tilde{Q}(k)\}_{k\in\mathbb{Z}_+}$ *such that*

$$0 \prec \mu_1 I_n \preceq \tilde{Q}(k) \preceq \mu_2 I_n, \quad k \in \mathbb{Z}_+$$

with the property that the corresponding non-homogeneous backward Lyapunov differential equation of type (24) has a unique bounded solution $\tilde{Y}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *such that*

$$0 \prec \nu_1 I_n \preceq \tilde{Y}(t) \preceq \nu_2 I_n,$$

for all $t \in \mathbb{R}_+$*;*

(iv) *there exists a function* $Z(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *with the properties:*

(α) $0 \prec \gamma_1 I_n \preceq Z(t) \preceq \gamma_2 I_n$*, for all* $t \in \mathbb{R}_+$*;*

(β) $Z(\cdot)$ *is differentiable on any interval* $(kh, (k+1)h) \subset \mathbb{R}_+$ *and it is right continuous in each* $t = kh$*,* $k \in \mathbb{Z}_+$*;*

(γ) $Z(\cdot)$ *solves the backward jump Lyapunov differential inequality:*

$$\dot{Z}(t) + \mathcal{L}^*(t)[Z(t)] \preceq 0, \quad kh \leq t < (k+1)h \tag{26a}$$

$$\sum_{i=0}^{1} \mathcal{A}_i^\top(k) Z(kh) \mathcal{A}_i(k) - Z(kh^-) \preceq -\gamma_3 I_n, \quad k \in \mathbb{Z}_+. \tag{26b}$$

Proof. The implication $(ii) \Rightarrow (iii)$ and the equivalence $(iii) \Leftrightarrow (iv)$ are straightforwardly obtained.

We prove $(i) \Rightarrow (ii)$. If (i) holds, then combining Proposition 2 and Proposition 3 we deduce that the DTLE (18)-(19) is exponentially stable. Let $H(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ be a bounded and piecewise continuous matrix valued function. Let, also, $\{Q(k)\}_{k\in\mathbb{Z}_+}$ be an arbitrary matrix valued sequence which is satisfying (23). We consider the discrete-time backward non-homogeneous equation

$$Z_k = \mathcal{L}_d^*(k)[Z_{k+1}] + \Xi_k \tag{27}$$

where

$$\Xi_k \triangleq Q(k) + \sum_{i=0}^{1} \mathcal{A}_i^\top(k) \int\limits_{kh}^{(k+1)h} \mathbf{T}^*(s,kh)[H(s)]ds\mathcal{A}_i(k). \tag{28}$$

Invoking again the Corollary 1 we may conclude that

$$\int\limits_{kh}^{(k+1)h} \mathbf{T}^*(s,kh)[H(s)]ds \succeq 0$$

because $\mathcal{S}_n^+$ is a convex closed cone. Furthermore we have that

$$0 \prec \hat{\mu}_1 I_n \preceq \Xi_k \preceq \hat{\mu}_2 I_n,$$

for all $k \in \mathbb{Z}_+$ for some positive constants $\hat{\mu}_j$, $j = 1, 2$.

The implication $(i) \Rightarrow (v)$ from Theorem 2.4 [6] applied in the case of the equation (27) allows us to deduce that this equation has a solution $\{\tilde{Z}_k\}_{k\in\mathbb{Z}_+}$ which is uniformly positive and bounded, that is, there exist positive constants c_j such that

$$0 \prec c_1 I_n \preceq \tilde{Z}_k \preceq c_2 I_n, \tag{29}$$

for all $k \in \mathbb{Z}_+$. We define $Y(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n$ as:

$$Y(t) \triangleq \mathbf{T}^*((k+1)h, t)[\tilde{Z}_{k+1}] + \int\limits_{t}^{(k+1)h} \mathbf{T}^*(s,t)[H(s)]ds \tag{30}$$

if $kh \le t < (k+1)h$, $k \in \mathbb{Z}_+$. From (30) one sees that $Y(\cdot)$ is differentiable on each interval $(kh, (k+1)h)$ and satisfies (24a) on each such interval. From (21), (27), (28) and (30) written for $t = kh$, we obtain:

$$\tilde{Z}_k = \sum_{i=0}^{1} \mathcal{A}_i^\top(k) Y(kh) \mathcal{A}_i(k) + Q(k), \quad k \in \mathbb{Z}_+. \tag{31}$$

On the other hand from (30) written for $t \in [(k-1)h, kh)$, we obtain that

$$Y(kh^-) \triangleq \lim_{\substack{t\to kh\\ t<kh}} Y(t) = \tilde{Z}_k.$$

So, (31) becomes

$$Y(kh^-) = \sum_{i=0}^{1} \mathcal{A}_i^\top(k) Y(kh) \mathcal{A}_i(k) + Q(k),$$

which confirms the fact that $Y(\cdot)$ introduced via (30) solves (24). Employing Corollary 2.1.7 (i) together with Theorem 2.6.1 (ii) from [7], we obtain via (23) and (30) that

$$Y(t) \succeq \check{\mu}_1 I_n \succ 0,$$

for all $t \in \mathbb{R}_+$. On the other hand (23) together with (30) allows us to conclude that under the considered assumptions there exists a constant $\check{\mu}_2$ such that

$$Y(t) \preceq \check{\mu}_2 I_n,$$

for all $t \in \mathbb{R}_+$. Hence, $Y(\cdot)$ defined in (30) is an uniform positive and bounded solution of (24) defined on the whole semiaxis $\mathbb{R}_+$. So the proof of implication $(i) \Rightarrow (ii)$ is complete.

Let us prove now the implication $(iii) \Rightarrow (i)$. Assume that for a piecewise continuous and bounded function $\tilde{H}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ and a bounded matrix valued sequence $\tilde{Q}(k)$, $k \in \mathbb{Z}_+$ satisfying a constraint of type (23), the non-homogeneous backward jump Lyapunov differential equation of type (24) has a unique and bounded solution $\tilde{Y}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ which is satisfying the constraints

$$0 \prec \tilde{\mu}_1 I_n \preceq \tilde{Y}(t) \preceq \tilde{\mu}_2 I_n,$$

for all $t \in \mathbb{R}_+$. By direct calculation one shows that the matrix valued sequence $\{\tilde{Z}_k\}_{k \in \mathbb{Z}_+}$ defined by $\tilde{Z}_k \stackrel{\Delta}{=} \tilde{Y}(kh)$, $k \in \mathbb{Z}_+$ solves a non-homogeneous discrete-time backward equation of type (27) - (28) and satisfies constraints of the form (29). Invoking the implication $(vi) \Rightarrow (i)$ from Theorem 2.4 in [6], we deduce that the DTLE (18)-(19) is exponentially stable. Then, Proposition 3 and Proposition 2 confirm the exponential stability in mean square of the JSLDE (3). Thus the proof ends. □

Remark 4. *Although Theorem 1 provides necessary and sufficient conditions for the exponential stability in mean square of a JSLDE of type (3) they are difficult to be applied because they ask for some global solutions of some jump matrix linear differential equations on unbounded time intervals. That is why, it remains as a challenge for future research the finding of some conditions easier feasible for testing the property of ESMS of a JSLDE (3) in the general time-varying case.*

4.2 The periodic case

Theorem 2. *Assume that the JSLDE (3) is in the periodic case in the sense of Definition 3. Under these conditions the following are equivalent:*

(i) *the JSLDE (3) is ESMS;*

(ii) *for each continuous and periodic function of period* θ_c, $H(\cdot):\mathbb{R}_+\to\mathcal{S}_n^+$, *and for each matrix valued sequence* $\{Q(k)\}_{k\in\mathbb{Z}_+}\subset\mathcal{S}_n^+$ *with the property* $Q(k+\tilde{\mathbf{k}})=Q(k)\succ 0$, *for all* $k\in\mathbb{Z}_+$, *the corresponding non-homogeneous backward jump Lyapunov equation of type (24) has a unique bounded solution* $Y(\cdot):\mathbb{R}_+\to\mathcal{S}_n^+$. *Additionally, this solution is a periodic function of period* $\tilde{\theta}$ *and satisfies* $Y(t)\succ 0$ *(*$\tilde{\theta}$ *and* $\tilde{\mathbf{k}}$ *being as in (7));*

(iii) *there exists a continuous and periodic function of period* θ_c, $\tilde{H}(\cdot):\mathbb{R}_+\to\mathcal{S}_n^+$ *and a matrix valued sequence* $\{\tilde{Q}(k)\}_{k\in\mathbb{Z}_+}$ *with the property that*

$$\tilde{Q}(k+\tilde{\mathbf{k}})=\tilde{Q}(k)\succ 0$$

for all $k\in\mathbb{Z}_+$ *such that the corresponding non-homogeneous jump Lyapunov equation (24) has a unique bounded solution* $\tilde{Y}(\cdot):\mathbb{R}_+\to\mathcal{S}_n^+$. *This solution is a periodic function of period* $\tilde{\theta}$ *and satisfies* $\tilde{Y}(t)\succ 0$, $t\in\mathbb{R}_+$;

(iv) *there exists a function* $Z(\cdot):\mathbb{R}_+\to\mathcal{S}_n^+$ *having the properties:*

- (α) $Z(\cdot)$ *is a periodic function of period* $\tilde{\theta}$;
- (β) $0\prec\gamma_1 I_n\preceq Z(t)\preceq\gamma_2 I_n$, *for all* $t\in\mathbb{R}_+$, *for some positive constants* γ_j, $j=1,2$;
- (γ) $Z(\cdot)$ *is differentiable on any interval* $(kh,(k+1)h)\subset\mathbb{R}_+$ *and right continuous in each* $t=kh$, $k\in\mathbb{Z}_+$;
- (δ) $Z(\cdot)$ *solves the backward jump Lyapunov differential inequality:*

$$\dot{Z}(t)+\mathcal{L}^*(t)[Z(t)]\preceq 0,\quad kh\le t<(k+1)h \tag{32a}$$

$$\sum_{i=0}^{1}\mathcal{A}_i^\top(k)Z(kh)\mathcal{A}_i(k)-Z(kh^-)\preceq 0,\quad 0\le k\le\tilde{\mathbf{k}}-1; \tag{32b}$$

$$Z(\tilde{\mathbf{k}}h)=Z(0); \tag{32c}$$

(v) *for each continuous and* θ_c-*periodic matrix valued function* $H(\cdot):\mathbb{R}_+\to\mathcal{S}_n^+$ *and any matrix valued sequence* $\{Q(k)\}_{k\in\mathbb{Z}_+}$ *having the properties from (ii) the non-homogeneous forward jump Lyapunov differential* *equation*

$$\dot{X}(t) = \mathcal{L}(t)[X(t)] + H(t), \quad kh < t \leq (k+1)h \tag{33a}$$

$$X(kh^+) = \sum_{i=0}^{1} \mathcal{A}_i(k)X(kh)\mathcal{A}_i^\top(k) + Q(k), \quad k \in \mathbb{Z}_+ \tag{33b}$$

has a unique bounded solution $X(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$. *This solution is a periodic function of period* $\tilde{\theta}$ *and satisfies* $X(t) \succ 0$, *for all* $t \in \mathbb{R}_+$;

(vi) *there exists a continuous and* θ_c*−periodic matrix valued function* $\tilde{H} : \mathbb{R}_+ \to \mathcal{S}_n^+$ *and a matrix valued sequence* $\{\tilde{Q}(k)\}_{k\in\mathbb{Z}_+}$ *as in (iii) for which the corresponding non-homogeneous forward jump Lyapunov differential equation (33) has a unique bounded solution* $\tilde{X}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *which is a periodic function of period* $\tilde{\theta}$ *and satisfies* $\tilde{X} \succ 0$, *for all* $t \in \mathbb{R}_+$;

(vii) *there exists a function* $\tilde{Z}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *having the properties:*

- **(α')** $\tilde{Z}(\cdot)$ *is a periodic function of period* $\tilde{\theta}$;
- **(β')** $0 \prec \hat{\gamma}_1 I_n \preceq \tilde{Z}(t) \preceq \hat{\gamma}_2 I_n$, *for all* $t \in [0, \tilde{\theta}]$, *for some positive constants* $\hat{\gamma}_j$, $j = 1, 2$;
- **(γ')** $\tilde{Z}(\cdot)$ *is differentiable on any interval* $(kh, (k+1)h)$ *and it is left continuous in each* $t = kh$, $0 < k \leq \tilde{\mathbf{k}}$;
- **(δ')** $\tilde{Z}(\cdot)$ *solves the following forward Lyapunov differential inequality with jumps:*

$$\mathcal{L}(t)[\tilde{Z}(t)] - \dot{\tilde{Z}}(t) \preceq 0, \quad kh < t \leq (k+1)h \tag{34a}$$

$$\sum_{i=0}^{1} \mathcal{A}_i(k)\tilde{Z}(kh)\mathcal{A}_i^\top(k) - \tilde{Z}(kh^+) \prec 0, \quad 0 \leq k \leq \tilde{\mathbf{k}} - 1; \tag{34b}$$

$$\tilde{Z}(\tilde{\mathbf{k}}h) = \tilde{Z}(0). \tag{34c}$$

Proof. The implications $(ii) \Rightarrow (iii)$, $(v) \Rightarrow (vi)$ as well as the equivalences $(iii) \Leftrightarrow (iv)$ and $(vi) \Leftrightarrow (vii)$, respectively, are straightforward. The proofs of the implications $(i) \Rightarrow (ii)$, $(iii) \Rightarrow (i)$, $(i) \Rightarrow (v)$ and $(vi) \Rightarrow (i)$ follows the same line as the proofs performed for the implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ from the Theorem 1. □

Remark 5. *Since the criteria for testing the property of ESMS provided by Theorem 2 are based on the existence of some periodic uniform positive* *definite solutions of some jump Lyapunov differential equations or jumps Lyapunov differential inequalities, to check if such solutions exist, may be used already existing numerical procedures for the calculus of periodic solutions (see for example Section 3.4 in [3]).*

4.3 The time-invariant case

Theorem 3. *Assume that the JSLDE (3) is time-invariant (in the sense of Definition 4 written for the case $\mathcal{B}_i = 0,\ i = 0, 1$). Under these conditions the following are equivalent:*

(i) *the JSLDE (3) is ESMS;*

(ii) *for any $H \in \mathcal{S}_n^+$ and $Q \succ 0$ the non-homogeneous backward matrix Lyapunov differential equation with jumps:*

$$-\dot{Y}(t) = A_0^\top Y(t) + Y(t)A_0 + A_1^\top Y(t)A_1 + H, \quad kh \leq t < (k+1)h \tag{35a}$$

$$Y(kh^-) = \sum_{i=0}^{1} \mathcal{A}_i^\top Y(kh)\mathcal{A}_i + Q, \quad k \in \mathbb{Z}_+ \tag{35b}$$

has a unique bounded solution $Y(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$. This solution is a periodic function of period h and satisfies $Y(t) \succ 0$, for all $t \in \mathbb{R}_+$;

(iii) *there exists $\tilde{H}$, $\tilde{Q}$ in $\mathcal{S}_n^+$ such that $\tilde{Q} \succ 0$ with the property that the corresponding non-homogeneous backward matrix Lyapunov equation with jumps of type (35) has a unique bounded solution $\tilde{Y}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$. This solution is a periodic function of period h and satisfies $\tilde{Y}(t) \succ 0$, $t \in \mathbb{R}_+$;*

(iv) *there exists a function $Z(\cdot) : [0, h] \to \mathcal{S}_n^+$ with the properties:*

- (α) *$Z(\cdot)$ is periodic of period h;*
- (β) *$Z(\cdot)$ is differentiable on $(0, h)$ and right continuous in $t = 0$;*
- (γ) *$Z(\cdot)$ solves the matrix Lyapunov differential inequality with jumps:*

$$\dot{Z}(t) + A_0^\top Z(t) + Z(t)A_0 + A_1^\top Z(t)A_1 \preceq 0, \quad 0 \leq t < h \tag{36a}$$

$$\sum_{i=0}^{1} \mathcal{A}_i^\top Z(0)\mathcal{A}_i - Z(h^-) \prec 0; \tag{36b}$$

(v) *for any* H, Q *from* $\mathcal{S}_n^+$ *with* $Q \succ 0$, *the non-homogeneous forward matrix Lyapunov differential equation with jumps:*

$$\dot{X}(t) = A_0 X(t) + X(t) A_0^\top + A_1 X(t) A_1^\top + H, \quad kh < t \leq (k+1)h \tag{37a}$$

$$X(kh^+) = \sum_{i=0}^{1} \mathcal{A}_i X(kh) \mathcal{A}_i^\top + Q, \quad k \in \mathbb{Z}_+ \tag{37b}$$

has a unique bounded solution $X(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *which is a periodic function of period* h *and satisfies* $X(t) \succ 0$, *for all* $t \in \mathbb{R}_+$;

(vi) *there exist matrices* $\tilde{H}$, $\tilde{Q}$ *in* $\mathcal{S}_n^+$ *with* $\tilde{Q} \succ 0$ *such that the corresponding non-homogeneous forward matrix Lyapunov differential equation with jumps of type (37) has a unique bounded solution* $\tilde{X}(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$. *This solution is a periodic function of period* h *and satisfies the sign condition* $\tilde{X}(t) \succ 0$, $t \in \mathbb{R}_+$;

(vii) *there exists a function* $\tilde{Z}(\cdot) : [0, h] \to \mathcal{S}_n$ *with the properties:*

(α') $\tilde{Z}(\cdot)$ *is a periodic function of period* h, *satisfying* $\tilde{Z}(t) \succ 0$;

(β') $\tilde{Z}(\cdot)$ *is differentiable on* $(0, h)$ *and left continuous in* $t = h$;

(γ') $\tilde{Z}(\cdot)$ *solves the linear matrix differential inequality with jumps:*

$$-\dot{\tilde{Z}}(t) + A_0 \tilde{Z}(t) + \tilde{Z}(t) A_0^\top + A_1 \tilde{Z}(t) A_1^\top \preceq 0, \quad 0 < t \leq h; \tag{38a}$$

$$\sum_{i=0}^{1} \mathcal{A}_i \tilde{Z}(h) \mathcal{A}_i^\top - \tilde{Z}(0^+) \prec 0. \tag{38b}$$

Proof. The proof follows the line of the previous theorems. □

5 Mean square stabilizability

In this section we shall use the criteria for exponential stability in mean-square for a JSLDE derived in Section 4 to provide necessary and sufficient conditions which allows us to test if an impulsive controlled linear stochastic system (ICLSS) of type (1) is stabilizable in mean-square by linear state feedback.

5.1 The general time-varying case

Based on the equivalences proved in Theorem 1 applied in the case of the resulting system (5), we obtain:

Theorem 4. *Assume:*

(a) $A_j(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $j = 0, 1$ *are bounded and continuous matrix valued functions;*

(b) $\{\mathcal{A}_j(k)\}_{k\in\mathbb{Z}_+} \subset \mathbb{R}^{n\times n}$, $\{\mathcal{B}_j(k)\}_{k\in\mathbb{Z}_+} \subset \mathbb{R}^{n\times m}$, $j = 0, 1$, *are bounded matrix valued sequences.*

Under these conditions, the following are equivalent:

(i) *the ICLSS (1) is mean square stabilizable by linear state feedback;*

(ii) *there exists a function* $Y(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ *and the matrices* $Z_d(k) \in \mathcal{S}_n^+$, $K(k) \in \mathbb{R}^{m\times n}$, $k \in \mathbb{Z}_+$ *satisfying*

$$0 \prec c_1 I_n \preceq Y(t) \preceq c_2 I_n \tag{39}$$

and solving the following system of linear matrix inequalities

$$\begin{pmatrix} \dot{Y}(t) + A_0(t)Y(t) + Y(t)A_0^\top(t) & Y(t)A_1^\top(t) \\ A_1(t)Y(t) & -Y(t) \end{pmatrix} \preceq -c_3 I_{2n}, \qquad kh \le t < (k+1)h \tag{40a}$$

$$\begin{pmatrix} -Z_d(k) & * & * \\ \mathcal{A}_0(k)Z_d(k) + \mathcal{B}_0(k)K(k) & -Y(kh) & 0 \\ \mathcal{A}_1(k)Z_d(k) + \mathcal{B}_1(k)K(k) & 0 & -Y(kh) \end{pmatrix} \preceq -c_4 I_{3n} \tag{40b}$$

$Y(kh^-) = Z_d(k)$, $k \in \mathbb{Z}_+$ *for some* c_j, $j = 1, 2, 3, 4$ *positive constants.*

If the triple $(Y(\cdot), \{Z_d(k)\}_{k\in\mathbb{Z}_+}, \{K(k)\}_{k\in\mathbb{Z}_+})$ *is a solution of (40), then the matrices* $F(k)$ *defined by*

$$F(k) := K(k)Z_d^{-1}(k), \quad k \in \mathbb{Z}_+ \tag{41}$$

have the property that the corresponding resulting system of type (5) is ESMS.

Proof. Employing the equivalence $(i) \Leftrightarrow (ii)$ from Theorem 1 in the case of the resulting system (5), taking $H(t) = I_n$, $Q(k) = I_n$, for all $(t, k) \in \mathbb{R}_+ \times \mathbb{Z}_+$ we deduce that a control law of type $u(k) = F(k)x(kh)$, $k \in$

$\mathbb{Z}_+$ stabilizes the ICLSS (1) if and only if the non-homogeneous backward Lyapunov differential equation with jumps

$$\dot{X}(t) + \mathcal{L}^*(t)[X(t)] + I_n = 0, \quad kh \le t < (k+1)h \tag{42a}$$

$$Y(kh^-) = \sum_{i=0}^{1} (\mathcal{A}_i(k) + \mathcal{B}_i(k)F(k))^\top X(kh)(\mathcal{A}_i(k) + \mathcal{B}_i(k)F(k)) + I_n, \quad k \in \mathbb{Z}_+ \tag{42b}$$

has a unique global solution $X(\cdot) : \mathbb{R}_+ \to \mathcal{S}_n^+$ satisfying

$$0 \prec \tilde{\gamma}_1 I_n \preceq X(t) \preceq \tilde{\gamma}_2 I_n, \tag{43}$$

for all $t \in \mathbb{R}_+$, with $\tilde{\gamma}_j$, $j = 1, 2$, are positive constants. From (42b), (43) one gets

$$I_n \prec X(kh^-) \preceq \tilde{\gamma}_3 I_n, \tag{44}$$

for all $k \in \mathbb{Z}_+$. Further, the Schur complement technique together with (39) and (40a), yield

$$\begin{pmatrix} \dot{X}(t) + A_0^\top(t)X(t) + X(t)A_0(t) & A_1^\top(t) \\ A_1(t) & -X^{-1}(t) \end{pmatrix} \preceq -\gamma I_{2n},$$

for all $t \in \mathbb{R}_+$, for some $\gamma > 0$. Pre and post-multiplying the last inequality by $diag\{X^{-1}(t), I_n\}$ and setting $Y(t) := X^{-1}(t)$ we obtain that (42a) is equivalent to

$$\begin{pmatrix} -\dot{Y}(t) + A_0(t)Y(t) + Y(t)A_0^\top(t) & Y(t)A_1^\top(t) \\ A_1(t)Y(t) & -Y(t) \end{pmatrix} \preceq -\xi I_{2n}, \tag{45}$$

for all $t \in \mathbb{R}_+$, for some $\xi > 0$. Thus (45) allows us to deduce that (42a) is equivalent to a matrix inequality of type (40a) with $c_3 = \xi$.

Using again the Schur complement technique we deduce that (42b) is equivalent to

$$\begin{pmatrix} -X(kh^-) & * & * \\ \mathcal{A}_0(k) + \mathcal{B}_0(k)F(k) & X^{-1}(kh) & 0 \\ \mathcal{A}_1(k) + \mathcal{B}_1(k)F(k) & 0 & -X^{-1}(kh) \end{pmatrix} \preceq -\xi_1 I_{3n}.$$

Pre and past-multiplying the last matrix inequality by $diag\{X^{-1}(kh), I_n, I_n\}$ we obtain via (44) that

$$\begin{pmatrix} -Z_d(k) & * & * \\ \mathcal{A}_0(k)Z_d(k) + \mathcal{B}_0(k)K(k) & -Y(kh) & 0 \\ \mathcal{A}_1(k)Z_d(k) + \mathcal{B}_1(k)K(k) & 0 & -Y(kh) \end{pmatrix} \preceq -\hat{\xi}_2 I_{3n}, \tag{46}$$

where we have denoted $Z_d(k) := X^{-1}(kh^-)$, $K(k) := F(k)X^{-1}(kh^-)$, $k \in \mathbb{Z}_+$. Comparing (46) and (40b) we may conclude that (40b) is equivalent to (42b). Hence, the system of inequalities (40) provides a set of necessary and sufficient conditions for the stabilizability of ICLSS (1). Moreover, the stabilizing matrices are provided by (41). Thus the proof is complete. □

Remark 6. *a) As in the case analysed in Theorem 1, the necessary and sufficient condition derived in Theorem 4 is also difficult to apply because it is based on the computation of the solution of a Lyapunov differential equation/inequation on an unbounded interval. It remains as a challenge for future research, the finding of some sufficient conditions for stabilizability of an ICLSS (1) in the general time varying case which be numerical tractable.*

b) The results similar to those of Proposition 3, Theorem 1 and Theorem 4 can be proved without difficulty in the case when the impulse time instances $\tau_k, k \in \mathbb{Z}_+$ are such that the sequence of the so called dwell times $\delta_k = \tau_{k+1} - \tau_k$ is not constant, but it satisfies a condition of the form:

$$0 < t_{min} \leq \tau_{k+1} - \tau_k \leq t_{max} < \infty$$

for all $k \in \mathbb{Z}_+$, t_{min}, t_{max} being given bounds.

In the remainder of this section, we consider two important cases of an ICLSS (1) for which we can provide necessary and sufficient conditions for stabilizability in mean square numerically feasible.

5.2 The periodic case

Theorem 5. *Assume that the ICLSS (1) is periodic in the sense of Definition 3. If, additionally, $A_j(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ are continuous matrix valued functions, then, the following are equivalent:*

(i) *the ICLSS (1) is stabilizable in mean square by linear state feedback;*

(ii) *there exists a function $Z(\cdot) : [0, \tilde{\theta}] \to \mathcal{S}_n^+$ and the matrices $(Z_d(k), K(k)) \in \mathcal{S}_n \times \mathbb{R}^{m\times n}$, $0 \leq k \leq \tilde{\mathbf{k}} - 1$ having the properties:*

- **(a)** *$Z(\cdot)$ is differentiable on each interval $(kh, (k+1)h)$ and it is left continuous in each $t = kh$, $0 < k \leq \tilde{\mathbf{k}}$;*
- **(b)** *$0 \prec c_1 I_n \preceq Z(t) \preceq c_2 I_n$, $t \in [0, \tilde{\theta}]$ and $Z(0) = Z(\tilde{\theta})$, $\tilde{\theta}$, $\tilde{\mathbf{k}}$ being defined in (7);*

(c) $(Z(\cdot), \{Z_d(k)\}_{k\in\mathbb{Z}_+}, \{K(k)\}_{k\in\mathbb{Z}_+})$ *solves the following system of inequalities:*

$$\mathcal{L}(t)[Z(t)] - \dot{Z}(t) \preceq 0, \quad kh < t \leq (k+1)h$$
$$Z(kh^+) = Z_d(k) \tag{47a}$$
$$\begin{pmatrix} -Z_d(k) & * & * \\ (\mathcal{A}_0(k)Z(kh) + \mathcal{B}_0(k)K(k))^\top & -Z(kh) & 0 \\ (\mathcal{A}_1(k)Z(kh) + \mathcal{B}_1(k)K(k))^\top & 0 & -Z(kh) \end{pmatrix} \prec 0,$$
$$0 \leq k \leq \tilde{\mathbf{k}} - 1. \tag{47b}$$

If $(Z(\cdot), \{Z_d(k)\}_{0\leq k\leq \tilde{\mathbf{k}}-1}, \{K(k)\}_{0\leq k\leq \tilde{\mathbf{k}}-1})$ *is a solution of (47), then the control law*

$$u(k) = K\left(k - \left[\frac{k}{\tilde{\mathbf{k}}}\right]\tilde{\mathbf{k}}\right) Z^{-1}\left(kh - \left[\frac{k}{\tilde{\mathbf{k}}}\right]\tilde{\mathbf{k}}h\right) X(kh), \quad k \in \mathbb{Z}_+ \tag{48}$$

achieves the exponential stability in mean square of the ICLSS (1). In (48), $\left[\frac{k}{\tilde{\mathbf{k}}}\right]$ *denotes the integer part of the number* $\frac{k}{\tilde{\mathbf{k}}}$.

Proof. The equivalence $(i) \Leftrightarrow (vii)$ from Theorem 2 applied in the case of the resulting system (5) shows that under the considered assumptions, the ICLSS (1) is stabilizable in mean square by linear state feedback if and only if there exists a function $\tilde{Z}(\cdot) : [0, \tilde{\theta}] \to \mathcal{S}_n^+$ which has the properties $(\alpha') - (\gamma')$ from the statement of Theorem 2 (vii) and solves the following forward Lyapunov differential inequality with jumps:

$$\mathcal{L}(t)[\tilde{Z}(t)] - \dot{\tilde{Z}}(t) \preceq 0, \quad kh < t \leq (k+1)h \tag{49a}$$
$$\sum_{i=0}^{1}(\mathcal{A}_i(k) + \mathcal{B}_i(k)F(k))\tilde{Z}(kh)(\mathcal{A}_i(k) + \mathcal{B}_i(k)F(k))^\top - \tilde{Z}(kh^-) \prec 0,$$
$$0 \leq k \leq \tilde{\mathbf{k}} - 1. \tag{49b}$$

Taking $Z(\cdot) = \tilde{Z}(\cdot)|_{[0,\tilde{\theta}]}$ and using the Schur complement technique one obtains that (49) is equivalent to (47). The details are omitted. □

Remark 7. *To feasibility testing of the matrix inequalities (47) one may use a discretization procedure as those described in Section 3.4 of [3].*

5.3 The invariant case

According to Remark 1 (b), we deduce that a criterion for the stabilizability in mean square by linear state feedback of a time-invariant ICLSS (1) is provided by the following theorem.

Theorem 6. *If the ICLSS (1) is time invariant in the sense of the Definition 4, the following are equivalent:*

(i) *the ICLSS (1) is stabilizable in mean square by linear state feedback;*

(ii) *there exists a function* $Z(\cdot) : [0, h] \to \mathcal{S}_n^+$ *and the matrices* $Z_d \in \mathcal{S}_n$, $Z_d \succ 0$ *and* $K \in \mathbb{R}^{m\times n}$ *having the properties:*

- **(a)** $Z(\cdot)$ *is differentiable on* $(0, h)$ *and left continuous in* $t = h$*;*
- **(b)** $0 \prec c_1 I_n \preceq Z(t) \preceq c_2 I_n$*, for all* $t \in [0, h]$ *and* $Z(0) = Z(h)$*;*
- **(c)** $(Z(\cdot), Z_d, K)$ *solves the following forward Lyapunov differential inequality with jumps:*

$$\dot{Z}(t) - A_0 Z(t) - Z(t) A_0^\top - A_1 Z(t) A_1^\top \succeq 0, \quad 0 < t \le h,$$
$$Z(0^+) = Z_d \tag{50a}$$
$$\begin{pmatrix} -Z_d & * & * \\ (\mathcal{A}_0 Z(h) + \mathcal{B}_0 K)^\top & -Z(h) & 0 \\ (\mathcal{A}_1 Z(h) + \mathcal{B}_1 K)^\top & 0 & -Z(h) \end{pmatrix} \prec 0. \tag{50b}$$

If $(Z(\cdot), Z_d, K)$ *is a solution of (50) then the control law*

$$u(k) = K Z^{-1}(h) X(kh), \quad k \in \mathbb{Z}_+$$

achieves the exponential stability in mean square of the time invariant ICLSS (1).

Appendix A

Proof of Proposition 2. To prove that the implication $(i) \Rightarrow (ii)$ holds, we take an arbitrary $(t_0, H) \in \mathbb{R}_+ \times \mathcal{S}_n$ but fixed. We have to show that the solution $Y(\cdot; t_0, H)$ of the JMLDE (13) satisfies (16). Let us assume that $H \in \mathcal{S}_n^+$. This means that there exist orthogonal vectors $e_i \in \mathbb{R}^n$, $|e_i| = 1$, $1 \le i \le n$ and non-negative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that

$$H = \sum_{i=1}^{n} \lambda_i e_i e_i^\top.$$

Hence,

$$Y(t;t_0,H)=\sum_{i=1}^{n}\lambda_i Y(t;t_0,e_ie_i^\top)=\sum_{i=1}^{n}\lambda_i\mathbb{E}[x(t;t_0,e_i)x^\top(t;t_0,e_i)].$$

This allows us to deduce that

$$\begin{aligned}\|Y(t;t_0,H)\} &\le \sum_{i=1}^{n}\lambda_i\|\mathbb{E}[x(t;t_0,e_i)x^\top(t;t_0,e_i)]\| \\ &\le \sum_{i=1}^{n}\lambda_i\mathbb{E}[|x(t;t_0,e_i)|^2]\end{aligned}$$

Employing (4) written for $x(\cdot;t_0,e_i)$ we deduce that

$$\|Y(t;t_0,H)\|\le\beta\left(\sum_{i=1}^{n}\lambda_i\right)e^{-\alpha(t-t_0)},$$

for all $t\ge t_0$. On the other hand, we have that

$$\sum_{i=1}^{n}\lambda_i\le n\|H\|.$$

Thus we have obtained that

$$\|Y(t;t_0,H)\|\le\hat{\beta}e^{-\alpha(t-t_0)}\|H\|, \tag{51}$$

for all $t\ge t_0\ge 0$, $H\in\mathcal{S}_n^+$. So, we have shown that the solutions of the JMLDE (13) which are starting from $\mathcal{S}_n^+$ have an exponential decay. If $H\in\mathcal{S}_n$, there exist the positive semidefinite matrices H_1, H_2 such that

$$H=H_1-H_2.$$

and

$$\|H_i\|\le\|H\|, i=1,2.$$

Hence,

$$Y(t;t_0,H)=Y(t;t_0,H_1)-Y(t;t_0,H_2)$$

which yields

$$\|Y(t;t_0,H)\|\le\|Y(t;t_0,H_1)\|+\|Y(t;t_0,H_2)\|. \tag{52}$$

From (51) and (52) we may conclude that all solutions of JMLDE (13) have an exponential decay as in (16). Thus, we have proved that (ii) holds if (i) is true.

To prove the converse implication, let us remark that

$$\mathbb{E}[|x(t;t_0,x_0)|^2] = Tr[\mathbb{E}[x(t;t_0,x_0)x^\top(t;t_0,x_0)]] = Tr[Y(t;t_0,x_0x_0^\top)],$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$. $Y(\cdot;t_0,x_0x_0^\top)$ being the solution of IVP (12). Since

$$0 \leq Tr[Y(t;t_0,x_0x_0^\top)] \leq n\|Y(t;t_0,x_0x_0^\top)\|$$

we deduce that

$$\mathbb{E}[|x(t;t_0,x_0)|^2] \leq n\|Y(t;t_0,x_0x_0^\top)\|.$$

Thus, we may conclude that $\mathbb{E}[|x(t;t_0,x_0)|^2]$ is satisfying (4) if $Y(t;t_0,x_0x_0^\top)$ is satisfying (16). Thus the proof ends.

Appendix B

Proof of Proposition 3. First we prove the implication $(i) \Rightarrow (ii)$. Let $\{Z_k\}_{k\geq k_0}$ be an arbitrary solution of the DTLE (18)-(19). For $t \in (kh,(k+1)h]$, $k \geq k_0$, we set

$$Y(t) \triangleq \sum_{j=0}^{1} \mathbf{T}(t,kh)[\mathcal{A}_j(k)Z_k\mathcal{A}_j^\top(k)]. \tag{53}$$

One sees that $Y(\cdot)$ is differentiable on each interval $(kh,(k+1)h)$ and we have

$$\dot{Y}(t) = \mathcal{L}(t)[Y(t)], \quad kh < t < (k+1)h. \tag{54}$$

From (53) written for k replaced by $k-1$ together with (19) we deduce that $Y(kh) = Z_k$, for all $k \geq k_0$. On the other hand (53) yields

$$\lim_{\substack{t\to kh\\ t>kh}} Y(t) = \sum_{j=0}^{1} \mathcal{A}_j(k)Z_k\mathcal{A}_j^\top(k)$$

which is equivalent to

$$Y(kh^+) = \sum_{j=0}^{1} \mathcal{A}_j(k)Z_k\mathcal{A}_j^\top(k). \tag{55}$$

From (54) and (55) we infer that $Y(\cdot)$ defined via (53) is a solution of the JMLDE (13). Since (i) is true, we deduce that there exist two constants $\beta \geq 1$, $\alpha > 0$ with the property that

$$\|Y(t)\| \leq \beta e^{-\alpha(t-k_1 h)} \|Y(k_1 h)\|, \tag{56}$$

for all $t \geq k_1 h$, $k_1 \geq k_0$. For $t = kh$, (56) becomes:

$$\|Z_k\| \leq \beta e^{-\alpha(k-k_1)} \|Z_{k_1}\|.$$

So, we have obtained that the solution $\{Z_k\}_{k \geq k_0}$ of the DTLE (18)-(19) has an exponential decay as in (20) with $\gamma = \beta$ and $\delta = e^{-\alpha h} \in (0,1)$ which are not depending upon the solution $\{Z_k\}_{k \geq k_0}$. This means that $(i) \Rightarrow (ii)$ is true.

We prove that the opposite implication holds. Let $Y(t; t_0, H)$ be an arbitrary solution of the JMLDE (13). Let $k(t_0) \in \mathbb{Z}_+$ be defined as in (2). We set $k_0 \stackrel{\Delta}{=} k(t_0)$ and we take

$$Z_{k_0} \stackrel{\Delta}{=} Y(k_0 h; t_0, H), \tag{57}$$

and for $k \geq k_0$, we set

$$Z_k \stackrel{\Delta}{=} Y(kh; k_0 h, Z_{k_0}) = Y(kh; t_0, H).$$

Since $\{Z_k\}_{k \geq k_0}$ solves the DTLE (18)-(19) it follows from (ii) that it satisfies (20). This fact together with (57) leads to

$$\|Y(kh; t_0, H)\| \leq \gamma \delta^{k-k_0} \|Y(k_0 h; t_0, H)\|, \tag{58}$$

for all $k \geq k_0$. We define $\alpha \stackrel{\Delta}{=} -\frac{1}{h} \ln \delta$. In this way, (58) may be rewritten as:

$$\|Y(kh; t_0, H)\| \leq \gamma e^{-\alpha(k-k_0)h} \|Y(k_0 h; t_0, H)\}, \tag{59}$$

for all $k \geq k_0$. Employing Gronwall's Lemma one may show that

$$\|\mathbf{T}(t, kh)\| \leq e^{\mu h},$$

for all $kh \leq t \leq (k+1)h$, $k \in \mathbb{Z}_+$, where $\mu \stackrel{\Delta}{=} \sup_{t \in \mathbb{R}_+} \|\mathcal{L}(t)\| < \infty$. This allows us to deduce via (59) that

$$\|Y(t; t_0, H)\| \leq \beta_1 e^{-\alpha(t-k_0 h)} \|Y(k_0 h; t_0, H)\|, \tag{60}$$

for all $t \geq k_0 h$, $\beta_1 = \gamma \nu e^{(\alpha+\mu)h}$ where $\nu = \max\{1, \sum_{j=0}^{1} \sup_{k \in \mathbb{Z}_+} \|\mathcal{A}_j(k)\|^2\}$. Further, from (60) together with

$$\|Y(k_0 h; t_0, H)\| \leq e^{\mu h} \|H\|$$

we deduce that

$$\|Y(t; t_0, H)\| \leq \beta e^{-\alpha(t-t_0)} \|H\|,$$

for all $t \geq t_0 \geq 0$, where $\beta = \gamma \nu e^{2(\alpha+\mu)h}$. Thus we have shown that the JMLDE (13) is exponentially stable if the accompanying DTLE (18)-(19) is exponentially stable, too. So the proof is complete.

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