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THREE WEAK FORMULATIONS FOR AN OBSTACLE MODEL AND THEIR RELATIONSHIP*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

We consider an obstacle model mathematically described by means of a boundary value problem governed by PDE. Three possible variational formulations are highlighted. The first one is a variational inequality of the first kind and the other two are mixed variational formulations with Lagrange multipliers in dual spaces. After we discuss the solvability of the three variational formulations under consideration we focus on the relationship between them. Subsequently, we address the recovery of the formulation in terms of PDE starting from the mixed variational formulations.

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keywords: boundary value problem, nonlinear boundary condition, obstacle model, mixed variational formulations, Lagrange multipliers, weak solution.

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1 Introduction

Many real world phenomena can be mathematically modeled as unilateral contact problems. The first unilateral contact condition was introduced by Signorini, see [26], while the first rigorous mathematical approach for frictionless unilateral contact problems is due to Fichera, see [12]. The frictional unilateral contact problems were firstly addressed by Duvaut, see [9]; for a representative book devoted to inequalities in mechanics and physics we refer to [10]. More recently, a complete collection of results and techniques in the analysis of unilateral contact problems can be found in [11]. The variational formulations of the unilateral contact problems are strongly related to the theory of variational inequalities. In [28] Stampacchia coined the terminology *variational inequality*; for a representative book in the theory of variational inequalities we refer to [16]. The topic of the present paper is also related to the nice results described in [4, 29].

For " unfriendly data " it is hard or even impossible to obtain classical solutions for unilateral contact problems. Therefore, there is a high interest to efficiently approximate the weak solutions. And to do this, a convenient variational formulation of the model is required. Several variational formulations can be delivered for a unilateral contact model. This fact has been highlighted in the present paper by considering a simplified unilateral contact model, an obstacle model for membranes. We made this choice instead of the general 3D frictionless unilateral contact model in order to simplify the presentation. Thus, in the present work we address the following boundary value problem.

Problem 1 *Find $u : \bar{\Omega} \to \mathbb{R}$ such that*

$$-\xi \Delta u(\boldsymbol{x}) = f_0(\boldsymbol{x}) \qquad \text{in } \Omega, \tag{1}$$

$$u(\boldsymbol{x}) = 0 \qquad \text{on } \Gamma_1, \tag{2}$$

$$\xi \frac{\partial u}{\partial \nu}(\boldsymbol{x}) = f_2(\boldsymbol{x}) \qquad \text{on } \Gamma_2, \tag{3}$$

$$u(\boldsymbol{x}) \leq 0,\ \frac{\partial u}{\partial \nu}(\boldsymbol{x}) \leq 0,\ u(\boldsymbol{x}) \frac{\partial u}{\partial \nu}(\boldsymbol{x}) = 0 \qquad \text{on } \Gamma_3. \tag{4}$$

Herein and everywhere below in this paper $\Omega \subset \mathbb{R}^2$ denotes a bounded domain with smooth boundary $\partial\Omega$ partitioned in three measurable parts with positive surface measure, $\Gamma_i \subset \partial\Omega$, $i \in \{1, 2, 3\}$, f_0 and f_2 are given functions, ξ is a given positive parameter and as usual, $\boldsymbol{\nu}$ denotes the unit outward normal vector at $\partial\Omega$.

Problem 1 represents the classical formulation of a unilateral contact model for membranes, often called obstacle problem. Details on obstacle problems can be found in, e.g., [19].

The study of Problem 1 will be made under the following assumption.

$$\textbf{(H)}\quad \xi > 0,\ \ f_0 \in L^2(\Omega),\ \ f_2 \in L^2(\Gamma_2).$$

By a classical solution for Problem 1 we understand a function $u \in C^2(\bar{\Omega})$ that fulfills (1)-(4). Problem 1 can not have classical solutions if, for instance, f_0 or f_2 are discontinuous functions. Hence, under the assumption **(H)**, the solvability of Problem 1 will be governed by weak solutions i.e. solutions of variational formulations.

In this paper we attract attention to three possible variational formulations of Problem 1: the first formulation is a variational inequality of the first kind, while the other two are variational formulations via Lagrange multipliers in dual spaces. The primal variational formulation is not numerically convenient. This has motivated the interest in writing alternative variational formulations, such as the mixed variational formulations. It is worth emphasizing that the relationship we discuss in the present paper brings a contribution to the well-posedness of the mixed variational formulations. The first mixed variational formulation is interesting in its own right but at the same time it can be viewed as an auxiliary problem helping us to investigate the second mixed weak formulation (the third weak formulation) which is the most convenient variational formulation from the numerical point of view. Thus, the relationship we highlight is important especially for the well-posedness of the obstacle model having a variational formulation with Lagrange multipliers in the dual of a closed subspace of the Hilbert space $H^{1/2}(\partial\Omega)$.

We discuss on the weak solvability of our obstacle problem by using successively the aforementioned weak formulations, paying attention to the relationship between them. Then, we study the recovery of the classical formulation assuming enough smoothness. The present study is based on the saddle point theory in the convex analysis along with some techniques in the theory of variational inequalities. Moreover, we use elements of the operator theory, some properties involving the kernel and the polar being crucial. Important tools are related to the theory of Sobolev spaces including the trace theory. The present investigation leads to results which are important for the numerical analysis of the obstacle problems; as it is known, the weak solutions via mixed variational formulations with Lagrange multipliers in dual spaces can be approximated by means of modern numerical techniques like the primal-dual active set strategy, see, e.g., [15].

The present paper can be seen as a continuation of [20] by considering a different class of contact models; unlike [20], where a *bilateral frictional contact model* was considered, herein a particular *unilateral frictionless contact model* is under consideration. In [20], the first variational formulation was a variational inequality of the second kind involving a convex functional governed by a positive friction bound. In the present paper, the first formulation is a variational inequality of the first kind. As it is known, the variational inequality of the first kind can be seen as a variational inequality of the second kind governed by the indicator function of the set $\mathcal{K} = \{r \in \mathbb{R} \,|\, r \leq 0\}$. However, even that the primal variational formulations of both models rely on the theory of the variational inequalities of the second kind, the study in the present paper is not a particularization of the results obtained in [20]. The obstacle problem is a physical model completely different from the *bilateral frictional* contact model, the obstacle model being a *unilateral frictionless* contact problem. It is worth pointing that the mixed variational formulations related to the obstacle problem involves *unbounded* sets of Lagrange multipliers while the mixed variational formulations in [20] involves *bounded* sets of Lagrange multipliers. On the other hand, in order to establish the relationship between the three variational formulations of the obstacle problem, a crucial role is played by the inclusion $u \in \{v \in H^1(\Omega) \,|\, \gamma v = 0$ a.e. on Γ_1, $\gamma v \leq 0$ a.e. on $\Gamma_3\}$, where γ is the trace operator; and this is one of the difficulties of the present study. The recovering of the formulation in terms of PDE starting from the mixed variational formulations we propose, is another novelty trait of the present paper.

To end this introductory part, we recall below some useful mathematical tools in a framework governed by two Hilbert spaces $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$.

- a bilinear form $c : X \times Y \to \mathbb{R}$ is continuous (of rank M_c) if there exists $M_c > 0$ such that

$$|c(v,\mu)| \leq M_c \|v\|_X \|\mu\|_Y \quad \text{for all } v \in X,\ \mu \in Y;$$

- a bilinear form $c : X \times X \to \mathbb{R}$ is $X-$elliptic (of rank m_c) if there exists $m_c > 0$ such that

$$c(u,u) \geq m_c \|u\|_X^2 \qquad \text{for all } u \in X.$$

Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be two forms and let Λ be a subset such that the following hypotheses are fulfilled:

the form $a : X \times X \to \mathbb{R}$ is symmetric bilinear continuous (of rank M_a) and $X-$elliptic (of rank m_a); (5)

the form $b : X \times Y \to R$ is bilinear continuous (of rank M_b), (6)

and, in addition,

$$\exists\, \alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha; \tag{7}$$

Λ is a closed convex subset of Y that contains 0_Y. (8)

Let us consider the following mixed variational problem: given $f \in X$, find

$$(u, \lambda) \in X \times \Lambda, \quad \text{such that} \quad a(u, v) + b(v, \lambda) \;=\; (f, v)_X, \tag{9}$$
$$b(u, \mu - \lambda) \;\leq\; 0, \tag{10}$$

for all $v \in X$, $\mu \in \Lambda$.

In the study of this problem, the following existence and uniqueness result holds true.

Theorem 1 *Assume* (5)–(8). *Then, the problem (9)-(10) has a unique solution.*

For a proof, see, e.g., Corollary 2 in [8]. The proof is based on the saddle point theory; see, e.g., [6, 7].

Let $B : X \to Y'$ be the linear and continuous operator defined as follows: for each $v \in X$,

$$\langle Bv, \lambda \rangle_{Y',Y} = b(v, \lambda) \quad \text{for all } \lambda \in Y,$$

and let $B^t : Y \to X'$ be the linear and continuous operator defined as follows: for each $\lambda \in Y$,

$$\langle B^t \lambda, v \rangle_{X',X} = \langle Bv, \lambda \rangle_{Y',Y} \quad \text{for all } v \in X.$$

We observe that

$$\langle B^t \lambda, v \rangle_{X',X} = \langle Bv, \lambda \rangle_{Y',Y} = b(v, \lambda) \quad \text{for all } v \in X, \lambda \in Y; \tag{11}$$

see [3] (pages 210-213) and [5](page 131) for details.

Let
$$KerB = \{v \in X \mid Bv = 0\}$$
and let $(KerB)^0$ be the polar of $KerB$, i.e.,
$$(KerB)^0 = \{l \in X' \mid \langle l, v\rangle_{X',X} = 0 \text{ for all } v \in KerB\}.$$
Keeping in mind (6)-(7), according to, e.g., [3](see 4.1.61-4.1.62), the following equality holds true:
$$ImB^t = (KerB)^0.$$
It is worth to underline that, keeping in mind (11), we are led to
$$\begin{aligned} KerB &= \{v \in X \mid b(v, \lambda) = 0 \text{ for all } \lambda \in Y\} \\ &= \{v \in X \mid \langle B^t\lambda, v\rangle_{X',X} = 0 \text{ for all } \lambda \in Y\}; \end{aligned}$$
see, e.g., (4.1.52) in [3]. Furthermore,
$$B^t : Y \to (KerB)^0 \subset X' \text{ is an isomorphism;} \tag{12}$$
see, e.g., Theorem 3.6 page 125 and Lemma 4.2 page 131 in [5].

The rest of the paper has the following structure. Section 2 describes the functional setting. Section 3 indicates three alternative variational formulations, a variational inequality of the first kind and two mixed variational formulations, along with their analysis. Section 4 is devoted to the recovery of the classical formulation of the obstacle model assuming enough smoothness, starting from the mixed variational formulations.

2 Preliminaries and functional setting

In this section we describe the functional setting and recall some fundamental results related on it.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth enough boundary. We use standard notation for $L^2(\Omega)$, $L^2(\Omega)^2$, $L^2(\partial\Omega)$, $H^1(\Omega)$, see, e.g., [1, 2, 13, 17, 18]. According to the trace theorem, see, e.g., [23] page 34, there exists a unique linear continuous operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ such that

a) $\gamma\, u = u|_{\partial\Omega}$ if $u \in H^1(\Omega) \cap C(\overline{\Omega})$;

b) $\|\gamma\, u\|_{L^2(\partial\Omega)} \le c_{tr}\|u\|_{H^1(\Omega)}$ with $c_{tr} = c_{tr}(\Omega) > 0$;

c) $\gamma(H^1(\Omega)) = H^{1/2}(\partial\Omega)$;

d) $\gamma : H^1(\Omega) \to L^r(\partial\Omega)$ is compact for each $r \geq 1$.

The function $\gamma\, u$ is called the trace of the scalar function u on $\partial\Omega$. The trace operator is neither an injection, nor a surjection from $H^1(\Omega)$ to $L^2(\partial\Omega)$. The space $H^{1/2}(\partial\Omega)$ is a Hilbert space endowed with the inner product

$$(v,w)_{H^{1/2}(\partial\Omega)} = \\ (v,w)_{L^2(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{(v(\boldsymbol{x})-v(\boldsymbol{y}))(w(\boldsymbol{x})-w(\boldsymbol{y}))}{\|\boldsymbol{x}-\boldsymbol{y}\|^2} d\Gamma(\boldsymbol{x})d\Gamma(\boldsymbol{y})$$

and the corresponding Sobolev Slobodeckij norm

$$\|v\|_{H^{1/2}(\partial\Omega)} = \left(\|v\|^2_{L^2(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{(v(\boldsymbol{x})-v(\boldsymbol{y}))^2}{\|\boldsymbol{x}-\boldsymbol{y}\|^2} d\Gamma(\boldsymbol{x})d\Gamma(\boldsymbol{y}) \right)^{1/2},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. We observe that $H^{1/2}(\partial\Omega)$ is continuously embedded in $L^2(\partial\Omega)$.

Let $\mathcal{Z}$ be the right inverse of the operator γ,

$$\mathcal{Z} : H^{1/2}(\partial\Omega) \to H^1(\Omega);$$

see, e.g., [24]. As it is known, $\mathcal{Z}$ is a linear and continuous operator. Moreover,

$$\gamma(\mathcal{Z}(\zeta)) = \zeta \quad \text{for all } \zeta \in H^{1/2}(\partial\Omega).$$

Let us consider the following closed subspace of $H^1(\Omega)$,

$$X = \{v \in H^1(\Omega) \mid \gamma v = 0 \text{ a.e. on } \Gamma_1\}, \tag{13}$$

where Γ_1 is a relatively open subset of $\partial\Omega$ whose surface measure is positive. According to [25], there is $c_P = c_P(\Omega,\Gamma_1) > 0$ such that

$$\|v\|_{H^1(\Omega)} \leq c_P \|\nabla v\|_{L^2(\Omega)^2} \;\; \text{for all } v \in X. \tag{14}$$

Based on (14), the space X becomes a Hilbert space endowed with the inner product

$$(u,v)_X = (\nabla u, \nabla v)_{L^2(\Omega)^2}.$$

Notice that

$$\|\gamma v\|_{H^{1/2}(\partial\Omega)} \leq c\|v\|_X \quad \text{for all } v \in X, \tag{15}$$

where $c = c_{tr}(\Omega) c_P(\Omega,\Gamma_1)$.

Notice that

$$\mathcal{Z}(\zeta) \in X \quad \text{for all } \zeta \in \gamma(X).$$

Afterward, we consider

$$M = \gamma(X) \tag{16}$$

where

$$\gamma(X) = \{\widetilde{v} \in H^{1/2}(\partial\Omega) |\ \text{there exists}\ v \in X\ \text{such that}\ \widetilde{v} = \gamma\, v \text{ a.e. on } \partial\Omega\}.$$

According to [22], M is a Hilbert space. Let us denote by Y the dual of the space M,

$$Y = M'. \tag{17}$$

Notice that $\langle \cdot, \cdot \rangle_{Y,M}$ stands for the duality pairing between Y and M.

Following [22] we introduce the operator

$$R : \gamma(X) \to X \quad R(\zeta) = \mathcal{Z}(\zeta). \tag{18}$$

The operator R is a linear and continuous operator. Thus, there are some $c_R > 0$ such that

$$\|R(\gamma v)\|_X \leq c_R \|\gamma v\|_{H^{1/2}(\partial\Omega)} \quad \text{for all } v \in X. \tag{19}$$

We also need a bilinear continuous form $b : X \times Y \to \mathbb{R}$ defined as follows,

$$b(v, \mu) = \langle \mu, \gamma v \rangle_{Y,M} \quad \text{for all } v \in X,\ \mu \in Y. \tag{20}$$

The form $b(\cdot, \cdot)$ is bilinear continuous of rank $c = c_{tr}(\Omega) c_P(\Omega, \Gamma_1)$. Moreover, the form $b(\cdot, \cdot)$ satisfies the inf-sup property

$$\exists\, \alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$$

Indeed,

$$\begin{aligned}
\|\mu\|_Y &= \sup_{w \in M,\, w \neq 0_M} \frac{< \mu, w >}{\|w\|_M} \\
&\leq \sup_{z \in X,\, \gamma z \neq 0_M} \frac{b(z, \mu)}{\|\gamma z\|_{H^{1/2}(\partial\Omega)}} \\
&= \sup_{z \in X,\, R(\gamma z) \neq 0_X} \frac{b(R(\gamma z), \mu)}{\|\gamma z\|_{H^{1/2}(\partial\Omega)}} \\
&\leq c_R \sup_{z \in X,\, R(\gamma z) \neq 0_X} \frac{b(R(\gamma z), \mu)}{\|R(\gamma z)\|_X} \\
&\leq c_R \sup_{v \in X,\, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X}.
\end{aligned}$$

Herein R is the operator introduced in (18). Let us take $\alpha = \frac{1}{c_R}$, c_R being the constant in (19). As a result,

$$\alpha\|\mu\|_Y \quad \leq \sup_{v\in X,\, v\neq 0_X} \frac{b(v,\mu)}{\|v\|_X}.$$

And from this, the inf-sup property of the form $b(\cdot,\cdot)$ is a straightforward consequence. Such a bilinear form was used in some previous works, see, e.g., [21].

Let $B : X \to Y'$ be the linear and continuous operator such that,

$$\langle Bv, \lambda\rangle_{Y',Y} = b(v,\lambda) = \langle \lambda, \gamma v\rangle_{Y,M} \quad \text{for all } v \in X, \lambda \in Y. \tag{21}$$

According to, e.g., Theorem 3.6 page 125 and Lemma 4.2 page 131 in [5], as well as Lemma 1 in [20],

$$B^t : Y \to (Ker\gamma)^0 \subset X' \text{ is an isomorphism.} \tag{22}$$

In (22), $(Ker\gamma)^0$ denotes the polar of $Ker\gamma$, i.e.,

$$(Ker\gamma)^0 = \{l \in X' \,|\, \langle l, v\rangle_{X',X} = 0 \text{ for all } v \in Ker\gamma\}.$$

Everywhere below in this paper

$$a : X \times X \to \mathbb{R} \quad a(u,v) = \xi(u,v)_X; \tag{23}$$

$$(f,v)_X = \int_\Omega f_0\, v\, dx + \int_{\Gamma_2} f_2\, \gamma v\, d\Gamma; \tag{24}$$

$$K = \{v \in X \,|\, \gamma v \leq 0 \text{ a.e on } \Gamma_3\}, \tag{25}$$

where X is the space defined in (13) and f_0, f_2 and ξ are the given data, see **(H)**. We point out that the form $a(\cdot,\cdot)$ defined in (23) is symmetric, bilinear continuous of rank ξ and $X-$elliptic of rank ξ.

3 Weak formulations and their analysis

In this section we present three variational formulations of Problem 1 along with their analysis.

3.1 First variational formulation

We firstly highlight a variational formulation of Problem 1 in terms of variational inequalities of the first kind.

Problem 2 *Find $u_0 \in K$ such that*

$$a(u_0, v - u_0) \geq (f, v - u_0)_X \quad \text{for all } v \in K. \tag{26}$$

By applying Corollary 3.4 in [27] we are led to the following existence and uniqueness result.

Theorem 2 *Admit (**H**). Then, Problem 2 has a unique solution $u_0 \in K$.*

3.2 The second variational formulation

Let us denote by X the space given by (13), by X' its dual and by $\langle \cdot, \cdot \rangle_{X',X}$ the duality pairing.

Let $\bar{u}$ be a regular enough function which verifies Problem 1. By introducing the Lagrange multiplier $\bar{\lambda} \in X'$,

$$\langle \bar{\lambda}, v \rangle_{X',X} = -\int_{\Gamma_3} \xi \frac{\partial u}{\partial \nu} \gamma v \, d\Gamma \quad \text{for all } v \in X,$$

we can write the following mixed variational formulation.

Problem 3 *Find $\bar{u} \in X$ and $\bar{\lambda} \in \bar{\Lambda} \subset X'$ such that*

$$\begin{aligned} a(\bar{u}, v) + \langle \bar{\lambda}, v \rangle_{X',X} &= (f, v)_X && \text{for all } v \in X, && (27)\\ \langle \bar{\mu} - \bar{\lambda}, \bar{u} \rangle_{X',X} &\leq 0 && \text{for all } \bar{\mu} \in \bar{\Lambda}, && (28) \end{aligned}$$

where a and f are given by (23) and (24), respectively. Furthermore, the set of the Lagrange multipliers is

$$\bar{\Lambda} = \{\bar{\mu} \in X' \mid \langle \bar{\mu}, v \rangle_{X',X} \leq 0 \text{ for all } v \in K\}, \tag{29}$$

where K is given by (25).

Theorem 3 *Admit (**H**). Then, Problem 3 has a unique solution $(\bar{u}, \bar{\lambda}) \in X \times \bar{\Lambda}$.*

Proof. By standard arguments we deduce that the form a defined in (23) fulfills (5) with $M_a = m_a = \xi$.

Let us introduce

$$\bar{b} : X \times X' \to \mathbb{R} \quad \bar{b}(v, \mu) = \langle \mu, v \rangle_{X',X}.$$

We observe that this form fulfills (6) with $M_b = 1$. In addition, as

$$\|\mu\|_{X'} \quad = \quad \sup_{v \in X, v \neq 0_X} \frac{\langle \mu, v \rangle_{X',X}}{\|v\|_X},$$

we observe that (7) is fulfilled with $\alpha = 1$. And finally, it is obvious that $\bar{\Lambda}$ is a closed convex subset of X' containing $0_{X'}$. Thus, all hypotheses of Theorem 1 are fulfilled. As a consequence, we get the conclusion by applying Theorem 1. ■

Subsequently, we deliver a characterization of the solution of Problem 3 by means of the unique solution of Problem 2, $u_0 \in K$.

Proposition 1 *Let $(\bar{u}, \bar{\lambda}) \in X \times \bar{\Lambda}$ be the unique pair assured by Theorem 3, let $u_0 \in K$ be the unique solution of Problem 2 and let $\lambda_0 \in X'$ be defined as follows*

$$\langle \lambda_0, v \rangle_{X',X} = (f, v)_X - a(u_0, v). \tag{30}$$

Then, $\bar{u} = u_0$ and $\bar{\lambda} = \lambda_0$.

Proof. By (30) we immediately observe that the pair $(u_0, \lambda_0) \in K \times X'$ verifies (27). On the other hand, keeping in mind (25), as u_0 is the unique solution of Problem 2, setting in (26) $v = 0_X \in K$ and $v = 2\, u_0 \in K$, we obtain

$$a(u_0, u_0) = (f, u_0)_X. \tag{31}$$

Using now (26) and (31), we get

$$a(u_0, v) \geq (f, v)_X \quad \text{ for all } v \in K. \tag{32}$$

By (32) and (30) we obtain

$$\langle \lambda_0, v \rangle_{X',X} \leq 0 \text{ for all } v \in K,$$

and so,

$$\lambda_0 \in \bar{\Lambda}.$$

In particular, due to (31) and (30),

$$\langle \lambda_0, u_0 \rangle_{X',X} = 0. \tag{33}$$

According to (29), since $u_0 \in K$,

$$\langle \bar{\mu}, u_0 \rangle_{X',X} \leq 0 \quad \text{for all } \bar{\mu} \in \bar{\Lambda}. \tag{34}$$

Therefore, by (33) and (34), we observe that $(u_0, \lambda_0) \in K \times \bar{\Lambda}$ verifies also (28). As $(\bar{u}, \bar{\lambda}) \in X \times \bar{\Lambda}$ is the unique pair verifying (27) and (28), we immediately get the conclusion. ■

Corollary 1 *Admit (**H**). Then, Problem 3 has a unique solution $(\bar{u}, \bar{\lambda}) \in K \times \bar{\Lambda}$.*

Proof. The proof of this corollary is a straightforward consequence of Theorem 3 and Proposition 1. ■

Remark 1 *The unique solution of Problem 2 coincides with the first component of the pair solution of Problem 3.*

Problem 3 is interesting in its own but at the same time it can be viewed as an auxiliary problem helping us to investigate the third weak formulation which is the most convenient variational formulation from the numerical point of view.

3.3 The third variational formulation

Let X, K, M, Y, a, b, f be given in Section 2, see (13), (25), (16), (17), (23), (20) and (24).

Assuming that u is a regular enough function which verifies Problem 1 and introducing a Lagrange multiplier $\lambda \in Y$ as follows,

$$\langle \lambda, \widetilde{v} \rangle_{Y,M} = -\int_{\Gamma_3} \xi \frac{\partial u}{\partial \nu} \widetilde{v} \, d\Gamma \quad \text{for all } \widetilde{v} \in M,$$

we are driven to the following mixed variational formulation.

Problem 4 *Find $u \in X$ and $\lambda \in \Lambda \subset Y$ such that*

$$a(u, v) + b(v, \lambda) = (f, v)_X \quad \textit{for all } v \in X, \tag{35}$$

$$b(u, \mu - \lambda) \leq 0 \quad \textit{for all } \mu \in \Lambda. \tag{36}$$

Herein

$$\Lambda = \{\mu \in Y \mid \langle \mu, \gamma v\rangle_{Y,M} \leq 0 \text{ for all } v \in K\}. \tag{37}$$

Theorem 4 *Admit (**H**). Then, Problem 4 has a unique solution* $(u, \lambda) \in X \times \Lambda$.

Proof. As it is already known, the form a defined in (23) fulfills (5) with $M_a = m_a = \xi$. On the other hand, the form $b(\cdot,\cdot)$ defined in (20) fulfills (6) with $M_b = c$, see (15) and (7) with $\alpha = \frac{1}{c_R}$. Finally, by standard argument we can justify that Λ is a closed convex subset of Y containing 0_Y. Hence, all hypotheses of Theorem 1 are fulfilled. As a result, we can apply Theorem 1. ■

Below, we focus on a characterization of the unique solution of Problem 4 by means of the unique solution of Problem 2. To this end in view, we firstly prove an auxiliary result.

Lemma 1 *Let* $(\bar{u}, \bar{\lambda}) \in K \times \bar{\Lambda}$ *be the unique solution of Problem 3. Then*

$$\bar{\lambda} \in (Ker\gamma)^0. \tag{38}$$

Proof. Obviously, $Ker\,\gamma \subset K$. Let $v \in Ker\,\gamma$. As $\bar{\lambda} \in \bar{\Lambda}$ we obtain

$$\langle \bar{\lambda}, v\rangle_{X',X} \leq 0.$$

Since $Ker\gamma$ is a linear subspace, we also have $-v \in Ker\gamma$. Thus,

$$-\langle \bar{\lambda}, v\rangle_{X',X} \leq 0.$$

Consequently,

$$\langle \bar{\lambda}, v\rangle_{X',X} = 0 \text{ for all } v \in Ker\gamma,$$

i.e. (38) holds true. ■

According to (22), there exists a unique $\widetilde{\lambda} \in Y$ such that

$$B^t\widetilde{\lambda} = \bar{\lambda}. \tag{39}$$

Proposition 2 *Let* $(u, \lambda) \in X \times \Lambda$ *be the unique pair assured by Theorem 4, let* $(\bar{u}, \bar{\lambda}) \in K \times \bar{\Lambda}$ *be the unique solution of Problem 3, and let* $\widetilde{\lambda}$ *be the unique element in* Y *such that (39) holds true. Then,* $u = \bar{u}$ *and* $\lambda = \widetilde{\lambda}$.

Proof. Firstly, keeping in mind (37), (29) and (21), we easily observe that $\widetilde{\lambda}$ defined by (39) is an element of Λ. Next, by (21) and (27) we obtain that $(\bar{u}, \widetilde{\lambda}) \in K \times \Lambda$ verifies the first line of Problem 4.

On the other hand, for all $\mu \in \Lambda$, since $\bar{u} \in K$, we have

$$\langle \mu, \gamma \bar{u} \rangle_{Y,M} \leq 0. \tag{40}$$

Moreover, due to (33) (with $u_0 = \bar{u}$ and $\lambda_0 = \bar{\lambda}$) we can write:

$$\langle \widetilde{\lambda}, \gamma \bar{u} \rangle_{Y,M} = \langle B^t \widetilde{\lambda}, \bar{u} \rangle_{X',X} = \langle \bar{\lambda}, \bar{u} \rangle_{X',X} = 0. \tag{41}$$

By (40) and (41) we observe that $(\bar{u}, \widetilde{\lambda})$ verifies the second line of Problem 4. As Problem 4 has a unique solution $(u, \lambda) \in X \times \Lambda$ we get the conclusion. ∎

Corollary 2 *Admit (**H**). Then, Problem 4 has a unique solution $(u, \lambda) \in K \times \Lambda$.*

Proof. This corollary is a straightforward consequence of Theorem 4 and Proposition 2. ∎

Remark 2 *According to Proposition 2, the unique solution of Problem 2 coincides with the first component of the unique solution of Problem 4. Actually, we have $u = \bar{u} = u_0 \in K$.*

Remark 3 *If $(u, \lambda) \in X \times \Lambda$ is the unique pair assured by Theorem 4, then we immediately obtain that*

$$a(u, v - u) \geq (f, v - u)_X \quad \text{for all } v \in K. \tag{42}$$

Indeed, by (35) we can write

$$a(u, v - u) + \langle \lambda, \gamma v \rangle_{Y,M} - \langle \lambda, \gamma u \rangle_{Y,M} = (f, v - u)_X \quad \text{for all } v \in X. \tag{43}$$

Because $\lambda \in \Lambda$,

$$\langle \lambda, \gamma v \rangle_{Y,M} \leq 0 \quad \text{for all } v \in K.$$

On the other hand, using (36), setting successively $\mu = 0_Y$ and $\mu = 2\lambda$, we obtain

$$\langle \lambda, \gamma u \rangle_{Y,M} = 0. \tag{44}$$

By (43)-(44) we obtain the inequality (42). However, this procedure doesn't allow us to say that the first component of the pair solution assured by Theorem 4 is the unique solution of Problem 2 because, by this procedure, we can not justify that $u \in K$.

4 The recovery of the formulation in terms of PDE

Let $(u,\lambda) \in K \times \Lambda$ be the unique solution of Problem 4 and let $(u,\bar{\lambda}) \in K \times \bar{\Lambda}$ be the unique solution of Problem 3. Assuming enough smoothness for the data and for the weak solutions, by a classical procedure, one can obtain that u verifies the pointwise relations (1)-(3); see, e.g., Section 5 in [20] and the references therein. Herein, we focus on getting the unilateral conditions (4).

As $u \in K$ then

$$u(\boldsymbol{x}) \leq 0 \text{ on } \Gamma_3, \tag{45}$$

which represents the first part of the unilateral conditions (4). Notice that everywhere in this section $u = \gamma u$ on Γ_3 assuming that the weak solution is smooth enough.

Keeping in mind (23), (24), using the first Green's formula and (1)-(3) we obtain,

$$\langle \bar{\lambda}, v\rangle_{X',X} = \langle \lambda, \gamma v\rangle_{Y,M} = -\int_{\Gamma_3} \xi \frac{\partial u}{\partial \nu}(\boldsymbol{x})\gamma v(\boldsymbol{x})\, d\Gamma \quad \text{for all } v \in K. \tag{46}$$

Since $\lambda \in \Lambda$ and $\bar{\lambda} \in \bar{\Lambda}$, then,

$$\langle \lambda, \gamma v\rangle_{Y,M} \leq 0 \text{ and } \langle \bar{\lambda}, v\rangle_{X',X} \leq 0 \quad \text{for all } v \in K. \tag{47}$$

By (47) and (46) it results

$$-\int_{\Gamma_3} \xi \frac{\partial u}{\partial \nu}(\boldsymbol{x})\gamma v(\boldsymbol{x})\, d\Gamma \leq 0 \quad \text{for all } v \in K.$$

And from this, by classical arguments of mathematical analysis, we deduce that

$$\xi \frac{\partial u}{\partial \nu}(\boldsymbol{x}) \leq 0 \text{ on } \Gamma_3 \tag{48}$$

which represents the second part of the unilateral conditions (4).

Let us establish the last part of (4). According to (41) with $\bar{u} = u$ and $\widetilde{\lambda} = \lambda$ we have to write the relations

$$\langle \lambda, \gamma u\rangle_{Y,M} = \langle \bar{\lambda}, u\rangle_{X',X} = 0. \tag{49}$$

By (46) and (49) we obtain that

$$\int_{\Gamma_3} \xi \frac{\partial u}{\partial \nu}(\boldsymbol{x}) u(\boldsymbol{x})\, d\Gamma = 0. \tag{50}$$

Taking into account (45) and (48) we deduce that

$$\xi \frac{\partial u}{\partial \nu}(\boldsymbol{x}) u(\boldsymbol{x}) \geq 0 \text{ on } \Gamma_3. \tag{51}$$

Therefore, (50) and (51) yield

$$\left\| \xi \frac{\partial u}{\partial \nu} u \right\|_{L^1(\Gamma_3)} = 0.$$

This leads us immediately to

$$\xi u(\boldsymbol{x}) \frac{\partial u}{\partial \nu}(\boldsymbol{x}) = 0 \text{ on } \Gamma_3. \tag{52}$$

By (45), (48) and (52) we conclude that the unilateral condition (4) is recovered.

5 Conclusions and final comments

In the present paper we address an obstacle model that is mathematically described by means of a boundary value problem governed by PDE, focusing on the relationship between three possible weak formulations: a variational inequality of the first kind (the primal variational formulation) and two mixed variational formulations with Lagrange multipliers in dual spaces. The relationship we highlight is important especially for the well-posedness of the obstacle model having a variational formulation with Lagrange multipliers in the dual of a closed subset of the Hilbert space $H^{1/2}(\partial\Omega)$. Such a variational formulation is very convenient for the numerical point of view allowing the use of modern numerical techniques like the primal-dual active set strategy.

The primal variational formulation is not convenient for the numerical point of view but, it has the advantage that its unique solution u_0 belongs to the set $\{v \in H^1(\Omega) \,|\, \gamma v = 0 \text{ a.e. on } \Gamma_1,\, \gamma v \leq 0 \text{ a.e. on } \Gamma_3\}$, this inclusion fitting well with (2) and (4); see, e.g., [14] pages 168-170, for the primal variational formulation of a 3D unilateral contact problem. The study we perform in the present paper allows us to prove that $u_0 = \bar{u} = u$. Hence $\bar{u}$ and u belongs to the set $\{v \in H^1(\Omega) \,|\, \gamma v = 0 \text{ a.e. on } \Gamma_1,\, \gamma v \leq 0 \text{ a.e. on } \Gamma_3\}$ as well. It is worth to underline that $\gamma u = \gamma \bar{u} = \gamma u_0 \leq 0$ a.e. on Γ_3. Moreover, assuming enough smoothness, we can also discuss the recovery of the strong formulation in terms of PDE starting from the mixed variational formulations.

The results of the present paper along with the results in the paper [20] can be seen as first steps in order to reinforce the well-posedness of a class of contact models with mixed variational formulations. Both works are based on "experimental models": in [20] it was used a simplified bilateral frictional contact model while in the present work, it was used a simplified frictionless unilateral contact. In a future work it could be interesting to explore how these results can be extended to general 3D contact models and how these results can be improved delivering a boot-strap procedure.

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