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ON AN IMPORTANT OBSERVATION REGARDING SOME HYDROMAGNETIC MOTIONS OF MAXWELL FLUIDS AND ITS APPLICATIONS*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

The problem of exact solutions for isothermal motions of non-Newtonian fluids is of interest yet and a new way to get them is welcome. In this work an important observation regarding the governing equations corresponding to some isothermal hydromagnetic unidirectional motions of incompressible Maxwell fluids is brought to light. It allows us to easily determine exact solutions for motions with shear stress or velocity on the boundary when similar solutions for motions with velocity, respectively shear stress on the boundary are know. To exemplify, the solutions of some hydromagnetic motion problems of Maxwell fluids with velocity on the boundary are used to generate exact steady state solutions for similar motions of same fluids with shear stress on the boundary. These solutions are very important for the experimental researchers who want to know the required time to reach the steady state.

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1 Introduction

The motion of a fluid over an oscillating flat plate, as well as the motion between parallel plates, is not only of fundamental theoretical interest but it also appears in many applied problems [1]. It is called Stokes second problem by Schlichting [2]. The same motion is also termed as Stokes or Rayleigh problem in the existing literature. The fluid motion between parallel plates is also termed as the modified Stokes second problem by Rajagopal et al. [3] if one of plates oscillates. They are some of the most important motion problems near moving bodies having multiple applications in engineering and science in general. Both motions have been extensively studied in the literature and the obtained solutions are important both for theoreticians and experimental researchers. Although the numerical integration of governing equations can be realized by computers, the accuracy of results can be established by a comparison with an exact solution.

The first exact starting solutions of the second problem of Stokes for incompressible Newtonian fluids seem to be those of Erdogan [1]. New equivalent exact solutions for the same problem and their extension to incompressible Maxwell fluids have been established by Corina Fetecau et al. [4], respectively [5]. On the other hand, the interaction between an electrical conducting fluid and the magnetic field produces important effects with many applications in physics, chemistry, engineering, horticulture and hydrology. In addition, the hydromagnetic motions of fluids have multiple applications in polymer technology, petroleum industry, nuclear reactors and so on. More recent results regarding hydromagnetic motions of incompressible Newtonian fluids have been obtained by Kiema et al. [6], Onyango et al. [7] and Dash and Ojha [8]. However, the first general solutions for such motions of same fluids over an infinite plate or between infinite parallel plates have been obtained by Fetecau et al. [9], respectively Fetecau and Narahary [10] using an important remark concerning the governing equations of velocity and shear stress.

The main purpose of this work is to show that the respective remark is also valid for the same isothermal hydromagnetic unidirectional motions of the incompressible Maxwell fluids. This observation, with direct applications in obtaining new exact solutions for fluid motions, says that the governing equations of the velocity and shear stress fields corresponding to some isothermal hydromagnetic unidirectional flows of incompressible Maxwell fluids are identical as form. As an immediate consequence, it results that exact solutions for some hydromagnetic motions of incompressible Maxwell fluids with velocity or shear stress on the boundary can be easily obtained if exact solutions for similar motions of same fluids with shear stress, respectively velocity on the boundary are known. In order to bring to light the advantages of this important observation, some hydromagnetic motions of Maxwell fluids with velocity on the boundary are considered and their solutions are used to determine exact solutions for similar motions of same fluids with shear stress on the boundary.

2 Constitutive and governing equations

The constitutive equations of incompressible upper-convected Maxwell fluids (IUCMFs) are given by the following relations [11]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \left(1 + \lambda\,\frac{\delta}{\delta t}\right)\mathbf{S} = \mu\mathbf{A}, \tag{1}$$

in which $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ is the undetermined spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the constitutively determined extra-stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor ($\mathbf{L}$ being the gradient of the velocity vector $\mathbf{v}$), μ is the fluid viscosity and λ is its relaxation time. The upper-convected derivative $\delta/\delta t$ is defined by the relation

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^T \quad \text{with} \quad \frac{d\mathbf{S}}{dt} = \frac{\partial \mathbf{S}}{\partial t} + [\text{grad } \mathbf{S}]\mathbf{v}. \tag{2}$$

In the following we shall consider isothermal unidirectional motions of IUCMFs whose velocity field in a suitable Cartesian coordinate system x, y and z has the form

$$\mathbf{v} = \mathbf{v}(y,t) = u(y,t)\mathbf{e}_x, \tag{3}$$

where $\mathbf{e}_x$ is the unit vector along the x-direction. If the fluid is at rest at the initial moment $t = 0$, it results that

$$\mathbf{v}(y,0) = \left.\frac{\partial \mathbf{v}(y,t)}{\partial t}\right|_{t=0} = \mathbf{0}, \ \mathbf{S}(y,0) = \mathbf{0}. \tag{4}$$

We also assume that the extra-stress tensor $\mathbf{S}$, as well as the velocity vector $\mathbf{v}$, is also a function of y and t only.

Substituting $\mathbf{v}(y,t)$ from Eq. (3) in $(1)_2$ and bearing in mind the last condition from Eqs. (4), it is easy to prove that the components S_{yy}, S_{yz}, S_{zz} and S_{zx} of the extra-stress tensor $\mathbf{S}$ are zero. The other two non-trivial components $\sigma(y,t) = S_{xx}(y,t)$ and $\tau(y,t) = S_{xy}(y,t)$ of $\mathbf{S}$ have to satisfy the following partial differential equations

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma(y,t) &= 2\lambda\tau(y,t)\,\frac{\partial u(y,t)}{\partial y},\\ \left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t) &= \mu\,\frac{\partial u(y,t)}{\partial y}.\end{aligned} \tag{5}$$

The balance of linear momentum for isothermal hydrodynamic unsteady fluid motions is given by the following relation [12]

$$\rho\,\frac{d\mathbf{v}}{dt} = \operatorname{div}\,\mathbf{T} + \rho\mathbf{b} + \mathbf{J}\times\mathbf{B}, \tag{6}$$

where ρ is the fluid density, $\mathbf{b}$ is the body force and $\mathbf{J}\times\mathbf{B}$ represents the Lorentz force due to the interaction between the current density $\mathbf{J}$ and the magnetic induction $\mathbf{B}$. We also assume that the fluid is finitely conducting so that the Joule heat due to the presence of magnetic field is negligible. Furthermore, there exists no surplus electric charge distribution present in the fluid and the magnetic Reynolds number is assumed to be small enough. Consequently, the induced magnetic field can be neglected and [12]

$$\mathbf{J}\times\mathbf{B} = -\tilde{\sigma}B^2\mathbf{v}(y,t), \tag{7}$$

where $\tilde{\sigma}$ is the electrical conductivity of the fluid and B represents the magnitude of the applied magnetic field.

In these conditions, in the case of conservative body forces and in the absence of a pressure gradient in the flow direction, Eqs. (3), (6) and (7) lead to the relation

$$\rho\,\frac{\partial u(y,t)}{\partial t} = \frac{\partial\tau(y,t)}{\partial y} - \tilde{\sigma}B^2u(y,t). \tag{8}$$

Eliminating $\tau(y,t)$ between Eqs. $(5)_2$ and (8), one obtains the governing equation

$$\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} = \mu\,\frac{\partial^2 u(y,t)}{\partial y^2} - \tilde{\sigma}B^2\left(1+\lambda\frac{\partial}{\partial t}\right)u(y,t), \tag{9}$$

for the dimensional velocity field $u(y,t)$.

Now, deriving the equality (8) with respect to y one obtains the equation

$$\rho \frac{\partial}{\partial t}\left[\frac{\partial u(y,t)}{\partial y}\right] = \frac{\partial^2 \tau(y,t)}{\partial y^2} - \tilde{\sigma} B^2 \frac{\partial u(y,t)}{\partial y}. \tag{10}$$

Substituting $\partial u(y,t)/\partial y$ from Eq. $(5)_2$ in (10), one finds the next governing equation for the dimensional shear stress $\tau(y,t)$, namely

$$\rho\left(1+\lambda \frac{\partial}{\partial t}\right)\frac{\partial \tau(y,t)}{\partial t} = \mu \frac{\partial^2 \tau(y,t)}{\partial y^2} - \tilde{\sigma} B^2 \left(1+\lambda \frac{\partial}{\partial t}\right)\tau(y,t), \tag{11}$$

Consequently, the governing equations (9) and (11) for the velocity $u(y,t)$, respectively the corresponding non-trivial shear stress $\tau(y,t)$ are identical as form.

3 Applications

In the previous section it was proved that governing equations for the fluid velocity $u(y,t)$ and the shear stress $\tau(y,t)$ corresponding to isothermal hydromagnetic unidirectional motions of IUCMFs, whose velocity vector is given by Eq. (3), are identical as form. In order to bring to light the advantages of this observation, some important applications will be considered in the following and new exact steady state solutions will be provided.

3.1 Motions over an infinite flat plate

3.1.1 The second problem of Stokes

Consider the isothermal hydromagnetic unsteady motion of an IUCMF over an infinite flat plate which oscillate in its plane according to one of the relations

$$\mathbf{v} = U\cos(\omega t)\mathbf{e}_x \quad \text{or} \quad \mathbf{v} = U\sin(\omega t)\mathbf{e}_x. \tag{12}$$

In the above relations, U and ω are the amplitude, respectively the frequency of oscillations. The velocity vector $\mathbf{v}$ corresponding to such motions is given by Eq. (3) and the dimensional velocity $u(y,t)$ has to satisfy the governing equation (9) with the boundary conditions

$$u(0,t) = U\cos(\omega t), \ \lim_{y\to\infty} u(y,t) = 0, \tag{13}$$

or

$$u(0,t) = U\sin(\omega t), \ \lim_{y\to\infty} u(y,t) = 0. \tag{14}$$

The second condition of Eqs. (13) and (14) says us that the fluid is quiescent at infinity. We also assume that there is no shear in the free stream, i.e.

$$\lim_{y\to\infty} \tau(y,t) = 0. \tag{15}$$

On the other hand, the normal and shear stresses $\sigma(y,t)$, respectively $\tau(y,t)$ have to satisfy the governing equations (5). The initial conditions (4) will not be here used. The boundary conditions (13) or (14) and the fact that the fluid was at rest at the initial moment tell us that the two unsteady motions become steady or permanent in time. An important problem for such motions refers to the required time to touch the steady state. To determine this time, the steady state solutions have to be known. This is the reason that, in the following, only steady state solutions will be determined. These solutions are independent of the initial conditions but satisfy the boundary conditions and governing equations.

Introducing the following non-dimensional functions, variables and parameters

$$\begin{gathered} u^* = \frac{u}{U},\ \tau^* = \frac{1}{\rho U^2}\,\tau,\ \sigma^* = \frac{1}{\rho U^2}\,\sigma,\ y^* = \frac{U}{v}\,y,\ t^* = \frac{U^2}{v}\,t, \\ \lambda^* = \frac{U^2}{v}\,\lambda,\ \omega^* = \frac{v}{U^2}\,\omega \end{gathered} \tag{16}$$

and dropping out the star notation, one obtains the next boundary value problem

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2} - M\left(1+\lambda\frac{\partial}{\partial t}\right)u(y,t);\ y>0,\ t>0, \tag{17}$$

$$u(0,t) = \cos(\omega t),\quad \lim_{y\to\infty} u(y,t) = 0;\ t>0, \tag{18}$$

or again the governing equation (17) with the boundary conditions

$$u(0,t) = \sin(\omega t),\quad \lim_{y\to\infty} u(y,t) = 0;\ t>0, \tag{19}$$

for the dimensionless velocity field $u(y,t)$. In Eqs. (16), $v = \mu/\rho$ is the kinematic viscosity of the fluid while in Eq. (17)

$$M = \frac{\sigma B^2}{\rho}\,\frac{v}{U^2}, \tag{20}$$

is the magnetic parameter. The dimensionless forms of Eqs. (5) are

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma(y,t) &= 2\lambda\tau(y,t)\frac{\partial u(y,t)}{\partial y},\\ \left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\tau(y,t)}{\partial t} &= \frac{\partial u(y,t)}{\partial y}.\end{aligned}\tag{21}$$

In the following, for differentiate, we denote by $u_c(y,t)$, $\tau_c(y,t)$, $\sigma_c(y,t)$ and and $u_s(y,t)$, $\tau_s(y,t)$, $\sigma_s(y,t)$ the dimensionless starting solutions of the two motion problems. These solutions can be presented as sums of steady state (permanent or long time) and transient components, i.e.

$$\begin{aligned}&u_c(y,t)=u_{cp}(y,t)+u_{ct}(y,t),\\ &\tau_c(y,t)=\tau_{cp}(y,t)+\tau_{ct}(y,t),\ \sigma_c(y,t)=\sigma_{cp}(y,t)+\sigma_{ct}(y,t),\end{aligned}\tag{22}$$

$$\begin{aligned}&u_s(y,t)=u_{sp}(y,t)+u_{st}(y,t),\\ &\tau_s(y,t)=\tau_{sp}(y,t)+\tau_{st}(y,t),\ \sigma_s(y,t)=\sigma_{sp}(y,t)+\sigma_{st}(y,t),\end{aligned}\tag{23}$$

The dimensionless steady state velocity fields $u_{cp}(y,t)$ and $u_{sp}(y,t)$, as it was previously mentioned, are independent of the initial conditions but they have to satisfy the governing equation (17) and the boundary conditions (18), respectively (19).

Direct computations show that the steady state solutions $u_{cp}(y,t)$ and $u_{sp}(y,t)$ of these motion problems can be presented in the simple forms

$$u_{cp}(y,t)=e^{-my}\cos(\omega t-ny),\ u_{sp}(y,t)=e^{-my}\sin(\omega t-ny),\tag{24}$$

or equivalent

$$u_{cp}(y,t)=\Re e\{e^{-\delta y+i\omega t}\},\ u_{sp}(y,t)=\mathrm{Im}\{e^{-\delta y+i\omega t}\},\tag{25}$$

where $\delta=\sqrt{(M+i\omega)(1+i\lambda\omega)}$ and

$$\begin{aligned}m&=\sqrt{\frac{\sqrt{(M-\lambda\omega^2)^2+\omega^2(1+\lambda M)^2}+M-\lambda\omega^2}{2}},\\ n&=\sqrt{\frac{\sqrt{(M-\lambda\omega^2)^2+\omega^2(1+\lambda M)^2}-(M-\lambda\omega^2)}{2}}\end{aligned}\tag{26}$$

The equivalence of the expressions of $u_{cp}(y,t)$ and $u_{sp}(y,t)$ given by Eqs. (24) and (25) is graphically proved by Figs. 1. Taking $\lambda=0$ in Eqs. (24) and neglecting magnetic effects, the steady state solutions obtained by Erdogan [1, Eqs. (12) and (17)] are recovered.

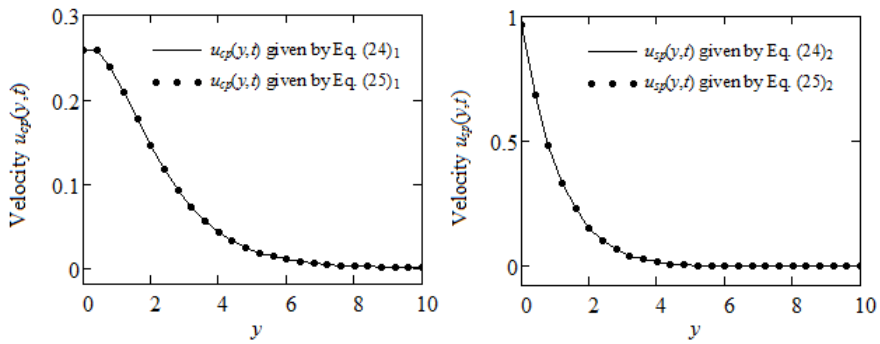

Figure 1: Profiles of velocities $u_{cp}(y,t)$ and $u_{sp}(y,t)$ given by equations $(24)_1$ and $(25)_1$, respectively $(24)_2$ and $(25)_2$ for $\lambda = 0.8$, $\omega = \pi/12$, $M = 0.6$ and $t = 5$.

The corresponding dimensionless steady state shear stresses, namely

$$\begin{aligned} \tau_{cp}(y,t) &= -\sqrt{p^2+q^2}\; e^{-my}\cos(\omega t - ny - \phi), \\ \tau_{sp}(y,t) &= -\sqrt{p^2+q^2}\; e^{-my}\sin(\omega t - ny - \phi), \end{aligned} \tag{27}$$

or equivalently

$$\begin{aligned} \tau_{cp}(y,t) &= -\Re e\left\{\frac{\delta}{1+i\lambda\omega}\, e^{-\delta y + i\omega t}\right\}, \\ \tau_{sp}(y,t) &= -\mathrm{Im}\left\{\frac{\delta}{1+i\lambda\omega}\, e^{-\delta y + i\omega t}\right\} \end{aligned} \tag{28}$$

have been obtained using the relations $(21)_2$, (24) and (25). Into Eqs. (27) $\phi = \mathrm{arctg}(q/p)$ and the constants p and q are given by the relations

$$p = -\frac{m+\lambda\omega n}{1+(\lambda\omega)^2}, \quad q = \frac{n-\lambda\omega m}{1+(\lambda\omega)^2}. \tag{29}$$

The equivalence of the expressions of $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ given by Eqs. (27) and (28) is graphically proved by Figs. 2.

Introducing the expressions of $u_{cp}(y,t)$, $u_{sp}(y,t)$ and $\tau_{cp}(y,t)$, $\tau_{sp}(y,t)$, from Eqs. (25) and (28) in $(21)_1$, one obtains the corresponding exact expressions for the dimensionless normal stresses $\sigma_{cp}(y,t)$ and $\sigma_{sp}(y,t)$, namely

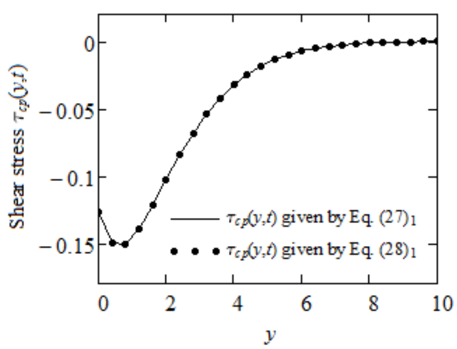

Figure 2: Profiles of shear stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ given by equations $(27)_1$ and $(28)_1$, respectively $(27)_2$ and $(28)_2$ for $\lambda = 0.8$, $\omega = \pi/12$, $M = 0.6$ and $t = 5$.

$$\begin{aligned} \sigma_{cp}(y,t) &= 2\lambda \Re e \left\{ \frac{\delta^2}{(1+i\lambda\omega)^2}\ e^{2(-\delta y + i\omega t)} \right\}, \\ \sigma_{sp}(y,t) &= 2\lambda \operatorname{Im} \left\{ \frac{\delta^2}{(1+i\lambda\omega)^2}\ e^{2(-\delta y + i\omega t)} \right\}. \end{aligned} \tag{30}$$

3.1.2 Motions induced by oscillatory shear stresses on the boundary

Let us now consider the isothermal hydromagnetic motions of same fluids generated by the flat plate that applies a shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid. Here S and ω are the amplitude and the frequency of oscillations. The velocity vector $\mathbf{v}$ corresponding to these motions is again given by the equality (3) and the corresponding governing equations have the same forms as in the previous section. Instead, the boundary conditions are

$$\tau(0,t) = S\cos(\omega t), \quad \lim_{y\to\infty} \tau(y,t) = 0, \tag{31}$$

or

$$\tau(0,t) = S\sin(\omega t), \quad \lim_{y\to\infty} \tau(y,t) = 0. \tag{32}$$

Introducing the next non-dimensional functions, variables and parameters

$$u^* = u\sqrt{\frac{\rho}{S}},\ \tau^* = \frac{\tau}{S},\ \sigma^* = \frac{\sigma}{S},\ y^* = \frac{y}{v}\sqrt{\frac{S}{\rho}},$$
$$t^* = \frac{S}{\mu}t,\ \lambda^* = \frac{S}{\mu}\lambda,\ \omega^* = \frac{\mu}{S}\omega \tag{33}$$

and again giving up the star notation, one obtains for the dimensionless velocity, shear stress and normal stress fields $u(y,t), \tau(y,t)$, respectively $\sigma(y,t)$ the same non-dimensional equations (17) and (21). However, as we have a problem with shear stress on the boundary, the non-dimensional form of the governing equation (11), namely

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial \tau(y,t)}{\partial t} = \frac{\partial^2 \tau(y,t)}{\partial y^2} - M\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t), \tag{34}$$

will be used. In this equation, the magnetic parameter M is defined by

$$M = \frac{\sigma B^2}{\rho}\ \frac{\mu}{S}. \tag{35}$$

The corresponding dimensionless boundary conditions are

$$\tau(0,t) = \cos(\omega t),\quad \lim_{y\to\infty}\tau(y,t) = 0, \tag{36}$$

or

$$\tau(0,t) = \sin(\omega t),\quad \lim_{y\to\infty}\tau(y,t) = 0. \tag{37}$$

Because the non-dimensional governing equation (34) and the boundary conditions (36) and (37) for the dimensionless shear stress $\tau(y,t)$ are identical as form to the governing equation (17) and the boundary conditions (18) and (19) for the velocity $u(y,t)$, it results that the dimensionless shear stresses $\tau_{cp}(y,t)$ and $\tau_{sp}(y,t)$ corresponding to the new motion problems are given by the following equalities

$$\tau_{cp}(y,t) = e^{-my}\cos(\omega t - ny),\ \tau_{sp}(y,t) = e^{-my}\sin(\omega t - ny), \tag{38}$$

or equivalent

$$\tau_{cp}(y,t) = \Re e\{e^{-\delta y + i\omega t}\},\quad \tau_{sp}(y,t) = \mathrm{Im}\{e^{-\delta y + i\omega t}\}, \tag{39}$$

in which the constants m, n and δ have been previously defined.

The corresponding velocity fields, namely

$$u_{cp}(y,t) = -\sqrt{p_1^2+q_1^2}\ e^{-my}\cos(\omega t - ny - \psi),$$
$$u_{sp}(y,t) = -\sqrt{p_1^2+q_1^2}\ e^{-my}\sin(\omega t - ny - \psi), \tag{40}$$

or equivalent

$$u_{cp}(y,t) = -\Re e\left\{\frac{1+i\lambda\omega}{\delta}\ e^{-\delta y+i\omega t}\right\},$$
$$u_{sp}(y,t) = -\mathrm{Im}\left\{\frac{1+i\lambda\omega}{\delta}\ e^{-\delta y+i\omega t}\right\}, \tag{41}$$

have been obtained using Eqs. $(21)_2$ and (38), respectively $(21)_2$ and (39). In Eqs. (40) $\psi = \mathrm{arctg}(q_1/p_1)$ and

$$p_1 = -\frac{m+\lambda\omega n}{m^2+n^2},\quad q_1 = \frac{\lambda\omega m - n}{m^2+n^2}. \tag{42}$$

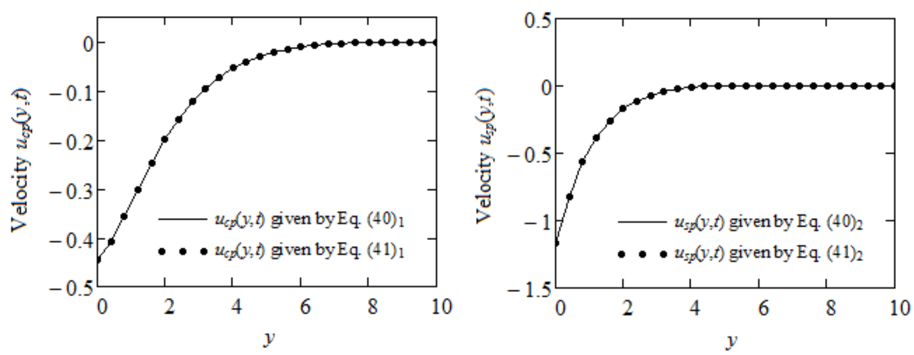

Figure 3: Profiles of velocities $u_{cp}(y,t)$ and $u_{sp}(y,t)$ given by equations $(40)_1$ and $(41)_1$, respectively $(40)_2$ and $(41)_2$ for $\lambda = 0.8$, $\omega = \pi/12$, $M = 0.6$ and $t = 5$.

The exact expressions of the dimensionless steady state normal stresses $\sigma_{cp}(y,t)$ and $\sigma_{sp}(y,t)$ corresponding to $u_{cp}(y,t)$, $\tau_{cp}(y,t)$, respectively $u_{sp}(y,t)$, $\tau_{sp}(y,t)$ given by the relations (39) and (41) are given by the equalities

$$\sigma_{cp}(y,t) = 2\lambda\Re e\left\{\frac{1+i\lambda\omega}{1+2i\lambda\omega}\ e^{2(-\delta y+i\omega t)}\right\},$$
$$\sigma_{sp}(y,t) = 2\lambda\,\mathrm{Im}\left\{\frac{1+i\lambda\omega}{1+2i\lambda\omega}\ e^{2(-\delta y+i\omega t)}\right\}, \tag{43}$$

Simple computations show that the governing equation $(21)_1$ is identically satisfied if $\tau_{cp}(y,t)$, $\tau_{sp}(y,t)$, $u_{cp}(y,t)$, $u_{sp}(y,t)$, and $\sigma_{cp}(y,t)$, $\sigma_{sp}(y,t)$, are given by the equalities (39), (41), respectively (43).

Finally, making $\lambda = 0$ into above relations, the solutions corresponding to isothermal hydrodynamic motions of incompressible Newtonian fluids performing the same motions are obtained. It is also worth to point out the fact that the dimensionless solutions given by Eqs. (38), (39), (41) and (43) are the first exact solutions obtained for motions of incompressible rate type fluids induced by oscillatory shear stresses $S\cos(\omega t)$ or $S\sin(\omega t)$ on the boundary when magnetic effects are taken into consideration.

3.2 Motions between parallel plates

Now, in order to provide new exact solutions for hydromagnetic motions of IUCMFs, let us consider such a fluid at rest between two infinite horizontal parallel plates. At the moment $t = 0^+$, both plates begin to move in their planes with the velocity $U\cos(\omega t)$ or $U\sin(\omega t)$ or to apply a shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid. The velocity vector corresponding to such a motion is also of the form (3) and the corresponding dimensional governing equations are identical to those from Section 2.

3.2.1 Motions with velocity on the boundary

Using the same notations as in the previous section, it results that the dimensional steady state velocity, shear stress and normal stress fields $u_{cp}(y,t)$, $\tau_{cp}(y,t)$, $\sigma_{cp}(y,t)$ and $u_{sp}(y,t)$, $\tau_{sp}(y,t)$, $\sigma_{sp}(y,t)$ corresponding to these motions have to satisfy the governing equations (5) and (9) and the boundary conditions

$$u(0,t) = U\cos(\omega t), \quad u(d,t) = U\cos(\omega t), \tag{44}$$

respectively

$$u(0,t) = U\sin(\omega t), \quad u(d,t) = U\sin(\omega t), \tag{45}$$

where d is the distance between plates.

Introducing the following non-dimensional functions, variables and parameters

$$\begin{gathered} u^* = \frac{u}{U},\ \tau^* = \frac{1}{\rho U^2}\,\tau,\ \sigma^* = \frac{1}{\rho U^2}\,\sigma,\ y^* = \frac{y}{d},\ t^* = \frac{U}{d}\,t, \\ \lambda^* = \frac{U}{d}\,\lambda,\ \omega^* = \frac{d}{U}\,\omega, \end{gathered} \tag{46}$$

in Eqs. (5) and (9) and again abandoning the star notation, one obtains the following partial differential equations

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma(y,t)&=2\lambda\tau(y,t)\,\frac{\partial u(y,t)}{\partial y},\\ \left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t)&=\frac{1}{\mathrm{Re}}\,\frac{\partial u(y,t)}{\partial y},\end{aligned}\tag{47}$$

$$\mathrm{Re}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t}=\frac{\partial^2 u(y,t)}{\partial y^2}-M\left(1+\lambda\frac{\partial}{\partial t}\right)u(y,t);\quad y>0,\ t>0,\tag{48}$$

for dimensionless velocity, shear stress and normal stress fields $u(y,t)$, $\tau(y,t)$ and $\sigma(y,t)$. In these relations the Reynolds number Re and the magnetic parameter M are defined by

$$\mathrm{Re}=\frac{Ud}{v},\ M=\frac{\sigma B^2}{\rho}\ \frac{d^2}{v}.\tag{49}$$

The corresponding dimensionless boundary conditions are

$$u(0,t)=\cos(\omega t),\quad u(1,t)=\cos(\omega t),\tag{50}$$

respectively

$$u(0,t)=\sin(\omega t),\quad u(1,t)=\sin(\omega t).\tag{51}$$

Direct computations show that the dimensionless steady state velocities $u_{cp}(y,t)$ and $u_{sp}(y,t)$ which satisfy the governing equation (48) and the boundary conditions (50), respectively (51) are given by the relations

$$\begin{aligned}u_{cp}(y,t)&=\Re e\left\{\frac{\sinh(\tilde{\delta}y)+\sinh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\,e^{i\omega t}\right\},\\ u_{sp}(y,t)&=\mathrm{Im}\left\{\frac{\sinh(\tilde{\delta}y)+\sinh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\,e^{i\omega t}\right\},\end{aligned}\tag{52}$$

where $\tilde{\delta}=\sqrt{(M+i\omega\,\mathrm{Re})(1+i\lambda\omega)}$. Similar solutions for motions of Newtonian fluids induced by the lower plate that oscillates in its plane have been recently established by Fetecau and Agop [13, Eqs. (48)]. As expected, making $\omega=0$ in Eq. $(52)_1$ (which means that both plates move in their planes with the constant velocity U) and taking the limit of the obtained result for M and λ going to zero, the steady velocity field obtained by Erdogan [14, Eq. (12)] is recovered.

The corresponding shear stresses, namely

$$\begin{aligned}\tau_{cp}(y,t) &= \frac{1}{\mathrm{Re}}\,\Re e\left\{\frac{\cosh(\tilde{\delta}y)-\cosh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\;\frac{\tilde{\delta}e^{i\omega t}}{1+i\lambda\omega}\right\},\\ \tau_{sp}(y,t) &= \frac{1}{\mathrm{Re}}\,\mathrm{Im}\left\{\frac{\cosh(\tilde{\delta}y)-\cosh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\;\frac{\tilde{\delta}e^{i\omega t}}{1+i\lambda\omega}\right\},\end{aligned} \tag{53}$$

have been determined using Eqs. $(47)_2$ and (52). Introducing $u_{cp}(y,t)$, $u_{sp}(y,t)$ and $\tau_{cp}(y,t)$, $\tau_{sp}(y,t)$ from the last relations (52) and (53) in $(47)_1$ one obtains the corresponding dimensionless normal stresses $\sigma_{cp}(y,t)$ and $\sigma_{sp}(y,t)$ under forms

$$\begin{aligned}\sigma_{cp}(y,t) &= \frac{2\lambda}{\mathrm{Re}}\,\Re e\left\{\frac{\{\cosh(\tilde{\delta}y)-\cosh(\tilde{\delta}(1-y)]\}^2}{\sinh^2(\tilde{\delta})}\;\frac{M+i\omega\,\mathrm{Re}}{1+2i\lambda\omega}\,e^{2i\omega t}\right\},\\ \sigma_{sp}(y,t) &= \frac{1}{\mathrm{Re}}\,\mathrm{Im}\left\{\frac{\{\cosh(\tilde{\delta}y)-\cosh(\tilde{\delta}(1-y)]\}^2}{\sinh^2(\tilde{\delta})}\;\frac{M+i\omega\,\mathrm{Re}}{1+2i\lambda\omega}\,e^{2i\omega t}\right\},\end{aligned} \tag{54}$$

Simple computations show that $u_{cp}(y,t)$, $\tau_{cp}(y,t)$, $\sigma_{cp}(y,t)$ and $u_{sp}(y,t)$, $\tau_{sp}(y,t)$, $\sigma_{sp}(y,t)$ given by the relations (52)–(54) satisfy the governing equations (47), (48) and the boundary conditions (50), respectively (51).

3.2.2 Motions induced by oscillatory shear stresses on the boundary

Let us now consider isothermal hydromagnetic motions of the same fluids between infinite parallel plates that applies a shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid. The velocity field corresponding to these motions is again of the form (3) and the corresponding governing equations are again identical to those from Section 2. The shear stresses corresponding to these motions have to satisfy the governing equation (11) with the boundary conditions

$$\tau(0,t) = S\cos(\omega t),\quad \tau(d,t) = S\cos(\omega t), \tag{55}$$

or

$$\tau(0,t) = S\sin(\omega t),\quad \tau(d,t) = S\sin(\omega t). \tag{56}$$

Using the non-dimensional functions, variables and parameters

$$\begin{gathered} u^* = \sqrt{\frac{\rho}{S}}\,u,\ \tau^* = \frac{\tau}{S},\ \sigma^* = \frac{\sigma}{S},\ y^* = \frac{y}{d},\ t^* = \frac{S}{\mu}\,t,\\ \lambda^* = \frac{S}{\mu}\,\lambda,\ \omega^* = \frac{\mu}{S}\,\omega, \end{gathered} \tag{57}$$

the equations (5) and (11) take the non-dimensional forms

$$\begin{aligned}&\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma(y,t)=\frac{2\lambda}{\sqrt{\mathrm{Re}}}\,\tau(y,t)\,\frac{\partial u(y,t)}{\partial y},\\&\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t)=\frac{1}{\sqrt{\mathrm{Re}}}\,\frac{\partial u(y,t)}{\partial y},\end{aligned}\tag{58}$$

respectively

$$\mathrm{Re}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial\tau(y,t)}{\partial t}=\frac{\partial^2\tau(y,t)}{\partial y^2}-M\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(y,t),\tag{59}$$

while the boundary conditions (55) and (56) become

$$\tau(0,t)=\cos(\omega t),\quad \tau(1,t)=\cos(\omega t),\tag{60}$$

or

$$\tau(0,t)=\sin(\omega t),\quad \tau(1,t)=\sin(\omega t).\tag{61}$$

In the above relations, the Reynolds number Re and the magnetic parameter M are defined by the next relations

$$\mathrm{Re}=\frac{Sd^2}{\mu v}=\frac{Vd}{v},\quad M=\frac{\sigma B^2}{\rho}\,\frac{d^2}{v},\tag{62}$$

where $V=Sd/\mu$ is a characteristic velocity.

Bearing in mind the fact that the form of the governing equation (59) for the shear stress $\tau(y,t)$ and the boundary conditions (60) and (61) are identical to those corresponding to $u(y,t)$ from the previous section, it results that

$$\begin{aligned}\tau_{cp}&=\Re e\left\{\frac{\sinh(\tilde{\delta}y)+\sinh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\;e^{i\omega t}\right\},\\\tau_{sp}&=\mathrm{Im}\left\{\frac{\sinh(\tilde{\delta}y)+\sinh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\;e^{i\omega t}\right\}.\end{aligned}\tag{63}$$

The corresponding velocity fields, namely

$$\begin{aligned}u_{cp}(y,t)&=\sqrt{\mathrm{Re}}\,\Re e\left\{\frac{\cosh(\tilde{\delta}y)-\cosh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\;\frac{1+i\lambda\omega}{\tilde{\delta}}\;e^{i\omega t}\right\},\\u_{sp}(y,t)&=\sqrt{\mathrm{Re}}\,\mathrm{Im}\left\{\frac{\cosh(\tilde{\delta}y)-\cosh[\tilde{\delta}(1-y)]}{\sinh(\tilde{\delta})}\;\frac{1+i\lambda\omega}{\tilde{\delta}}\;e^{i\omega t}\right\},\end{aligned}\tag{64}$$

have been obtained using Eqs. (58) and (64). Finally, the expressions of the dimensionless steady state normal stresses $\sigma_{cp}(y,t)$ and $\sigma_{sp}(y,t)$ are

$$\begin{aligned}\sigma_{cp}(y,t) &= 2\lambda \Re e\left\{\frac{\{\sinh(\tilde{\delta}y)-\sinh[\tilde{\delta}(1-y)]\}^2}{\sinh^2(\tilde{\delta})}\ \frac{1+i\lambda\omega}{1+2i\lambda\omega}\ e^{2i\omega t}\right\},\\ \sigma_{sp}(y,t) &= 2\lambda\,\mathrm{Im}\left\{\frac{\{\sinh(\tilde{\delta}y)-\sinh[\tilde{\delta}(1-y)]\}^2}{\sinh^2(\tilde{\delta})}\ \frac{1+i\lambda\omega}{1+2i\lambda\omega}\ e^{2i\omega t}\right\}.\end{aligned} \tag{65}$$

4 Conclusions

The problem of obtaining exact solutions for motions of Newtonian or non-Newtonian fluids is an important one. In order to facilitate the possibility of obtaining new exact solutions for hydromagnetic motions of the incompressible upper-convected Maxwell fluids, an important observation concerning the governing equations of velocity and the non-trivial shear stress has been brought to light. More exactly, it was showed that the governing equations of the two important entities corresponding to some isothermal hydromagnetic unidirectional motions of these fluids have identical forms. This observation allows us to easily obtain exact solutions for motions of these fluids with shear stress or velocity on the boundary if similar solutions for motions of same fluids with velocity, respectively shear stress on the boundary are known. As application, some hydromagnetic unsteady motions of the incompressible Maxwell fluids over an infinite plate or between two infinite horizontal parallel plates have been solved and new exact solutions have been provided for the dimensionless steady state velocity, shear stress and normal stress fields.

The obtained solutions are especially important for those who want to know the need time to touch the steady or permanent state. This is the time after which the fluid moves according to the steady state solutions. From mathematical point of view, it is the time after which the diagrams of starting solutions (numerical solutions) superpose over those of their steady state components. In addition, the exact solutions for different motions of fluids can be used to test the numerical schemes that are developed to study more complex unsteady flow problems. For a check of results that have been here provided, the velocity and shear stress fields corresponding to motions over an infinite flat plate have been presented in different forms whose equivalence was graphically proved.

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