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CONVERGENCE CRITERIA, WELL-POSEDNESS CONCEPTS AND APPLICATIONS*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

We consider an abstract problem $\mathcal{P}$ in a metric space X which has a unique solution $u \in X$. Our aim in this current paper is two folds: first, to provide a convergence criterion to the solution of Problem $\mathcal{P}$, that is, to give necessary and sufficient conditions on a sequence $\{u_n\} \subset X$ which guarantee the convergence $u_n \to u$ in the space X; second, to find a Tyknonov triple $\mathcal{T}$ such that a sequence $\{u_n\} \subset X$ is a $\mathcal{T}$-approximating sequence if and only if it converges to u. The two problems stated above, associated to the original Problem $\mathcal{P}$, are closely related. We illustrate how they can be solved in three particular cases of Problem $\mathcal{P}$: a variational inequality in a Hilbert space, a fixed point problem in a metric space and a minimization problem in a reflexive Banach space. For each of these problems we state and prove a convergence criterion that we use to define a convenient Tykhonov triple $\mathcal{T}$ which requires the condition stated above. We also show how the convergence criterion and the corresponding $\mathcal{T}$-well posedness concept can be used to deduce convergence and classical well-posedness results, respectively.

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1 Introduction

Convergence results represent an important topic in Functional Analysis, Numerical Analysis and Partial Differential Equation Theory. The continuous dependence of the solution of a partial differential equation with respect to the data, the convergence of the solution of a penalty problem to the solution of the original problem as the penalty parameter converges, the convergence of the discrete solution to the solution of the continuous problem when the time step or the discretization parameter converges to zero are some simple example, among many others. On the other hand, convergence results abound in the study of various mathematical models in Mechanics, Physics and Engineering Sciences. The convergence of the solution of a contact model with a deformable foundation to the solution of a contact model with a rigid foundation as the stiffness coefficient of the foundation converges to infinity, the convergence of the solution of a frictional problem to the solution of a frictionless problem as the coefficient of friction tends to zero represent two relevant examples, with potential real-world applications.

For all these reasons, a considerable effort was done to obtain convergence results in the study of various mathematical problems including nonlinear equations, inequality problems, inclusions, fixed point problems, optimization problems, among others. The literature in the field is extensive. The corresponding results have been obtained by using various methods and functional arguments, which differ from problem to problem and from paper to paper. Nevertheless, most of these convergence results can be casted in the abstract functional framework we describe below. Consider a mathematical object $\mathcal{P}$, called generic "problem", defined in a metric space (X, d). Problem $\mathcal{P}$ could be an equation, a minimization problem, a fixed point problem, an inclusion or an inequality problem, for instance. We associate to Problem $\mathcal{P}$ the concept of "solution" which follows from the context and we assume that $\mathcal{P}$ has a unique solution $u \in X$. Then, a convergence result is a result of the form $u_n \to u$ in X where $\{u_n\} \subset X$ represents a given sequence.

Note that in most of the cases, convergence results consists in sufficient conditions which guarantee the convergence of a specific sequence $\{u_n\}$ to the solution u of the corresponding problem $\mathcal{P}$. They do not describe all the sequences which have this property. Therefore, we naturally arrive to consider the following problem, associated to $\mathcal{P}$.

Problem $\mathcal{Q}_{\mathcal{P}}$. *Given a Problem $\mathcal{P}$ with a unique solution u, describe the convergence of a sequence $\{u_n\} \subset X$ to the solution u. In the words, provide necessary and sufficient conditions for the convergence $u_n \to u$ in X, i.e., provide a* convergence criterion.

Note that Problem $\mathcal{Q}_{\mathcal{P}}$ represents a major issue in the study of conveggence results. Its solution depends on the structure of Problem $\mathcal{P}$ and cannot be provided in this general framework. For this reason, we restrict ourselves to solve Problem $\mathcal{Q}_{\mathcal{P}}$ in the particular case when $\mathcal{P}$ is one of the problems $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ that we introduce in the next sections. Providing such a convergence criterion for problems $\mathcal{P}_i$ with $i = 1, 2, 3$ represents our first aim in this paper.

Well-posedness results for a given problem $\mathcal{P}$ represent another important topic in Analysis, Numerical Analysis, Partial Differential Equation Theory, with important applications in Experimental Sciences. The theory of well-posed problems grew up rapidly in the last decades, due to the very rich literature in the field. Thus, a concept of well-posedness for a minimization problem was considered in the paper of Tykhonov [19]. A version of this concept, known as the Levitin-Polyak well-posedness concept, was introduced in [9] in the study of constrainted minimization problems. Various extensions of the Tykhonov and Levitin-Polyak well-posedness concepts have been considered in [2, 3, 11, 12, 13] in the study of variational and hemivariational inequalities. Well-poseness results for fixed point and set-valued optimization problems have been obtained in [7, 8] and [4, 5, 6], respectively. Comprehensive references in the field are the books [1, 10] and, more recently, [14].

A careful analysis of the well-posedness concepts in the literature shows that they are based on two main ingredients: the existence of a unique solution for the problem considered and the convergence to it of a special class of sequences, the so-called approximating sequence. As a consequence of this remark, a new concept of well-posedness, based on the notion of Tykhonov triple, was introduced in [20], applied in [16, 17, 18] in the study of contact and heat transfer problems, and intensively studied in [14]. It has the merit to extend several classical well-posedness concepts and, for this reason, we shall use it in this paper. With the notation (X, d) and $\mathcal{P}$ above, together with notation 2^X for the set of parts of X, this concept is defined as follows.

Definition 1 a) *A Tykhonov triple is a mathematical object of the form* $\mathcal{T} = (I, \Omega, \mathcal{C})$ *where* I *is a given nonempty set,* $\Omega : I \to 2^X$ *is a set-valued mapping such that* $\Omega(\theta) \neq \emptyset$ *for each* $\theta \in I$ *and* $\mathcal{C}$ *is a nonempty subset of sequences with elements in* I*.*

b) *Given a Tykhonov triple* $\mathcal{T} = (I, \Omega, \mathcal{C})$*, a sequence* $\{u_n\} \subset X$ *is called a* $\mathcal{T}$*-approximating sequence if there exists a sequence* $\{\theta_n\} \in \mathcal{C}$ *such that* $u_n \in \Omega(\theta_n)$ *for each* $n \in \mathbb{N}$*.*

c) *Given a Tykhonov triple* $\mathcal{T} = (I, \Omega, \mathcal{C})$*, Problem* $\mathcal{P}$ *is said to be* $\mathcal{T}$*-well-posed (or, equivalently, well-posed with* $\mathcal{T}$*) if it has a unique solution and every* $\mathcal{T}$*-approximating sequence converges in* X *to this solution.*

Note that the concept of approximating sequence above depends on the Tykhonov triple $\mathcal{T}$ and, for this reason, everywhere in this paper we use the terminology "$\mathcal{T}$-approximating sequence". As a consequence, the concept of well-posedness for Problem $\mathcal{P}$ is not an intrinsic one, since it depends on the Tykhonov triple $\mathcal{T}$. For this reason we use the terminology "$\mathcal{T}$-well-posedness".

Assume now that Problem $\mathcal{P}$ has a unique solution $u \in X$. Then, Definition 1 c) shows that $\mathcal{P}$ is $\mathcal{T}$-well-posed if and only if the following implication holds:

$$\{u_n\} \text{ is a } \mathcal{T}\text{-approximating sequence} \quad \Longrightarrow \quad u_n \to u \quad \text{in} \quad X. \tag{1}$$

Some elementary examples could be easily constructed to show that, even if Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed, the converse of this implication is not valid. Nevertheless, it follows from (1) that a $\mathcal{T}$-well-posedness result allows us to identify a class of sequences $\{u_n\}$ which converge to the solution u and, therefore, it provides a partial answer to Problem $\mathcal{Q}_{\mathcal{P}}$. Moreover, $\mathcal{T}$-well-posedness results have the merit to indicate that a positive answer to Problem $\mathcal{Q}_{\mathcal{P}}$ can be obtained by using an appropriate Tykhonov triple. The comments above lead in a natural way to the following problem, which represent a major issue in the theory of $\mathcal{T}$-well-posed problems.

Problem $\widetilde{\mathcal{Q}}_{\mathcal{P}}$**.** *Given a Problem* $\mathcal{P}$ *which has a unique solution* $u \in X$*, find a Tykhonov triple* $\mathcal{T}$ *such that the following equivalence holds:*

$$\{u_n\} \text{ is a } \mathcal{T}\text{-approximating sequence} \quad \Longleftrightarrow \quad u_n \to u \quad \text{in} \quad X. \tag{2}$$

It is easy to see that the following example provides a Tykhonov triple which satisfies condition (2).

Example 1 *Let* $\mathcal{T}_{\mathcal{P}} = (I_{\mathcal{P}}, \Omega_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}})$ *where*

$$I_{\mathcal{P}} = \mathbb{R}_+ = [0, +\infty), \tag{3}$$

$$\Omega_{\mathcal{P}} : I_{\mathcal{P}} \to 2^X, \quad \Omega_{\mathcal{P}}(\theta) = \Big\{ \widetilde{u} \in X \ : \ d(\widetilde{u}, u) \le \theta \Big\} \quad \forall\, \theta \ge 0, \tag{4}$$

$$\mathcal{C}_{\mathcal{P}} = \Big\{ \{\theta_n\} \subset I_{\mathcal{P}} \ : \ \theta_n \to 0 \Big\}. \tag{5}$$

Nevertheless, note that the choice of the Tykhonov triple $\mathcal{T}_{\mathcal{P}}$ above is not convenient in applications, since definition (4) uses the solution u of Problem $\mathcal{P}$ which is a priori unknown. A reasonable definition of the approximating sets $\Omega(\theta)$ with $\theta \in I$ would use Problem $\mathcal{P}$ itself, or some of its perturbations, not its solution. For this reason it is important to introduce Tykhonov triples for which the equivalence (2) holds, but which are defined without any explicit mention to the solution u of Problem $\mathcal{P}$.

Note that finding such Tykhonov triples is useful for several reasons. First, it reformulates Problem $\mathcal{Q}_{\mathcal{P}}$ in a rigourous mathematical framework. Second, it provides an answer to Problem $\mathcal{Q}_{\mathcal{P}}$ stated above. Third, it could be used to unify the proofs of different convergence results. Finally, it allows us to deduce a number of classical well-posedness results previously obtained in the literature. Nevertheless, it is clear that the solution of Problem $\widetilde{\mathcal{Q}}_{\mathcal{P}}$ depends on the structure of the original problem $\mathcal{P}$ and cannot be provided in this general framework. For this reason, we restrict ourselves to solve Problem $\widetilde{\mathcal{Q}}_{\mathcal{P}}$ in the particular case of problems $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ mentionned above. Providing such a Tyknonov triple for these problems represents the second aim of this paper.

The rest of the manuscript is structured as follows. In Sections 2, 3 and 4 we provide answers to Problem $\mathcal{Q}_{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}_{\mathcal{P}}$, in the particular setting of problems $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, respectively. More precisely, for each of these problems we deduce a criterion which allows us to identify all the sequences $\{u_n\} \subset X$ which converge to the solution u of the corresponding problem $\mathcal{P}_i$, $i = 1, 2, 3$. We also provide Tyknonov triples which satisfy condition (2). We use these ingredients in order to present convergence results and to recover classical well-posedness results in the study of these problems. Finally, in Section 5 we provide some concluding remarks.

We end this Introduction with a description of the notation we shall use in this paper. First, in the next sections X will denote either a Hilbert space, a metric space or a reflexive Banach space. The symbol "$\to$" denotes both the convergence in X and the convergence in $\mathbb{R}$. All the limits, upper and lower limits below are considered as $n \to \infty$, even if we do not mention it explicitly. For a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ which converges to zero we use the short hand notation $0 \leq \varepsilon_n \to 0$. When X is a metric space we denote by $d(u,v)$ the distance between the elements u, $v \in X$ and by $d(u,K)$ the distance between an element $u \in X$ to the nonempty subset $K \subset X$, that is

$$d(u,K) = \inf_{v \in K} d(u,v). \tag{6}$$

When X is a reflexive Banach space we use $\|\cdot\|_X$, 0_X and I_X for the norm, the zero element and the identity operator of X, respectively. Moreover, recall that in this case $d(u,v) = \|u - v\|_X$ for all u, $v \in X$ and, therefore,

$$d(u,K) = \inf_{v \in K} \|u - v\|_X. \tag{7}$$

Finally, when X is a Hilbert space we use $(\cdot,\cdot)_X$ for the inner product of X. In addition, if K is a nonempty closed convex subset of X we have

$$d(u,K) = \|u - P_K u\|_X \tag{8}$$

where, here and below, $P_K : X \to K$ is the projector operator on K.

2 The inequality Problem $\mathcal{P}_1$

Everywhere in this section we assume that X is a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. We also assume that K is a nonempty closed convex subset of X and $f \in X$. Then, the problem we consider is stated as follows.

Problem $\mathcal{P}_1$. Find an element u such that

$$u \in K, \qquad (u, v - u)_X \geq (f, v - u)_X \qquad \forall\, v \in K. \tag{9}$$

It is well known that this problem has a unique solution, $u = P_K f$. Moreover, following [1] we recall the following definitions.

Definition 2 a) *A sequence $\{u_n\}$ is called an approximating sequence for inequality* (9) *if there exists a sequence $0 \leq \varepsilon_n \to 0$ such that*

$$u_n \in K, \quad (u_n, v - u_n)_X + \varepsilon_n \|v - u_n\|_X \geq (f, v - u_n)_X \quad \forall\, v \in K,\ n \in \mathbb{N}.$$

b) *Problem $\mathcal{P}_1$ is said to be well-posed in the sense of Tykhonov if any approximating sequence converges to the solution u.*

c) *A sequence* $\{u_n\} \subset X$ *is called a generalized (or LP) approximating sequence for inequality* (9) *if there exist two sequences* $\{w_n\} \subset X$ *and* $\{\varepsilon_n\} \subset \mathbb{R}_+$ *such that* $w_n \to 0_X$ *in* X, $\varepsilon_n \to 0$ *and, moreover,*

$$u_n + w_n \in K, \qquad (u_n, v - u_n)_X + \varepsilon_n \|v - u_n\|_X \geq (f, v - u_n)_X \quad \forall\, v \in K,\ n \in \mathbb{N}.$$

d) *Problem* $\mathcal{P}_1$ *is said to be well-posed in the sense of Levitin-Polyak if any generalized approximating sequence converges to the solution* u.

The definitions above show that if (9) is well-posedness in the sense of Levitin-Polyak then it is well-posedness in the sense of Tykhonov, too. Some simple examples can be constructed in order to see that the converse of this statement is not true.

Our main result in this section is the following.

Theorem 1 *The following statements are equivalent:*

$$u_n \to u \qquad \text{in } X. \tag{10}$$

$$\begin{cases} \text{(a)} \quad d(u_n, K) \to 0\ ; \\ \text{(b)} \quad \text{there exists } 0 \leq \varepsilon_n \to 0 \text{ such that} \\ \qquad (u_n, v - u_n)_X + \varepsilon_n(1 + \|v - u_n\|_X) \\ \qquad\qquad \geq (f, v - u_n)_X \quad \forall\, v \in K,\ n \in \mathbb{N}. \end{cases} \tag{11}$$

Proof. The proof of Theorem 1 is carried out in three steps, as follows.

Step i). We prove that any sequence $\{u_n\} \subset X$ *which satisfies condition* (11) (b) *is bounded.* Let $n \in \mathbb{N}$. We test in (11) with $v = u$ to see that

$$(u_n, u - u_n)_X + \varepsilon_n(1 + \|u - u_n\|_X) \geq (f, u - u_n)_X,$$

which implies that

$$(u_n - u, u_n - u)_X \leq (u, u - u_n)_X + \varepsilon_n(1 + \|u - u_n\|_X) + (f, u_n - u)_X$$

and, moreover,

$$\|u - u_n\|_X^2 \leq (\|u\|_X + \|f\|_X + \varepsilon_n)\|u - u_n\|_X + \varepsilon_n.$$

Next, we use inequality

$$x^2 \le ax + b \quad \Longrightarrow \quad x \le a + \sqrt{b} \qquad \forall\, x,\ a,\ b \ge 0 \tag{12}$$

and the convergence $\varepsilon_n \to 0$ to see that the sequence $\{\|u-u_n\|_X\}$ is bounded in $\mathbb{R}$. This implies that the sequence $\{u_n\}$ is bounded in X, which concludes the proof of this step.

Step ii) We prove that (10) *implies* (11). Assume that (10) holds. Then, since $u \in K$ it follows that $d(u_n, K) \le \|u_n - u\|_X$ for each $n \in \mathbb{N}$, which implies that (11)(a) holds. To prove (11)(b) we fix $n \in \mathbb{N}$ and $v \in K$. We write

$$(u_n, v - u_n)_X - (f, v - u_n)_X = (u_n - u, v - u_n)_X + (u, v - u)_X$$
$$+(u, u - u_n)_X - (f, v - u)_X + (f, u_n - u)_X$$

and, using (9), we deduce that

$$\begin{aligned} &(u_n, v - u_n)_X - (f, v - u_n)_X \\ &\qquad \ge (u_n - u, v - u_n)_X + (u, u - u_n)_X + (f, u_n - u)_X. \end{aligned} \tag{13}$$

We now use (13) and inequalities

$$(u_n - u, v - u_n)_X \ge -\|u_n - u\|_X \|v - u_n\|_X,$$
$$(u, u - u_n)_X \ge -\|u\|_X \|u - u_n\|_X,$$
$$(f, u_n - u)_X \ge -\|f\|_X \|u - u_n\|_X$$

to find that

$$(u_n, v - u_n)_X - (f, v - u_n)_X + \|u - u_n\|_X \|v - u_n\|_X$$
$$+\|u\|_X \|u - u_n\|_X + \|f\|_X \|u - u_n\|_X \ge 0.$$

Therefore, with notation

$$\varepsilon_n = \max\{\, \|u - u_n\|_X, (\|u\|_X + \|f\|_X)\|u - u_n\|_X \,\} \tag{14}$$

we see that

$$(u_n, v - u_n)_X + \varepsilon_n (1 + \|v - u_n\|_X) \ge (f, v - u_n)_X. \tag{15}$$

On the other hand, (14) and assumption (10) show that

$$\varepsilon_n \to 0. \tag{16}$$

We now combine (15) and (16) to see that condition (11)(b) is satisfied.

Step iii) We prove that (11) *implies* (10). Assume that (11) holds. Then, (11)(a) and definition (7) of the distance function show that for each $n \in \mathbb{N}$ there exist two elements v_n and w_n such that

$$u_n = v_n + w_n, \quad v_n \in K, \quad w_n \in X, \quad \|w_n\|_X \to 0. \tag{17}$$

We fix $n \in \mathbb{N}$ and use (11)(b) with $v = u \in K$ to see that

$$(u_n, u - u_n)_X + \varepsilon_n(1 + \|u - u_n\|_X) \geq (f, u - u_n)_X. \tag{18}$$

On the other hand, we use the regularity $v_n \in K$ in (17) and test with $v = v_n$ in (9) to find that

$$(u, v_n - u)_X \geq (f, v_n - u)_X. \tag{19}$$

We now add inequalities (18), (19) to obtain that

$$(u_n, u - u_n)_X + (u, v_n - u)_X + \varepsilon_n(1 + \|u - u_n\|_X) \geq (f, v_n - u_n)_X. \tag{20}$$

Next, we use equality $u_n = v_n + w_n$ to see that

$$(u_n, u - u_n)_X + (u, v_n - u)_X = (u_n, u - v_n - w_n)_X + (u - v_n, v_n - u)_X$$
$$+(v_n, v_n - u)_X = (u - v_n, v_n - u)_X + (u_n - v_n, u - v_n)_X - (u_n, w_n)$$
$$= (u - v_n, v_n - u)_X + (w_n, u - v_n)_X - (u_n, w_n)$$

and, therefore, (20) implies that

$$(u - v_n, v_n - u)_X + (w_n, u - v_n)_X - (u_n, w_n)$$
$$+\varepsilon_n(1 + \|u - v_n - w_n\|_X) + (f, w_n)_X \geq 0.$$

Hence,

$$\|u - v_n\|_X^2 \leq \|w_n\|_X \|u - v_n\|_X + \|u_n\|_X \|w_n\|_X \tag{21}$$
$$+\varepsilon_n + \varepsilon_n \|u - v_n\|_X + \varepsilon_n \|w_n\|_X + \|f\|_X \|w_n\|_X.$$

On the other hand, assumption (11) and Step i) guarantee that the sequence $\{u_n\}$ is bounded in X. Therefore, there exists $D > 0$ such that

$$\|u_n\|_X \le D \qquad \forall\, n \in \mathbb{N}. \tag{22}$$

We now combine the bounds (21) and (22) to deduce that

$$\begin{aligned}\|u - v_n\|_X^2 \le (\|w_n\|_X + \varepsilon_n)\|u - v_n\|_X \\ +(D + \varepsilon_n + \|f\|_X)\|w_n\|_X + \varepsilon_n.\end{aligned} \tag{23}$$

Next, we use (23), inequality (12) and the convergences $\|w_n\|_X \to 0$, $\varepsilon_n \to 0$ to find that $\|u - v_n\|_X \to 0$. This implies that $v_n \to u$ in X and, using (17) we deduce that (10) holds, which concludes the proof. □

Note that Theorem 1 provides an answer to Problem $\mathcal{Q}_{\mathcal{P}}$, in the context of the variational inequality $\mathcal{P}_1$, i.e., in the case when the abstract problem $\mathcal{P}$ is replaced by Problem $\mathcal{P}_1$. We now illustrate its use in order to obtain a well-known convergence result. To this end, we recall that for each $\lambda > 0$ there exists an element $u \in X$ such that

$$u + \frac{1}{\lambda}(u - P_K u) = f. \tag{24}$$

Remark 1 *The proof of the previous statement is as follows. First, recall that the projection operator is monotone and nonexpansive, that is,*

$$(P_K v_1 - P_K v_2, v_1 - v_2)_X \ge 0, \quad \|P_K v_1 - P_K v_2\|_X \le \|v_1 - v_2\|_X$$

for all v_1, $v_2 \in X$. Using these inequalities it follows that for each $\lambda > 0$ the operator $A_\lambda : X \to X$ given by $A_\lambda v = v + \frac{1}{\lambda}(v - P_K v)$ for all $v \in X$ is strongly monotone and Lipschitz continuous, i.e.,

$$(A_\lambda v_1 - A_\lambda v_2, v_1 - v_2)_X \ge \|v_1 - v_2\|_X^2,$$

$$\|A_\lambda v_1 - A_\lambda v_2\|_X \le \Big(1 + \frac{2}{\lambda}\Big)\|v_1 - v_2\|_X$$

for all v_1, $v_2 \in X$. These two properties of A_λ allows us to use a well-known result (Theorem 1.24 in [15], for instance) to deduce the existence of a unique solution $u = u_\lambda$ to the nonlinear equation (24)*, for each $\lambda > 0$.*

Our convergence result in the study of equation (24) is as follows.

Corollary 1 *Let $\{\lambda_n\}$ be a sequence such that $0 < \lambda_n \to 0$, $u = P_K f$ and, for each $n \in \mathbb{N}$, let $u_n \in X$ be the unique element of X such that*

$$u_n + \frac{1}{\lambda_n}(u_n - P_K u_n) = f. \tag{25}$$

Then, $u_n \to u$ in X.

Proof. Fix $n \in \mathbb{N}$ and $v \in K$. We use (25) to see that

$$(u_n, v - u_n)_X + \frac{1}{\lambda_n}(u_n - P_K u_n, v - u_n)_X = (f, v - u_n)_X. \tag{26}$$

Then, since

$$\begin{aligned}(u_n - P_K u_n, v - u_n)_X &= (u_n - P_K u_n - v + P_K v, v - u_n)_X \\ &= (P_K v - P_K u_n, v - u_n) - \|v - u_n\|_X^2 \\ &\le \|P_K v - P_K u_n\|_X \|v - u_n\|_X - \|v - u_n\|_X^2 \le 0,\end{aligned}$$

we deduce from (26) that

$$(u_n, v - u_n)_X \ge (f, v - u_n)_X.$$

This implies that condition (11)(b) holds with $\varepsilon_n = 0$. Moreover, Step i) in the proof of Theorem 1 implies that there exists $M > 0$ which does not depend on n such that

$$\|u_n\|_X \le M. \tag{27}$$

We now use (25) and the bound (27) to deduce that

$$\|u_n - P_K u_n\|_X = \lambda_n \|f - u_n\|_X \le \lambda_n\big(\|f\|_X + \|u_n\|_X\big) \le \lambda_n\big(\|f\|_X + M\big).$$

This inequality, combined with (8) and the convergence $\lambda_n \to 0$, shows that $d(u_n, K) \to 0$. We deduce from here that condition (11)(a) holds, too. It follows now from Theorem 1 that $u_n \to u$ in X, which ends the proof. □

We now turn to the $\mathcal{T}$-well-posedness of Problem $\mathcal{P}_1$ and, to this end, we consider the Tykhonov triple $\mathcal{T}_1 = (I_1, \Omega_1, \mathcal{C}_1)$ where

$$I_1 = \mathbb{R}_+ = [0, +\infty),$$

$$\begin{aligned}\Omega : I_1 \to 2^X, \quad \Omega_1(\theta) = \Big\{ &\widetilde{u} \in X \,:\, d(\widetilde{u}, K) \le \theta, \\ &(\widetilde{u}, v - \widetilde{u})_X + \theta(1 + \|v - \widetilde{u}\|_X) \ge (f, v - \widetilde{u})_X \quad \forall\, v \in K\Big\} \quad \forall\, \theta \ge 0,\end{aligned}$$

$$\mathcal{C}_1 = \Big\{ \{\theta_n\} \,:\, 0 \le \theta_n \to 0 \Big\}.$$

We use Theorem 1 and Definition 1 to see that, with the choice $\mathcal{T}_1$ above, equivalence (2) holds. This shows that the Tykhonov triple $\mathcal{T}_1$ solves Problem $\widetilde{Q}_{\mathcal{P}_1}$. It represents an alternative to the Tykhonov triple introduced in Example 1 and has the advantage that no reference to the solution u of (9) is made in its statement.

A direct consequence of these results is the following.

Corollary 2 *Problem $\mathcal{P}_1$ is Tyknonov and Levitin-Polyak well-posed.*

Proof. We use Definition 2 to see that any approximating sequence as well as any generalized approximating sequence is a $\mathcal{T}_1$-approximating sequence. Corollary 2 is now a direct consequence of the $\mathcal{T}_1$-well posedness of Problem $\mathcal{P}_1$, guaranteed by equivalence (2). □

We end this section with two elementary examples which show that, in the study of Problem $\mathcal{P}_1$, there exist generalized approximating sequences which are not approximating sequences and $\mathcal{T}_1$-approximating sequences which are not generalized approximating sequences.

Example 2 *Consider Problem $\mathcal{P}$ in the particular case $X = \mathbb{R}$, $K = [0,1]$, $f = 1$ and note that its solution is $u = P_K f = 1$. Let $\{u_n\} \subset \mathbb{R}$ be the sequence given by $u_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ is not an approximating sequence since condition $u_n \in K$ for each $n \in \mathbb{N}$ is not satisfied. Nevertheless, $\{u_n\}$ is a generalized approximating sequence since conditions in Definition* 2 c) *hold with $w_n = -\frac{1}{n}$ and $\varepsilon_n = \frac{1}{n}$, for all $n \in \mathbb{N}$.*

Example 3 *Consider Problem $\mathcal{P}$ in the particular case $X = \mathbb{R}$, $K = [0,1]$ and $f = 2$. The solution of this problem is $u = P_K f = 1$. Let $\{u_n\} \subset \mathbb{R}$ be the sequence given by $u_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $u_n \to u$ and, therefore $\{u_n\}$ is a $\mathcal{T}_1$–approximating sequence. Nevertheless, $\{u_n\}$ is not a generalized approximating sequence. Indeed assume that there exists $0 \leq \varepsilon_n \to 0$ such that, for all $n \in \mathbb{N}$, the inequality below holds:*

$$u_n(v - u_n) + \varepsilon_n |v - u_n| \geq f(v - u_n) \qquad \forall\, v \in [0,1].$$

We now fix $n \in \mathbb{N}$, take $v = 1 - \frac{1}{2n}$ in the previous inequality and use equalities $u_n = 1 - \frac{1}{n}$, $f = 2$ to deduce that $\varepsilon_n \geq 1 + \frac{1}{n}$, which contradicts the convergence $\varepsilon_n \to 0$.

3 The fixed point problem $\mathcal{P}_2$

Unless it is specified explicitely, everywhere in this section we assume that (X,d) is a complete metric space, K a closed subset of X and $\Lambda : K \to K$. Then, the problem we consider is stated as follows.

Problem $\mathcal{P}_2$. *Find an the element u such that*

$$u \in K, \qquad \Lambda u = u. \tag{28}$$

We assume that Problem $\mathcal{P}_2$ has a unique solution. It is well-known that this is the case when Λ is a contraction, i.e.,

$$\text{there exists } \alpha \in [0,1) \text{ such that } \ d(\Lambda u, \Lambda v) \le \alpha\, d(u,v) \ \ \forall\, u,\, v \in X. \tag{29}$$

We now follow [1, 7] and recall the following definitions.

Definition 3 *A sequence $\{u_n\}$ is called an approximating sequence for Problem $\mathcal{P}_2$ if $\{u_n\} \subset K$ and $d(\Lambda u_n, u_n) \to 0$. Problem $\mathcal{P}_2$ is well-posed in the sense of Tykhonov (or, equivalently, is Tykhonov well-posed) if there exists a unique element $u \in K$ such that $\Lambda u = u$ and every approximating sequence converges to u in X.*

We now provide two elementary examples.

Example 4 *Let $X = K = \mathbb{R}$, $p \in \mathbb{N}$ and let $\Lambda u = u^{2p+1} + u - 1$ for any $u \in \mathbb{R}$. Then, Problem $\mathcal{P}_2$ is Tykhonov well-posed.*

Example 5 *Let $X = K = \mathbb{R}$ and let*

$$\Lambda u = \begin{cases} u^4 & \text{if} \quad u < 0, \\ 2 - 3u & \text{if} \quad u \ge 0. \end{cases}$$

Then, problem $\mathcal{P}_2$ is ill-posed. Indeed, the only fixed point of Λ is $u = \frac{1}{2}$, but the sequence $\{-\frac{1}{n}\}$ is an approximating sequence which does not converge to u.

Our main result in this section is the following.

Theorem 2 *Assume that* (29) *holds and let $\{u_n\} \subset X$. Then, the following statements are equivalent:*

$$u_n \to u \qquad \text{in } X. \tag{30}$$

$$\begin{cases} \text{there exists a sequence } \{v_n\} \subset K \text{ such that} \\ d(u_n, v_n) \to 0 \ \text{ and } \ d(\Lambda v_n, v_n) \to 0, \quad \text{as } n \to \infty. \end{cases} \tag{31}$$

Proof. Assume that (30) holds. Then, is easy to see that condition (31) holds with the sequence $\{v_n\}$ given by $v_n = u$, for each $n \in \mathbb{N}$.

Conversely, assume that (31) holds. We use equality $\Lambda u = u$ in (28) and assumption (29) to see that

$$d(v_n, u) \le d(v_n, \Lambda v_n) + d(\Lambda v_n, \Lambda u) \le d(v_n, \Lambda v_n) + \alpha\, d(v_n, u),$$

which implies that

$$d(v_n, u) \le \frac{1}{1-\alpha}\, d(v_n, \Lambda v_n).$$

Therefore, since

$$d(u_n, u) \le d(u_n, v_n) + d(v_n, u),$$

we deduce that

$$d(u_n, u) \le d(u_n, v_n) + \frac{1}{1-\alpha}\, d(v_n, \Lambda v_n).$$

We now use assumption (31) to obtain the convergence (30), which ends the proof. □

Note that Theorem 2 provides an answer to Problem $\mathcal{Q}_{\mathcal{P}_2}$. We now illustrate its use in order to obtain a well-known convergence result.

Corollary 3 *Let X be a normed space, $K \subset X$ a closed convex set, $\Lambda : K \to X$ a contraction, $\{a_n\} \subset [\alpha_0, 1]$ with $\alpha_0 \in (0, 1]$ and let $\{u_n\}$ be the sequence of Mann iterations defined by*

$$u_{n+1} = (1 - a_n)u_n + a_n \Lambda u_n \qquad \forall\, n \in \mathbb{N}. \tag{32}$$

Then $u_n \to u$ in X.

Proof. We use (29) to write

$$\begin{aligned} \|\Lambda u_n - u_n\|_X &\le \|\Lambda u_n - \Lambda u_{n-1}\|_X + \|\Lambda u_{n-1} - u_n\|_X \\ &\le \alpha \|u_n - u_{n-1}\|_X + \|\Lambda u_{n-1} - u_n\|_X, \end{aligned}$$

then we use (32) to substitute u_n, twice, in order to see that

$$\|\Lambda u_n - u_n\|_X \le \big(1 + (\alpha - 1)a_{n-1}\big)\|\Lambda u_{n-1} - u_{n-1}\|_X,$$

for any $n \in \mathbb{N}$. This implies that

$$\|\Lambda u_n - u_n\|_X \le \big(1 + (\alpha - 1)a_{n-1}\big) \cdots \big(1 + (\alpha - 1)a_0\big)\|\Lambda u_0 - u_0\|_X$$

for any $n \in \mathbb{N}$. Using now the inequalities $0 \leq 1 + (\alpha - 1)a_k \leq 1 + (\alpha - 1)\alpha_0$ with $k = 0, 1, 2, \ldots, n-1$ we find that

$$\|\Lambda u_n - u_n\|_X \leq \big(1 + (\alpha - 1)\alpha_0\big)^n \|\Lambda u_0 - u_0\|_X,$$

for any $n \in \mathbb{N}$. Note that $0 \leq 1 + (\alpha - 1)\alpha_0 < 1$ and, therefore, the previous inequality implies that $\|\Lambda u_n - u_n\|_X \to 0$. We now use Theorem 2 with $v_n = u_n$ to conclude the proof. □

Remark 2 *Note that for $a_n = a \in (0, 1]$ the Mann iteration* (32) *reduces to Krasnoselski iteration*

$$u_{n+1} = (1 - a)u_n + a\Lambda u_n \qquad \forall\, n \in \mathbb{N}. \tag{33}$$

and for $a = 1$ the Krasnoselski iteration (33) *reduces to Picard iteration*

$$u_{n+1} = \Lambda u_n \qquad \forall\, n \in \mathbb{N}. \tag{34}$$

We conclude from Corollary 3 *the convergence of all sequences defined by iterations* (33) *and* (34)*.*

We now turn to the $\mathcal{T}$-well-posedness of Problem $\mathcal{P}_2$ and, to this end, we consider the Tykhonov triple $\mathcal{T}_2 = (I_2, \Omega_2, \mathcal{C}_2)$ where

$$I_2 = K \times \mathbb{R}_+,$$

$$\Omega : I_2 \to 2^X, \quad \Omega(\theta) = \Big\{\, \widetilde{u} \in X \,:\, d(\widetilde{u}, v) \leq \varepsilon,\ \ d(\Lambda v, v) \leq \varepsilon \,\Big\} \quad \forall\, \theta = (v, \varepsilon) \in I_2,$$

$$\mathcal{C}_2 = \Big\{\, \{\theta_n\} = \{(v_n, \varepsilon_n)\} \subset I_2 \ :\ d(\Lambda v_n, v_n) \to 0,\ 0 \leq \varepsilon_n \to 0 \,\Big\}.$$

We use Theorem 2 and Definition 1 to see that that, with the choice $\mathcal{T}_2$ above, equivalence (2) holds. This shows that the Tykhonov triple $\mathcal{T}_2$ solves Problem $\widetilde{Q}_{\mathcal{P}_2}$. It represents an alternative to the Tykhonov triple introduced in Example 1 and has the advantage that no reference to the solution u of Problem $\mathcal{P}_2$ is made in its statement.

We end this section with a direct consequence of these results.

Corollary 4 *Problem $\mathcal{P}_2$ is Tyknonov well-posed.*

Proof. We use Definition 3 to see that any approximating sequence is a $\mathcal{T}_2$-approximating sequence. Corollary 4 is now a direct consequence of the $\mathcal{T}_2$-well posedness of Problem $\mathcal{P}_2$, guaranteed by equivalence (2). □

4 The minimization problem $\mathcal{P}_3$

Unless specified explicitly, everywhere below we assume that $(X, \|\cdot\|_X)$ is a reflexive Banach space and we use the symbol "$\rightharpoonup$" to represent the weak convergence in X. Let $K \subset X$ and $J : X \to \mathbb{R}$. Then, the problem we consider in this section is stated as follows.

Problem $\mathcal{P}_3$. Find an element u such that

$$u \in K, \qquad J(u) \le J(v) \qquad \forall\, v \in K. \tag{35}$$

We assume that Problem $\mathcal{P}_3$ has a unique solution. It is well-known that this is the case when

$$\begin{cases} K \text{ is a nonempty closed convex subset of } X \text{ and} \\ J : X \to \mathbb{R} \text{ a strongly convex lower semicontinuous function.} \end{cases} \tag{36}$$

The unique solvability of Problem $\mathcal{P}_3$ under condition (36) follows from a version of the Weierstrass theorem. Let

$$\omega = \min_{v \in K} J(v). \tag{37}$$

Then, we follow [1, 9, 19] and recall the following definitions.

Definition 4 a) *A sequence $\{u_n\}$ is called a minimizing sequence for Problem $\mathcal{P}_3$ if $\{u_n\} \subset K$ and $J(u_n) \to \omega$.*

b) *Problem $\mathcal{P}_3$ is well-posed in the sense of Tykhonov (or, equivalently, is Tykhonov well-posed) if there exists a unique element $u \in K$ such that $J(u) = \omega$ and every minimizing sequence $\{u_n\} \subset K$ converges in X to u, i.e,.*

$$J(u_n) \to \omega \quad \Longrightarrow \quad u_n \to u \quad \text{in} \quad X.$$

c) *The sequence $\{u_n\}$ is a generalized (or LP) minimizing sequence for Problem $\mathcal{P}_3$ if*

$$u_n \in X, \quad J(u_n) \to \omega, \quad d(u_n, K) \to 0.$$

d) *Problem $\mathcal{P}_3$ is well-posed in the sense of Levitin-Polyak (or, equivalently, is Levitin-Polyak well-posed) if it has a unique solution $u \in K$ and any LP-minimizing sequence $\{u_n\}$ converges in X to u.*

We now provide two elementary examples which illustrate the previous definition.

Example 6 *Let* $X = K = \mathbb{R}$ *and let* $J(u) = u^2$ *for all* $u \in \mathbb{R}$. *Then, Problem* $\mathcal{P}_3$ *(constructed with these data) is Tykhonov well-posed.*

Example 7 *Let* $X = K = \mathbb{R}$ *and let*

$$J(u) = \begin{cases} (u+1)^2 & \text{if} \quad u \leq 0, \\ u^2 & \text{if} \quad u > 0. \end{cases}$$

Then, the corresponding problem $\mathcal{P}_2$ *is ill-posed. Indeed, the only minimum point of* J *is* $u = -1$ *but the sequence* $\{\frac{1}{n}\}$ *is a minimizing sequence which does not converge to* u.

It is easy to see that every minimizing sequence for Problem $\mathcal{P}_3$ is an LP-minimizing sequence. Therefore, the Levitin-Polyak well-posedness of Problem $\mathcal{P}_3$ implies its Tykhonov well-posedness. The converse fails to be true as it follows from the following example.

Example 8 *Let* $X = \mathbb{R}^2$, $K = \mathbb{R} \times \{0\}$, $J(x,y) = x^2 - (x^4 + x)y^2$ *and consider the generalized minimizing sequence* $\{u_n\} \subset \mathbb{R}^2$ *where* $u_n = (n, \frac{1}{n})$, *for each* $n \in \mathbb{N}$. *Then, it is easy to see that the corresponding Problem* $\mathcal{P}_3$ *is Tykhonov well-posed but is not Letivin-Polyak well-posed.*

Our main result in this section is the following.

Theorem 3 *Assume* (36), *let* $\{u_n\} \subset X$ *and denote by* u *the solution of Problem* $\mathcal{P}_3$, *that is* $u \in K$ *and* $J(u) = \omega$. *Then, the following statements are equivalent :*

$$u_n \to u \qquad \text{in } X. \tag{38}$$

$$d(u_n, K) \to 0 \quad \text{and} \quad J(u_n) \to \omega, \quad \text{as } n \to \infty. \tag{39}$$

Proof. Assume (38). Then (7) and the continuity of J, guaranteed by assumption (36), imply that (39) holds.

As now that (39) holds. Then, the convergence $d(u_n, K) \to 0$ guarantees that there exists two sequences $\{v_n\} \subset X$ and $\{w_n\} \subset X$ such that

$$v_n \in K, \quad u_n = v_n + w_n \quad \forall\, n \in \mathbb{N}, \quad \|w_n\|_X \to 0 \quad \text{as} \quad n \to \infty. \tag{40}$$

Moreover,

$$J(u_n) \to J(u). \tag{41}$$

We claim that the sequence $\{v_n\}$ is bounded in X. Arguing by contradiction, if $\{v_n\}$ is not bounded then, passing to a subsequence still denoted by $\{v_n\}$, we can assume that $\|v_n\|_X \to \infty$. Using now (40) we write $\|v_n\|_X = \|u_n - w_n\|_X \leq \|u_n\|_X + \|w_n\|_X$, which implies that $\|u_n\|_X \to \infty$. Therefore, using the coercivity of J guaranteed by assumption (36), we deduce that $J(u_n) \to \infty$, which contradicts (41).

Now, since the sequence $\{v_n\}$ is bounded in X, using the reflexivity of X we deduce that there exist an element $\widetilde{u} \in X$ and a subsequence of the sequence $\{v_n\}$, again denoted by $\{v_n\}$, such that

$$v_n \rightharpoonup \widetilde{u} \quad \text{in} \quad X. \tag{42}$$

Recall that K is a closed convex subset of X. Then K is weakly closed and, since $\{v_n\} \subset K$, the convergence (42) implies that

$$\widetilde{u} \in K. \tag{43}$$

Moreover, (40) and (42) show that

$$u_n \rightharpoonup \widetilde{u} \quad \text{in} \quad X. \tag{44}$$

We now use inequality $J(u) \leq J(v)$ for all $v \in K$, regularity (43), the weak lower semicontinuity of J and the convergences (44), (41) to see that

$$J(u) \leq J(\widetilde{u}) \leq \liminf J(u_n) = J(u).$$

We conclude from there that $\widetilde{u}$ is a solution to Problem $\mathcal{P}_3$ and, by the uniqueness of the solution, we have $\widetilde{u} = u$.

A careful analysis based on the arguments above reveals the fact that any weakly convergent subsequence of the sequence $\{v_n\}$ converges to the same limit u. On the other hand, the sequence $\{v_n\}$ is bounded in X. Therefore, using a standard argument we find that the whole sequence $\{v_n\}$ converges weakly in X to u. This implies that the whole sequence $\{u_n\}$ converges weakly in X to u. Therefore, for each $n \in \mathbb{N}$ we have

$$J(u) \leq \liminf J\Big(\frac{u_n + u}{2}\Big) \leq \limsup J\Big(\frac{u_n + u}{2}\Big). \tag{45}$$

On the other hand, the convexity of J guarantees that

$$J\Big(\frac{u_n + u}{2}\Big) \leq \frac{1}{2} J(u_n) + \frac{1}{2} J(u)$$

and, using (41), we deduce that

$$\limsup J\Big(\frac{u_n+u}{2}\Big) \le J(u). \tag{46}$$

We now combine the inequalities (45) and (46) to find that

$$J\Big(\frac{u_n+u}{2}\Big) \to J(u). \tag{47}$$

Finally, by the strong convexity of J we deduce that

$$\frac{m}{4}\|u_n-u\|_X^2 \le \frac{1}{2}\Big(J(u_n)-J\Big(\frac{u_n+u}{2}\Big)\Big)+\frac{1}{2}\Big(J(u)-J\Big(\frac{u_n+u}{2}\Big)\Big)$$

with some $m>0$. We pass to the limit in this inequality and use the convergences (41) and (47) to deduce that $u_n\to u$ in X which concludes the proof. □

Note that Theorem 3 provides an answer to Problem $\mathcal{Q}_{\mathcal{P}_3}$ and, therefore, it can be used to prove various convergence results.

We now turn to the $\mathcal{T}$-well-posedness of Problem $\mathcal{P}_3$ and, to this end, we consider the Tykhonov triple $\mathcal{T}_3=(I_3,\Omega_3,\mathcal{C}_3)$ where

$$I_3=\mathbb{R}_+=[0,+\infty),$$

$$\Omega_3: I_3\to 2^X,\quad \Omega_3(\theta)=\Big\{\,\widetilde{u}\in X\ :\ d(\widetilde{u},K)\le\theta,\ |J(\widetilde{u})-\omega|\le\theta\,\Big\}\quad \forall\,\theta\ge 0,$$

$$\mathcal{C}_3=\Big\{\,\{\theta_n\}\ :\ 0\le\theta_n\to 0\,\Big\}.$$

We use Theorem 3 and Definition 1 to see that that equivalence (2) holds. This shows that, in the context of Problem $\mathcal{P}_3$, the Tykhonov triple $\mathcal{T}_3$ solves Problem $\widetilde{Q}_{\mathcal{P}_3}$. It represents an alternative to the Tykhonov triple introduced in Example 1.

We end this section with a direct consequence of these results.

Corollary 5 *Assume* (36)*. Then, Problem $\mathcal{P}_3$ is Tyknonov and Levitin-Polyak well-posed.*

Proof. We use Definition 4 to see that any minimizing sequence for Problem $\mathcal{P}_3$ is an LP-minimizing sequence for $\mathcal{P}_3$ and any LP-minimizing sequence for $\mathcal{P}_3$ is a $\mathcal{T}_3$-approximating sequence. Corollary 4 is now a direct consequence of the $\mathcal{T}_3$-well posedness of Problem $\mathcal{P}_3$. $\Box$

We end of this section with the remark that any $\mathcal{T}_3$-approximating sequence is an LP-minimizing sequence for Problem $\mathcal{P}_3$ and conversely. This proves that, in the study of Problem $\mathcal{P}_3$, the $\mathcal{T}_3$- and the Levitin-Polyak well-posedness concepts are equivalent. Therefore, Theorem 3 shows that, under assumption (36), the Levitin-Polyak well-posedness concept is an optimal well-posedness concept in the study of Problem $\mathcal{P}_3$, in the sense that a sequence $\{u_n\} \subset X$ converges to the solution of Problem $\mathcal{P}_3$ if and only if it is an LP–minimization sequence. Note that this property does not hold if we skip assumption (36). An evidence of this statement is provided by Example 8.

5 Conclusions

In this paper we were interested to convergence and well-posedness results in the study of an abstract problem $\mathcal{P}$, asumed to have a unique solution. We identified two problems related to this issues: the problem of finding a convergence criterion to the solution u of Problem $\mathcal{P}$ and the problem of finding a Tykhonov triple $\mathcal{T}$ such that the set of $\mathcal{T}$-approximating sequences coincides with the set of convergences sequences to u. We solved these problems in the particular setting of a variational inequality, a fixed point problem and a minimization problem. To this end we used various arguments, depending on the structure of each problem. Our conclusion is that the obtained results strongly depend on the framework and the assumption we use. We also presented exemples and applications in the study of the considered problems.

Our results in this paper deserve to be extended in the study of a large class of nonlinear problems $\mathcal{P}$, including elliptic and evolutionary variational inequalities, hemivariational inequalities, inclusions and saddle point, for instance. Considering such problems under various assumptions would give rise to interesting convergence and well-posedness results, with various applications in Mechanics and Enginering Sciences. A first step on this direction was made in the recent book [14].

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