Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 15, No. 1-2/2023

CONVERGENCE CRITERIA, WELL-POSEDNESS CONCEPTS AND APPLICATIONS*

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DOI https://doi.org/10.56082/annalsarscimath.2023.1-2.308

Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

We consider an abstract problem $\mathcal{P}$ in a metric space X which has a unique solution $u \in X$. Our aim in this current paper is two folds: first, to provide a convergence criterion to the solution of Problem $\mathcal{P}$, that is, to give necessary and sufficient conditions on a sequence $\{u_n\} \subset X$ which guarantee the convergence $u_n \to u$ in the space X; second, to find a Tyknonov triple $\mathcal{T}$ such that a sequence $\{u_n\} \subset X$ is a $\mathcal{T}$-approximating sequence if and only if it converges to u. The two problems stated above, associated to the original Problem $\mathcal{P}$, are closely related. We illustrate how they can be solved in three particular cases of Problem $\mathcal{P}$: a variational inequality in a Hilbert space, a fixed point problem in a metric space and a minimization problem in a reflexive Banach space. For each of these problems we state and prove a convergence criterion that we use to define a convenient Tykhonov triple $\mathcal{T}$ which requires the condition stated above. We also show how the convergence criterion and the corresponding $\mathcal{T}$-well posedness concept can be used to deduce convergence and classical well-posedness results, respectively.

MSC: 49J40, 47H09, 47H10, 49J45.

*Accepted for publication on April 25-th, 2023

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keywords: convergence criterion, Tykhonov triple, well-posedness, projection operator, contraction mapping, minimization problem, unilateral constraint.

1 Introduction

Convergence results represent an important topic in Functional Analysis, Numerical Analysis and Partial Differential Equation Theory. The continuous dependence of the solution of a partial differential equation with respect to the data, the convergence of the solution of a penalty problem to the solution of the original problem as the penalty parameter converges, the convergence of the discrete solution to the solution of the continuous problem when the time step or the discretization parameter converges to zero are some simple example, among many others. On the other hand, convergence results abound in the study of various mathematical models in Mechanics, Physics and Engineering Sciences. The convergence of the solution of a contact model with a deformable foundation to the solution of a contact model with a rigid foundation as the stiffness coefficient of the foundation converges to infinity, the convergence of the solution of a frictional problem to the solution of a frictionless problem as the coefficient of friction tends to zero represent two relevant examples, with potential real-world applications.

For all these reasons, a considerable effort was done to obtain convergence results in the study of various mathematical problems including nonlinear equations, inequality problems, inclusions, fixed point problems, optimization problems, among others. The literature in the field is extensive. The corresponding results have been obtained by using various methods and functional arguments, which differ from problem to problem and from paper to paper. Nevertheless, most of these convergence results can be casted in the abstract functional framework we describe below. Consider a mathematical object $\mathcal{P}$, called generic "problem", defined in a metric space (X, d). Problem $\mathcal{P}$ could be an equation, a minimization problem, a fixed point problem, an inclusion or an inequality problem, for instance. We associate to Problem $\mathcal{P}$ the concept of "solution" which follows from the context and we assume that $\mathcal{P}$ has a unique solution $u \in X$. Then, a convergence result is a result of the form $u_n \to u$ in X where $\{u_n\} \subset X$ represents a given sequence.

Note that in most of the cases, convergence results consists in sufficient conditions which guarantee the convergence of a specific sequence $\{u_n\}$ to the solution u of the corresponding problem $\mathcal{P}$. They do not describe all