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STRONG STATIONARITY FOR THE CONTROL OF VISCOUS HISTORY-DEPENDENT EVOLUTIONARY VIS ARISING IN APPLICATIONS*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

This paper addresses optimal control problems governed by history-dependent EVIs with viscosity. One of the prominent properties of the state system is its nonsmooth nature, so that the application of standard adjoint calculus is excluded. We extend previous results by showing that history-dependent EVIs with viscosity can be formulated as nonsmooth ODEs in Hilbert space in a general setting. The Hadamard directional differentiability of the solution map is then investigated. This allows us to establish strong stationary conditions for two different viscous damage models with fatigue.
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1 Introduction

The study of history-dependent evolutionary variational inequalities (EVIs) has attracted much attention lately, see e.g. [28, 3] (frictional contact), [29] (sweeping processes) and the references therein. Most of the papers on this topic address the existence and the regularity of solutions. For a comprehensive study of history-dependent EVIs *with viscosity* we refer to [34, Chp. 4.4] where the applications focus on total slip-dependent frictional contact problems involving viscoelastic materials [34, Chp. 10.4].

When it comes to the *optimal control* of history-dependent variational inequalities, there are only a few papers available, most of which are concerned with the existence of optimal solutions, cf. for instance [27, 37]. Because of the nonsmooth nature of EVIs, the derivation of optimality conditions is a challenging issue. Indeed, the literature on differentiability properties of EVIs is rather scarce, see [21, 12, 14] (EVIs of obstacle type) and [36, 8, 10] (viscous EVIs). We point out that these contributions do not take a history operator into account, except [10], where a concrete application is considered. To the best of our knowledge, the sensitivity analysis of viscous history-dependent EVIs in a general framework has not been examined so far, let alone the *strong stationarity* for the control thereof.

This paper aims at addressing this particular aspect. We establish strong stationary optimality conditions for the control of two damage models with fatigue. Both these models fall into the category of viscous history-dependent EVIs. In a general framework, this type of evolution is described as follows

$$\begin{aligned} &R(\mathcal{H}(y)(t), \eta) - R(\mathcal{H}(y)(t), \dot{y}(t)) + \langle \mathcal{V}\dot{y}(t), \eta - \dot{y}(t)\rangle_Y \\ &\geq \langle g(y(t), \ell(t)), \eta - \dot{y}(t)\rangle_Y \quad \forall\, \eta \in Y, \end{aligned} \tag{EVI}$$

a.e. in $(0, T)$, where Y is a Hilbert space, $\mathcal{H}$ is the history operator and $\mathcal{V}$ denotes the viscosity. In the present paper, the dissipation R may take infinite values and g is a directionally differentiable mapping. The precise assumptions on the data are stated in Assumption 7 below. The existence of solutions in a slightly less general framework has already been addressed in [34, Chp. 4.4]. In this paper we go one step further, by formulating (EVI) as a nonsmooth ODE in Hilbert space. This facilitates the investigation of the directional differentiability of the solution map associated to (EVI). By resorting to previous findings [8], we are then able to establish strong stationarity for the control of the two applications mentioned above.

From the point of view of optimal control, the essential feature of the problem under consideration is that it has a *nonsmooth* character, so that the standard methods for the derivation of qualified optimality conditions are not applicable here. The key novelties of the present paper are:

- the equivalence of (EVI) to a concrete nonsmooth ODE in Hilbert space; in this context we give an explicit formula for the underlying nonsmooth non-linearity
- sensitivity analysis for viscous *history-dependent* evolutionary VIs in a general framework
- optimal control for damage models with fatigue in terms of strong stationarity

Deriving necessary optimality conditions is a challenging issue even in finite dimensions, where a special attention is given to MPCCs (mathematical programs with complementarity constraints). In [31] a detailed overview of various optimality conditions of different strength was introduced, see also [19] for the infinite-dimensional case. The most rigorous stationarity concept is strong stationarity. Roughly speaking, the strong stationarity conditions involve an optimality system, which is equivalent to the purely primal conditions saying that the directional derivative of the reduced objective in feasible directions is nonnegative (which is referred to as B stationarity).

When it comes to establishing optimality conditions for the control of EVIs (and of nonsmooth processes in general), most of the authors resort to smoothening procedures. This approach is meanwhile standard [4] when dealing with the control of nonsmooth evolutions see e.g. [4, 5, 39, 20, 38, 7] and the references therein. The optimality systems derived in this way are of intermediate strength and are not expected to be of strong stationary type, since one always loses information when passing to the limit in the regularization scheme see e.g. [9, Rem. 3.11] and [22, Subsec 7.2]. Thus, proving strong stationarity for the optimal control of nonsmooth problems requires direct approaches, which employ the limited differentiability properties of the control-to-state map. Based on the pioneering work [25] (strong stationarity for optimal control of elliptic VIs of obstacle type), most of them focus on elliptic VIs, see e.g. [26, 41, 15] and the references therein. Regarding strong stationarity for optimal control of nonsmooth evolution processes, the literature is very scarce and the only papers known to the author addressing this issue so far are [13, 14, 8, 12] (EVIs) and [22, 8, 2] (time-dependent PDEs/ODEs). We point out that, in contrast to our problem, all the above mentioned contributions on the topic of strong stationarity for EVIs do not take a history operator into account.

Let us give an overview of the main contributions in this paper. After introducing the notation, we recall in section 2 a result from [8] concerning strong stationarity for the optimal control of nonsmooth coupled systems. Together with our main finding from section 3, this will allow us to establish strong stationarity for two concrete applications (section 4).

In section 3, we show that viscous history-dependent EVIs are nonsmooth ODEs in Hilbert space (Theorem 8). Here we extend the results from the previous work [8], where such a characterization was introduced for viscous EVIs which do not involve history. We give an explicit formula for the nonsmooth non-linearity in the ODE in terms of a projection operator (Definitions 1, 2). By contrast to [8], the nonsmoothness in the present paper has two arguments instead of one. Its concrete description provides multiple advantages. Firstly, it simplifies the solvability theory and allows us to easily examine the existence and the regularity of solutions for discontinuous right-hand sides by means of a classical fix point argument (cf. also Remark 5). Secondly, the characterization of (EVI) in terms of a nonsmooth ODE plays an essential role in the context of sensitivity analysis and optimal control. In fact, with the explicit formulation of the nonsmoothness at hand, we can state conditions on the projection operator such that the Hadamard *directional differentiability* of the solution map to (EVI) is guaranteed (Theorem 9).

Section 4 focuses on proving strong stationarity for the optimal control of two viscous gradient damage models with *fatigue*. Here we are concerned with the application of the above mentioned results. We first employ the main findings from section 3 to show that these concrete applications can be rewritten as nonsmooth ODEs, after which we make use of the result from section 2 to derive strong stationary optimality conditions. In subsection 4.1, the viscosity is expressed in terms of the H_0^1 norm and it describes the evolution of a single damage variable. In subsection 4.2, a penalization approach is employed such that the model becomes two-field. This allows us to work with L^2 viscosity. It has the advantage that the nonsmoothness appearing in the ODE is then expressed by means of the Nemytskii operator associated to $\max(\cdot, 0)$. This is not the case in the single-field model, where the underlying nonsmooth function features the projection onto a subset of $H^{-1}(\Omega)$. As we will see, the more accurate description of the damage evolution in the two-field model carries over to the associated directional differentiability and the strong stationarity conditions.

Notation

Throughout the paper, $T > 0$ is a fixed final time. If X and Y are linear normed spaces, then the space of linear and bounded operators from X to Y is denoted by $\mathcal{L}(X, Y)$, and $X \overset{d}{\hookrightarrow} Y$ means that X is densely embedded in Y. The dual space of X will be denoted by X^*, except the dual of the space $H_0^1(\Omega)$ which is denoted by $H^{-1}(\Omega)$. For the dual pairing between X and X^* we write $\langle ., . \rangle_X$. The closed ball in X around $x \in X$ with radius $\alpha > 0$ is denoted by $B_X(x, \alpha)$. If X is a Hilbert space, we write $(\cdot, \cdot)_X$ for the associated scalar product. The following abbreviations will be used throughout the paper:

$$H_0^1(0, T; X) := \{z \in H^1(0, T; X) : z(0) = 0\},$$
$$H_T^1(0, T; X) := \{z \in H^1(0, T; X) : z(T) = 0\},$$

where X is a Banach space. For the polar cone of a set $M \subset X$ we use the notation $M^\circ := \{x^* \in X^* : \langle x^*, x \rangle_X \leq 0 \quad \forall\, x \in M\}$. By χ_M we denote the characteristic function associated to the set M. Given $x \in X^*$, we denote its annihilator by $[x]^\perp := \{\mu \in X : \langle x, \mu \rangle_X = 0\}$. Derivatives w.r.t. time (weak derivatives of vector-valued functions) are frequently denoted by a dot. The symbol ∂f stands for the convex subdifferential and by $\operatorname{dom}(f)$ we denote the domain of the functional $f : X \to (-\infty, \infty]$, see e.g. [30]. For a mapping $\mathcal{R} : X \times Y \to (-\infty, \infty]$, the set $\partial_2 \mathcal{R}(x, y) \subset Y^*$ describes the convex subdifferential of the functional $\mathcal{R}(x, \cdot) : Y \to (-\infty, \infty]$ in y. The Nemystkii-operators associated with the mappings considered in this paper will be described by the same symbol, even when considered with different domains and ranges. By $\max(\cdot, 0)$ we denote the positive part function, while $\max'(x; h)$ indicates its directional derivative in the point x in direction h. Similarly, $\min(\cdot, 0)$ stands for the negative part function. With a little abuse of notation, we use in the paper the Laplace symbol for the operator $\Delta : H^1(\Omega) \to H^1(\Omega)^*$ defined by

$$\langle \Delta\eta, \psi \rangle_{H^1(\Omega)} := -\int_\Omega \nabla\eta \nabla\psi \, dx \quad \forall\, \psi \in H^1(\Omega).$$

The same symbol is used for the Laplace operator with domain $H_0^1(\Omega)$ and range $H^{-1}(\Omega)$.

2 A strong stationarity result

In this section we recall a known result [8] concerning the optimal control of nonsmooth coupled systems (Theorem 5 below). This states the strong stationarity optimality conditions for this particular type of state system and it will play an essential role later on in section 4.

$$\left.\begin{aligned} \min_{\ell \in L^2(0,T;V)} \quad & J(y,u,\ell) \\ \text{s.t.} \quad & \dot{y}(t) = f(\Phi(y,u)(t)) \quad \text{a.e. in } (0,T), \quad y(0) = 0, \\ & \Psi(y,u)(t) = \ell(t) \quad \text{a.e. in } (0,T), \\ & y \in H^1(0,T;Y), \quad u \in L^2(0,T;U). \end{aligned}\right\} \tag{1}$$

We begin by gathering all the necessary assumptions that are needed for the main result in Theorem 5 to be true.

Assumption 1 ([8, Assumption 2.1])**.** *For the quantities in* (1) *we require the following:*

1. *V, Y, and U are real reflexive Banach spaces, such that $V \overset{d}{\hookrightarrow} U^*$.*
2. *The mappings $\Phi : L^2(0,T;Y \times U) \to L^2(0,T;Y^*)$ and $\Psi : L^2(0,T;Y \times U) \to L^2(0,T;U^*)$ are Gâteaux-differentiable operators.*
3. *The nonsmooth function $f : Y^* \to Y$ is assumed to be Lipschitz continuous and directionally differentiable, i.e.,*
$$\left\| \frac{f(x+\tau h) - f(x)}{\tau} - f'(x;h) \right\|_Y \overset{\tau \searrow 0}{\longrightarrow} 0 \quad \forall\, x,h \in Y^*.$$
4. *The objective $J : L^2(0,T;Y) \times L^2(0,T;U) \times L^2(0,T;V) \to \mathbb{R}$ is Fréchet-differentiable.*

Remark 1. *Note that in contrast to [8, Assump. 2.1.2], we work with operators Φ and Ψ mapping between abstract function spaces, so that these operators are not necessary Nemytskii operators as in [8, Sec. 2]. This allows us to apply the findings in this section to applications which feature e.g. integral operators such as history operators, cf. Assumption 10.1 below. A short inspection of [8, Sec. 2] shows that the entire analysis can be carried on in the same manner for our slightly more general setting without affecting the main result in Theorem 5 below.*

As in [8, Sec. 2], the properties we need from the control-to-state map in order to prove the main result (Theorem 5) are just assumed to be true. To keep the demonstration concise, we do not discuss the unique solvability of the state system nor its differentiability properties. These issues will be addressed in detail for the applications considered in section 4 below.

Assumption 2 (Control-to-state operator, [8, Assumption 2.3])**.** *Throughout this section, we assume that*

1. *For every* $\ell \in L^2(0,T;V)$*, the state equation*

$$\left.\begin{array}{l}\dot{y} = f(\Phi(y,u)) \quad a.e.\ in\ (0,T),\ y(0)=0,\\ \Psi(y,u) = \ell \quad a.e.\ in\ (0,T)\end{array}\right\} \tag{2}$$

admits a unique solution $(y,u) \in H_0^1(0,T;Y) \times L^2(0,T;U)$ *and denote the associated solution operator by*

$$\mathcal{S} : L^2(0,T;V) \ni \ell \mapsto (y,u) \in H_0^1(0,T;Y) \times L^2(0,T;U).$$

2. *The mapping* $\mathcal{S} : L^2(0,T;V) \to L^2(0,T;Y) \times L^2(0,T;U)$ *is directionally differentiable, i.e.,*

$$\Big\| \frac{\mathcal{S}(\ell + \tau\,\delta\ell) - \mathcal{S}(\ell)}{\tau} - \mathcal{S}'(\ell;\delta\ell) \Big\|_{L^2(0,T;Y)\times L^2(0,T;U)} \stackrel{\tau\searrow 0}{\longrightarrow} 0$$
$$\forall\ \ell, \delta\ell \in L^2(0,T;V).$$

Moreover, we suppose that for any $\ell, \delta\ell \in L^2(0,T;V)$*, the pair* $(\delta y, \delta u) := \mathcal{S}'(\ell;\delta\ell) \in H_0^1(0,T;Y) \times L^2(0,T;U)$ *is the unique solution of*

$$\dot{\delta y} = f'\big(\Phi(y,u); \Phi'(y,u)(\delta y, \delta u)\big)\ a.e.\ in\ (0,T),\ \delta y(0) = 0, \tag{3a}$$
$$\Psi'(y,u)(\delta y, \delta u) = \delta\ell \quad a.e.\ in\ (0,T), \tag{3b}$$

where we abbreviate $(y,u) := \mathcal{S}(\ell)$*.*

3. *For any* $\ell \in L^2(0,T;V)$*, there exists a constant* $K > 0$ *so that*

$$\|\mathcal{S}'(\ell;\delta\ell)\|_{L^2(0,T;Y)\times L^2(0,T;U)} \le K\,\|\delta\ell\|_{L^2(0,T;U^*)} \quad \forall\,\delta\ell \in L^2(0,T;V).$$

If $(\hat{\delta y}, \hat{\delta u}) \in H_0^1(0,T;Y) \times L^2(0,T;U)$ *solves* (3) *with r.h.s.* $\hat{\delta\ell} \in L^2(0,T;U^*)$ *and if there exists a sequence* $\{\delta\ell_n\}_n \subset L^2(0,T;V)$ *with* $\delta\ell_n \to \hat{\delta\ell}$ *in* $L^2(0,T;U^*)$*, then* $\mathcal{S}'(\ell;\delta\ell_n) \to (\hat{\delta y}, \hat{\delta u})$ *in* $L^2(0,T;Y) \times L^2(0,T;U)$*.*

Throughout this section one tacitly assumes that Assumptions 1 and 2 always hold true, without mentioning them everytime.

Lemma 1 (B-stationarity, [8, Lemma 2.5])**.** *If* $\bar{\ell} \in L^2(0,T;V)$ *is locally optimal for* (1)*, then there holds*

$$\partial_{(y,u)} J(\mathcal{S}(\bar{\ell}), \bar{\ell})\mathcal{S}'(\bar{\ell};\delta\ell) + \partial_\ell J(\mathcal{S}(\bar{\ell}), \bar{\ell})\delta\ell \ge 0 \quad \forall\,\delta\ell \in L^2(0,T;V). \tag{4}$$

Assumption 3 ([8, Assumption 2.6]). *For any local optimum $\bar{\ell}$ of* (1)*, we assume that the range of $\partial_u \Phi(\bar{y}, \bar{u})$ is dense in $L^2(0,T;Y^*)$, where $(\bar{y}, \bar{u}) := \mathcal{S}(\bar{\ell})$.*

Assumption 4 ([8, Assumption 2.9]). *For any local optimum $\bar{\ell}$ of* (1)*, we assume that there exists $\lambda \in L^2(0,T;Y)$ so that*

$$-\partial_u \Phi(\bar{y}, \bar{u})^* \lambda = \partial_u J(\bar{y}, \bar{u}, \bar{\ell}) + \partial_u \Psi(\bar{y}, \bar{u})^* \partial_\ell J(\bar{y}, \bar{u}, \bar{\ell}),$$

where $(\bar{y}, \bar{u}) := \mathcal{S}(\bar{\ell})$.

Theorem 5 (Strong Stationarity, [8, Theorem 2.11]). *Suppose that Assumptions 3 and 4 are satisfied. Let $\bar{\ell} \in L^2(0,T;V)$ be locally optimal for* (1) *with associated state $(\bar{y}, \bar{u}) := \mathcal{S}(\bar{\ell})$. Then, there exist unique adjoint states*

$$\xi \in H^1_T(0,T;Y^*) \quad \text{and} \quad w \in L^2(0,T;U)$$

and a unique multiplier $\lambda \in L^2(0,T;Y)$ such that the following system is satisfied

$$-\dot{\xi} - \partial_y \Phi(\bar{y}, \bar{u})^* \lambda + \partial_y \Psi(\bar{y}, \bar{u})^* w = \partial_y J(\bar{y}, \bar{u}, \bar{\ell}) \quad \text{in } L^2(0,T;Y^*), \quad \xi(T) = 0, \tag{5a}$$

$$-\partial_u \Phi(\bar{y}, \bar{u})^* \lambda + \partial_u \Psi(\bar{y}, \bar{u})^* w = \partial_u J(\bar{y}, \bar{u}, \bar{\ell}) \quad \text{in } L^2(0,T;U^*), \tag{5b}$$

$$\langle \xi(t), f'(\Phi(\bar{y}, \bar{u})(t); v) \rangle_Y \geq \langle \lambda(t), v \rangle_{Y^*} \quad \forall\, v \in Y^*, \text{ a.e. in } (0,T), \tag{5c}$$

$$w + \partial_\ell J(\bar{y}, \bar{u}, \bar{\ell}) = 0 \quad \text{in } L^2(0,T;U). \tag{5d}$$

Theorem 6 (Equivalence between B- and strong stationarity, [8, Theorem 2.13]). *Assume that $\bar{\ell} \in L^2(0,T;V)$ together with its states $(\bar{y}, \bar{u}) \in H^1_0(0,T;Y) \times L^2(0,T;U)$, some adjoint states $(\xi, w) \in H^1_T(0,T;Y^*) \times L^2(0,T;U)$, and a multiplier $\lambda \in L^2(0,T;Y)$ satisfy the optimality system* (5a)–(5d)*. Then, it also satisfies the variational inequality* (4)*. If Assumptions 3 and 4 hold true,* (4) *is equivalent to* (5a)–(5d)*.*

3 Formulation of viscous history-dependent EVIs as nonsmooth ODEs

This section focuses on proving that the following viscous history-dependent evolution

$$\begin{aligned} &R(\mathcal{H}(y)(t), \eta) - R(\mathcal{H}(y)(t), \dot{y}(t)) + \langle \mathcal{V}\dot{y}(t), \eta - \dot{y}(t) \rangle_Y \\ &\geq \langle g(y(t), \ell(t)), \eta - \dot{y}(t) \rangle_Y \quad \forall\, \eta \in Y, \end{aligned} \tag{EVI}$$

a.e. in $(0,T)$, is equivalent to a nonsmooth ODE in the Hilbert space Y, cf. Theorem 8 below.

If the dependency of the dissipation R on the history operator $\mathcal{H}$ is dropped, (EVI) is just the viscous EVI from [8, Sec. 3]. By contrast to [8], the nonsmooth non-linearity $\mathcal{F}$ (Definition 2 below) appearing in our ODE (9) has two arguments (ζ,ω) such that for each ζ, $\mathcal{F}(\zeta)$ is the solution operator of an elliptic VI of the second kind, cf. (8). This can be described by means of an explicit formula featuring the projection operator [8, Sec. 3]. Such a formula allows us to state conditions under which the directional differentiability of the solution map associated to (EVI) is guaranteed (Theorem 9 below).

In all what follows, $\ell \in L^2(0,T;Z)$ is fixed. Here, Z is a real reflexive Banach space, while Y is a real Hilbert space.

Assumption 7. *For the operators in* (EVI) *we require:*

1. *The nonsmooth functional $R : X \times Y \to (-\infty,\infty]$ has the following properties:*
   - (a) *For each $\zeta \in X$, $R(\zeta,\cdot)$ is proper, convex, lower semicontinuous and positively homogeneous, i.e., $R(\zeta,\alpha\eta) = \alpha R(\zeta,\eta)$ for all $\alpha > 0$ and all $\eta \in Y$.*
   - (b) *There exists $L_R \geq 0$ such that*

$$\begin{aligned} R(\zeta_1,\eta_2)-R(\zeta_1,\eta_1) + R(\zeta_2,\eta_1) - R(\zeta_2,\eta_2) \\ \leq L_R \|\zeta_1-\zeta_2\|_X \|\eta_1-\eta_2\|_Y \\ \forall\, \zeta_1,\zeta_2 \in X,\ \forall\, \eta_1 \in \operatorname{dom} R(\zeta_1,\cdot), \eta_2 \in \operatorname{dom} R(\zeta_2,\cdot). \end{aligned}$$

2. *The history operator $\mathcal{H} : L^2(0,T;Y) \to L^2(0,T;X)$ satisfies*

$$\|\mathcal{H}(\eta_1)(t) - \mathcal{H}(\eta_2)(t)\|_X \leq L_{\mathcal{H}} \int_0^t \|\eta_1(s)-\eta_2(s)\|_Y\, ds \quad \text{a.e. in } (0,T),$$

   for all $\eta_1,\eta_2 \in L^2(0,T;Y)$, where $L_{\mathcal{H}} > 0$ is a positive constant. Moreover, $\mathcal{H} : L^2(0,T;Y) \to L^2(0,T;X)$ is supposed to be directionally differentiable.

3. *The viscosity operator $\mathcal{V} \in \mathcal{L}(Y,Y^*)$ is coercive, i.e., there exists $\vartheta > 0$ so that $\langle \mathcal{V}\eta,\eta\rangle_Y \geq \vartheta\|\eta\|_Y^2$ for all $\eta \in Y$. Moreover, $\mathcal{V}$ is self-adjoint, i.e., $\langle \mathcal{V}\eta, y\rangle_Y = \langle \mathcal{V}y,\eta\rangle_Y$ for all $\eta, y \in Y$.*

4. *The mapping $g : Y \times Z \to Y^*$ is directionally differentiable and Lipschitz continuous with Lipschitz constant $L_g > 0$.*

Remark 2. *The requirement in Assumption 7.1b corresponds to [34, (3.54)]. Note that this condition is needed in the proof of [34, Thm. 4.9, page 72] as well, in order to be able to apply a fix point argument, which in turn leads to the existence of a unique solution for* (EVI).
Assumption 7.2 is satisfied by the Volterra operator $\mathcal{H} : L^2(0,T;Y) \to C([0,T];X)$*, defined as*

$$[0,T] \ni t \mapsto \mathcal{H}(y)(t) := \int_0^t A(t-s)y(s)\,ds + y_0 \in X,$$

where $A \in C([0,T];\mathcal{L}(Y,X))$ *and* $y_0 \in X$*. This type of operator is often employed in the study of history-depedent evolutionary variational inequalities, see e.g.[34, Ch. 4.4].*

3.1 Preliminaries

In the sequel, Assumption 7 is tacitly assumed, without mentioning it every time. Note that, in view of Assumption 7.3, the operator $\mathcal{V}$ induces a norm on Y, which will be denoted by $\|\cdot\|_{\mathcal{V}} := \sqrt{\langle \mathcal{V}\cdot, \cdot\rangle_Y}$. Similarly, the operator $\mathcal{V}^{-1}$ induces a norm on Y^*, which we abbreviate $\|\cdot\|_{\mathcal{V}^{-1}} := \sqrt{\langle \mathcal{V}^{-1}\cdot, \cdot\rangle_{Y^*}}$ in the following. We remark that $\|\cdot\|_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{V}^{-1}}$ are equivalent to $\|\cdot\|_Y$ and $\|\cdot\|_{Y^*}$, respectively.

Definition 1 (The projection operator)**.** *Let us define the function* $B : X \times Y^* \to Y^*$ *as*

$$B(\zeta,\omega) := P_{\partial_2 R(\zeta,0)}\omega,$$

where, for each $\zeta \in X$*, the operator* $P_{\partial_2 R(\zeta,0)} : Y^* \to Y^*$ *is the (metric) projection onto the set*

$$\partial_2 R(\zeta,0) = \{\varphi \in Y^* : \langle \varphi, v\rangle_Y \leq R(\zeta,v) \quad \forall\, v \in Y\}$$

w.r.t. the inner product $\langle \mathcal{V}^{-1}\cdot, \cdot\rangle_{Y^*}$*, i.e.,* $P_{\partial_2 R(\zeta,0)}\omega$ *is the unique solution of*

$$\min_{\mu \in \partial_2 R(\zeta,0)} \frac{1}{2}\|\omega - \mu\|^2_{\mathcal{V}^{-1}} \tag{6}$$

for any $\omega \in Y^*$*.*

Remark 3. *Note that the projection operator in Definition 1 is well-defined. Indeed, since* $R(\zeta,\cdot)$ *is proper, convex and lower semicontinuous, cf. Assumption 7.1a, the subdifferential* $\partial_2 R(\zeta,0)$ *is a non-empty, convex and closed set, see for instance [6]. Hence,* (6) *admits a solution. Its uniqueness is due to the strict convexity of the norm squared.*

Definition 2 (The nonsmooth non-linearity). *Let us define the function* $\mathcal{F} : X \times Y^* \to Y$ *as*

$$\mathcal{F}(\zeta, \omega) := \mathcal{V}^{-1}(\omega - B(\zeta, \omega)). \tag{7}$$

Lemma 2. *For each* $\zeta \in X$, *the mapping* $\mathcal{F}(\zeta, \cdot) : Y^* \ni \omega \mapsto z \in Y$ *is the solution operator of the following elliptic VI of the second kind*

$$R(\zeta, \eta) - R(\zeta, z) + \langle \mathcal{V} z, \eta - z \rangle_Y \geq \langle \omega, \eta - z \rangle_Y \quad \forall\, \eta \in Y. \tag{8}$$

Thus, (8) *is equivalent to* $z = \mathcal{F}(\zeta, \omega) = \mathcal{V}^{-1}(\omega - B(\zeta, \omega))$ *for any* $(\zeta, \omega) \in X \times Y^*$.

Proof. The result follows by applying [8, Lemma 3.3] for $R(\zeta, \cdot)$ for each $\zeta \in X$. □

Lemma 3 (Lipschitz continuity of $\mathcal{F}$). *The function* $\mathcal{F} : X \times Y^* \to Y$ *is Lipschitz continuous with Lipschitz constant* $L_{\mathcal{F}} := \frac{\max\{1, L_R\}}{\vartheta}$.

Proof. Let $(\zeta_1, \omega_1), (\zeta_2, \omega_2) \in X \times Y^*$ be arbitrary but fixed and let us abbreviate $z_i := \mathcal{F}(\zeta_i, \omega_i)$, $i = 1, 2$. According to Lemma 2, z_i, $i = 1, 2$, solves the VI

$$R(\zeta_i, \eta) - R(\zeta_i, z_i) + \langle \mathcal{V} z_i, \eta - z_i \rangle_Y \geq \langle \omega_i, \eta - z_i \rangle_Y \quad \forall\, \eta \in Y.$$

Testing with z_j, $j \neq i$ and adding the resulting inequalities leads to

$$\begin{aligned} \langle \mathcal{V}(z_2 - z_1), z_2 - z_1 \rangle_Y &\leq \langle \omega_2 - \omega_1, z_2 - z_1 \rangle_Y \\ &\quad + R(\zeta_1, z_2) - R(\zeta_1, z_1) + R(\zeta_2, z_1) - R(\zeta_2, z_2) \\ &\leq \|\omega_2 - \omega_1\|_{Y^*} \|z_2 - z_1\|_Y + L_R \|\zeta_2 - \zeta_1\|_X \|z_2 - z_1\|_Y, \end{aligned}$$

where the last inequality is due to Assumption 7.1b; note that $z_i \in \operatorname{dom} R(\zeta_i, \cdot)$, $i = 1, 2$, as a result of Lemma 2. Now, the coercivity of $\mathcal{V}$, see Assumption 7.3, yields the desired assertion. □

As an immediate consequence of Lemma 2 and Assumption 7.3, we have the following

Corollary 1. *The solution operator* $\mathcal{F} : X \times Y^* \ni (\zeta, \omega) \mapsto z \in Y$ *of* (8) *is directionally differentiable at* $(\bar{\zeta}, \bar{\omega}) \in X \times Y^*$ *if and only if the mapping* $B : (\zeta, \omega) \mapsto P_{\partial_2 R(\zeta, 0)} \omega$ *is directionally differentiable at* $(\bar{\zeta}, \bar{\omega}) \in X \times Y^*$. *If this is the case, then*

$$\mathcal{F}'((\bar{\zeta}, \bar{\omega}); (\delta\zeta, \delta\omega)) = \mathcal{V}^{-1}\Big(\delta\omega - B'((\bar{\zeta}, \bar{\omega}); (\delta\zeta, \delta\omega))\Big) \quad \forall\, (\delta\zeta, \delta\omega) \in X \times Y^*.$$

Remark 4. *A criterion for the directional differentiability of the mapping B can be formulated in terms of polyhedricity [25, Thm. 2.1], cf. also Lemma 5 below. In Section 4, we rewrite B as a projection on a fixed set, i.e., a set independent of ζ, see Lemma 6 below. With the polyhedricity of the fixed set at hand, the directional differentiability of B is ensured by [25, Thm. 2.1].*

3.2 Main results

Theorem 8 (Viscous history-dependent EVIs are nonsmooth ODEs)**.** *The viscous history-dependent problem* (EVI) *is equivalent to the following ODE in Hilbert space*

$$\dot{y} = \mathcal{F}(\mathcal{H}(y), g(y,\ell)) \quad \text{a.e. in } (0,T), \tag{9}$$

where $\mathcal{F}$ is given by (7) *and $\ell : [0,T] \to Z$. If $y(0) = y_0$, $y_0 \in Y$, then* (EVI) *admits a unique solution $y \in H^1(0,T;Y)$ for every right-hand side $\ell \in L^2(0,T;Z)$.*

Proof. The first assertion is due to Lemma 2. To solve (9), we apply a fixed-point argument. For this, we take a look at the mapping

$$C([0,T];Y) \ni \eta \mapsto \mathcal{G}(\eta) \in H^1(0,T;Y) \hookrightarrow C([0,T];Y)$$

given by

$$\mathcal{G}(\eta)(\tau) := y_0 + \int_0^\tau \mathcal{F}(\mathcal{H}(\eta), g(\eta,\ell))(s)\, ds \quad \forall\, \tau \in [0,T].$$

For all $\eta_1, \eta_2 \in C([0,T];Y)$ the following estimate is true

$$\begin{aligned}
&\|\mathcal{G}(\eta_1)(\tau) - \mathcal{G}(\eta_2)(\tau)\|_Y \\
&\leq L_{\mathcal{F}} \int_0^\tau \|\mathcal{H}(\eta_1)(s) - \mathcal{H}(\eta_2)(s)\|_X + \|g(\eta_1(s),\ell(s)) - g(\eta_2(s),\ell(s))\|_{Y^*}\, ds \\
&\leq L_{\mathcal{F}} \int_0^\tau L_{\mathcal{H}} \int_0^s \|\eta_1(\zeta) - \eta_2(\zeta)\|_Y\, d\zeta + L_g\, \|\eta_1(s) - \eta_2(s)\|_Y\, ds \\
&\leq (L_{\mathcal{F}}\, L_{\mathcal{H}} T + L_{\mathcal{F}}\, L_g) \int_0^\tau \|\eta_1(s) - \eta_2(s)\|_Y\, ds \qquad \text{for all } \tau \in [0,T].
\end{aligned} \tag{10}$$

Here we used the fact that $\mathcal{F} : X \times Y^* \to Y$ is Lipschitzian according to Lemma 3, as well as Assumptions 7.2 and 7.4. From [34, Lem. 1.42] we now deduce that $\mathcal{G}$ has a unique fix point y in $C([0,T];Y)$. As a consequence of $\mathcal{G}(y) = y$, we have that $y \in H^1(0,T;Y)$ is the unique solution of the ODE (9) with initial condition $y(0) = y_0$. □

Lemma 4. *For all* $\eta, \delta\eta_1, \delta\eta_2 \in L^2(0,T;Y)$*, it holds*

$$\|\mathcal{H}'(\eta;\delta\eta_1)(t) - \mathcal{H}'(\eta;\delta\eta_2)(t)\|_X \leq L_{\mathcal{H}} \int_0^t \|\delta\eta_1(s) - \delta\eta_2(s)\|_Y \, ds \tag{11}$$

a.e. in $(0,T)$.

Proof. We observe that, in view of Assumption 7.2, it holds

$$\frac{1}{\tau}\|\mathcal{H}(\eta+\tau\delta\eta_1)(t) - \mathcal{H}(\eta+\tau\delta\eta_2)(t)\|_X \leq L_{\mathcal{H}} \int_0^t \|\delta\eta_1(s) - \delta\eta_2(s)\|_Y \, ds$$

a.e. in $(0,T)$, for all $\eta, \delta\eta_1, \delta\eta_2 \in L^2(0,T;Y)$ and all $\tau > 0$. Passing to the limit $\tau \searrow 0$, where one uses the directional differentiability of $\mathcal{H}$ and the fact that convergence in $L^2(0,T;X)$ implies a.e. in convergence in X for a subsequence, then yields the desired estimate. □

Theorem 9 (Hadamard directional differentiability)**.** *The solution map* $\mathcal{S} : L^2(0,T;Z) \ni \ell \mapsto y \in H_0^1(0,T;Y)$ *associated to* (EVI) *is Hadamard directionally differentiable [32, Def. 3.1.1] at* $\bar{\ell} \in L^2(0,T;Z)$*, if* $\mathcal{F} : X \times Y^* \to Y$ *is directionally differentiable at* $(\mathcal{H}(\bar{y})(t), g(\bar{y}(t), \bar{\ell}(t)))$ *f.a.a.* $t \in (0,T)$ *or, equivalently, if* $B : X \times Y^* \to Y^*$ *does so, where we abbreviate* $\bar{y} := \mathcal{S}(\bar{\ell})$*. Its directional derivative* $\delta y := \mathcal{S}'(\bar{\ell};\delta\ell)$ *at* $\bar{\ell}$ *in direction* $\delta\ell \in L^2(0,T;Z)$ *is the unique solution of*

$$\begin{aligned} &\dot{\delta y} = \mathcal{F}'\Big((\mathcal{H}(\bar{y}), g(\bar{y},\bar{\ell})); \big(\mathcal{H}'(\bar{y};\delta y), g'(\bar{y},\bar{\ell});(\delta y, \delta\ell)\big)\Big) \quad \text{a.e. in } (0,T), \\ &\delta y(0) = 0. \end{aligned} \tag{12}$$

Proof. By arguing as in the proof of Theorem 8, we get that, for any $\delta\ell \in L^2(0,T;Z)$, (12) admits a unique solution $\delta y \in H_0^1(0,T;Y)$. Note that in this case we rely on the Lipschitz continuity of the directional derivatives of $\mathcal{F}$ and g w.r.t. direction and on the estimate (11). Further, since $\mathcal{F}$ is Lipschitzian according to Lemma 3, we can apply Lebesgue's dominated convergence theorem to obtain that

$$\mathcal{F} : L^2(0,T;X) \times L^2(0,T;Y^*) \to L^2(0,T;Y)$$

is directionally differentiable at $(\mathcal{H}(\bar{y}), g(\bar{y},\bar{\ell}))$. Moreover, by relying again on Lemma 3, we obtain that $\mathcal{F}$ is even Hadamard directionally differentiable at $(\mathcal{H}(\bar{y}), g(\bar{y},\bar{\ell}))$ [32, Def. 3.1.1], as a result of [32, Lem. 3.1.2(b)]. Since

$$f : L^2(0,T;Y) \times L^2(0,T;Z) \ni (y,\ell) \mapsto (\mathcal{H}(y), g(y,\ell)) \in L^2(0,T;X) \times L^2(0,T;Y^*)$$

is directionally differentiable, by Assumptions 7.2 and 7.4, chain rule [33, Prop. 3.6(i)] implies that

$$\widehat{G} := \mathcal{F} \circ f$$

is (Hadamard) directionally differentiable at $(\bar{y}, \bar{\ell})$ with

$$\begin{aligned}\widehat{G}'((\bar{y},\bar{\ell}); h) &= \mathcal{F}'\Big((\mathcal{H}(\bar{y}), g(\bar{y},\bar{\ell})); \big(\mathcal{H}'(\bar{y}; h_1), g'((\bar{y},\bar{\ell}); h)\big)\Big) \\ &\forall\, h = (h_1, h_2) \in L^2(0,T; Y \times Z). \end{aligned} \tag{13}$$

For simplicity, in the following we abbreviate $\bar{y}^\tau := \mathcal{S}(\bar{\ell} + \tau\,\delta\ell)$, where $\tau > 0$ and $\delta\ell \in L^2(0,T;Z)$ are arbitrary, but fixed. Due to (13) and by combining the equations for $\bar{y}^\tau$, $\bar{y}$ and (12), we obtain

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\bar{y}^\tau - \bar{y}}{\tau} - \delta y\Big) &= \frac{\widehat{G}(\bar{y}^\tau, \bar{\ell} + \tau\,\delta\ell) - \widehat{G}(\bar{y},\bar{\ell})}{\tau} \\ &\qquad - \widehat{G}'\big((\bar{y},\bar{\ell}); (\delta y, \delta\ell)\big) \quad \text{a.e. in } (0,T), \\ \Big(\frac{\bar{y}^\tau - \bar{y}}{\tau} - \delta y\Big)(0) &= 0. \end{aligned} \tag{14}$$

This implies

$$\begin{aligned} &\Big\|\Big(\frac{\bar{y}^\tau - \bar{y}}{\tau} - \delta y\Big)(t)\Big\|_Y \\ &\quad \le \int_0^t \Big\|\frac{\widehat{G}(\bar{y}^\tau, \bar{\ell} + \tau\,\delta\ell)(s) - \widehat{G}\big((\bar{y},\bar{\ell}) + \tau(\delta y, \delta\ell)\big)(s)}{\tau}\Big\|_Y\, ds \\ &+ \int_0^t \underbrace{\Big\|\frac{\widehat{G}\big((\bar{y},\bar{\ell}) + \tau(\delta y, \delta\ell)\big)(s) - \widehat{G}(\bar{y},\bar{\ell})(s)}{\tau} - \widehat{G}'\big((\bar{y},\bar{\ell}); (\delta y, \delta\ell)\big)(s)\Big\|_Y}_{=: A_\tau(s)}\, ds \\ &\quad \le \frac{L_{\mathcal{F}}}{\tau} \int_0^t \Big\| f(\bar{y}^\tau, \bar{\ell} + \tau\,\delta\ell)(s) - f\big((\bar{y},\bar{\ell}) + \tau(\delta y, \delta\ell)\big)(s)\Big\|_{X \times Y^*} \\ &\qquad + \int_0^t A_\tau(s)\, ds \qquad \forall\, t \in [0,T], \end{aligned} \tag{15}$$

where we employed the definitions of $\widehat{G}$ and the fact that $\mathcal{F} : X \times Y^* \to Y$ is Lipschitzian according to Lemma 3. In view of the definition of f, (16) can be continued as

$$\begin{aligned}
&\Big\|\Big(\frac{\bar y^\tau-\bar y}{\tau}-\delta y\Big)(t)\Big\|_Y\\
&\quad\le \frac{L_{\mathcal F}}{\tau}\int_0^t \|\mathcal H(\bar y^\tau)(s)-\mathcal H(\bar y+\tau\delta y)(s)\|_X\,ds\\
&\qquad+\frac{L_{\mathcal F}}{\tau}\int_0^t \|g(\bar y^\tau(s),(\bar\ell+\tau\,\delta\ell)(s))-g((\bar y+\tau\delta y)(s),(\bar\ell+\tau\,\delta\ell)(s))\|_{Y^*}\,ds\\
&\qquad+\int_0^t A_\tau(s)\,ds\qquad \forall\, t\in[0,T],
\end{aligned}\tag{16}$$

Using Assumptions 7.2 and 7.4 gives in turn

$$\begin{aligned}
\Big\|\Big(\frac{\bar y^\tau-\bar y}{\tau}-\delta y\Big)(t)\Big\|_Y &\le \frac{L_{\mathcal F}}{\tau}\int_0^t L_{\mathcal H}\int_0^s \|\bar y^\tau(\zeta)-\bar y(\zeta)-\tau\delta y(\zeta)\|_Y\,d\zeta\\
&+L_g\,\|\bar y^\tau(s)-\bar y(s)-\tau\delta y(s)\|_Y\,ds+\int_0^t A_\tau(s)\,ds\\
&\le (T\,L_{\mathcal F}L_{\mathcal H}+L_g)\int_0^t \Big\|\frac{\bar y^\tau(s)-\bar y(s)-\tau\delta y(s)}{\tau}\Big\|_Y\,ds\\
&\quad+\int_0^t A_\tau(s)\,ds\ \ \forall\, t\in[0,T].
\end{aligned}$$

Applying Gronwall's inequality then yields

$$\Big\|\Big(\frac{\bar y^\tau-\bar y}{\tau}-\delta y\Big)(t)\Big\|_Y\le c\int_0^t A_\tau(s)\,ds\qquad \forall\, t\in[0,T],\tag{17}$$

where $c>0$ is a constant dependent only on the given data. Now, (14) and estimating as in (16), in combination with (17), leads to

$$\Big\|\frac{\bar y^\tau-\bar y}{\tau}-\delta y\Big\|_{H^1(0,T;Y)}\le c\,\|A_\tau\|_{L^2(0,T)}\quad \forall\,\tau>0.\tag{18}$$

On the other hand, we recall the definition of A_τ in (16) and the fact that $\widehat G: L^2(0,T;Y)\times L^2(0,T;Z)\to L^2(0,T;Y)$ is directionally differentiable at $(\bar y,\bar\ell)$, from which we deduce

$$\|A_\tau\|_{L^2(0,T)}\to 0\quad \text{as } \tau\searrow 0.$$

Finally, the desired assertion follows from (18). The Hadamard directional differentiability [32, Def. 3.1.1] is due to Proposition 1 and [32, Lem. 3.1.2(b)]. This completes the proof. □

Remark 5. *All the results established in this subsection are valid for right-hand sides $\ell \in L^p(0,T;Z)$, where $1 \leq p \leq \infty$, in which case the unique solution to* (9) *belongs to $W_0^{1,p}(0,T;Y)$ (provided that the history operator $\mathcal{H}$ maps between $L^p(0,T)$ spaces).*

4 Strong stationarity for the control of viscous damage models with fatigue

Based on the results from the previous sections, we next derive strong stationary optimality conditions for the control of two viscous damage models with fatigue.

The underlying *non-viscous* damage problem with fatigue reads as follows:

$$-\partial_q \mathcal{E}(t, q(t)) \in \partial_2 \mathcal{R}(H(q)(t), \dot{q}(t)) \quad \text{in } H^1(\Omega)^*, \quad q(0) = 0 \tag{19}$$

a.e. in $(0,T)$, cf [1]. Here, $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, is a bounded Lipschitz domain. In (19), $\mathcal{E} : [0,T] \times H^1(\Omega) \to \mathbb{R}$ is the stored energy; this will be specified in the upcoming subsections, depending on the setting. The non-viscous dissipation $\mathcal{R} : L^2(\Omega) \times H^1(\Omega) \to (-\infty, \infty]$ is defined as

$$\mathcal{R}(\zeta, \eta) := \begin{cases} \displaystyle\int_\Omega \kappa(\zeta)\, \eta \, dx, & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{20}$$

The differential inclusion appearing in (19) describes the evolution of the damage variable q under *fatigue* effects. Therein, H is a so-called *history operator* that models how the damage experienced by the material affects its fatigue level. The fatigue degradation mapping $\kappa : \mathbb{R} \to \mathbb{R}$ appearing in (20) indicates in which measure the fatigue affects the fracture toughness of the material. Whereas usually the toughness of the material is described by a fixed (nonnegative) constant [18, 17], in the present model it changes at each point in time and space, depending on $H(q)$. To be more precise, the value of the fracture toughness of the body at (t,x) is given by $\kappa(H(q))(t,x)$, cf. (20). Hence, the model (19) takes into account the following crucial aspect: the occurrence of damage is favoured in regions where fatigue accumulates.

Assumption 10. *For the mappings associated with fatigue in* (19) *we require the following:*

1. *The history operator* $H : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *satisfies*

$$\|H(y_1)(t)-H(y_2)(t)\|_{L^2(\Omega)} \leq L_H \int_0^t \|y_1(s)-y_2(s)\|_{L^2(\Omega)}\, ds \quad \text{a.e. in } (0,T),$$

for all $y_1, y_2 \in L^2(0,T;L^2(\Omega))$*, where* $L_H > 0$ *is a positive constant. Moreover,* $H : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *is supposed to be Gâteaux-differentiable.*

2. *The non-linear function* $\kappa : \mathbb{R} \to \mathbb{R}$ *is assumed to be Lipschitz continuous with Lipschitz constant* $L_\kappa > 0$ *and differentiable.*

4.1 H_0^1-viscosity

The model we intend to examine describes the evolution of damage under the influence of a time-dependent load $\ell : [0,T] \to H^{-1}(\Omega)$ (control) acting on a body occupying the bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$. The induced damage is expressed in terms of $q : [0,T] \to H_0^1(\Omega)$. The problem we consider is a viscous version of the fatigue damage model addressed in [1]; for simplicity reasons, we do not take a displacement variable into account. Let us mention that H_0^1- viscosity has been used in the context of optimal control in [35] as well, see [35, Eq. (5)], where a sweeping process is considered. For more details on the viscous approximation of damage models we refer the reader to [24, Sec. 4.4].

The viscous (single-field) damage problem with fatigue reads as follows:

$$-\partial_q \mathcal{E}(t,q(t)) \in \partial_2 \mathcal{R}_\epsilon(H(q)(t),\dot{q}(t)) \quad \text{in } H^{-1}(\Omega), \quad q(0) = 0 \tag{21}$$

a.e. in $(0,T)$. In (21), the stored energy $\mathcal{E} : [0,T] \times H_0^1(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{E}(t,q) := \frac{\alpha}{2}\|\nabla q\|^2_{L^2(\Omega)} - \langle \ell(t), q\rangle_{H^1(\Omega)}, \tag{22}$$

where $\alpha > 0$ is a fixed parameter. The viscous dissipation $\mathcal{R}_\epsilon : L^2(\Omega) \times H_0^1(\Omega) \to (-\infty,\infty]$ is defined as

$$\mathcal{R}_\epsilon(\zeta,\eta) := \begin{cases} \displaystyle\int_\Omega \kappa(\zeta)\,\eta\, dx + \frac{\epsilon}{2}\|\nabla\eta\|^2_{L^2(\Omega)}, & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise}, \end{cases} \tag{23}$$

where $\epsilon > 0$ is the viscosity parameter.

Definition 3. *The set* $\mathcal{C} \subset H_0^1(\Omega)$ *is defined as*

$$\mathcal{C} := \{v \in H_0^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}.$$

Lemma 5. *The set $\mathcal{C}$ is polyhedric, that is,*

$$\overline{\mathbb{R}^+(\mathcal{C}-\eta)} \cap [\omega]^\perp = \overline{\mathbb{R}^+(\mathcal{C}-\eta) \cap [\omega]^\perp} \quad \forall\, (\eta,\omega) \in \mathcal{C} \times \overline{\mathbb{R}^+(\mathcal{C}-\eta)}^\circ.$$

Proof. See for instance [11, Cor. 6.46]. □

In all what follows, the (metric) projections onto a subset of $H^{-1}(\Omega)$ are considered w.r.t. the inner product $\langle \frac{1}{\epsilon}(-\Delta)^{-1}\cdot, \cdot \rangle_{H^{-1}(\Omega)}$, unless otherwise specified; cf. also (6).

Lemma 6. *For each $(\zeta,\omega) \in L^2(\Omega) \times H^{-1}(\Omega)$ it holds*

$$P_{\partial_2 \mathcal{R}(\zeta,0)}\omega = P_{\partial I_{\mathcal{C}}(0)}(\omega - \kappa(\zeta)) + \kappa(\zeta) \quad \text{in } H^{-1}(\Omega).$$

Proof. We observe that

$$\begin{aligned} \partial_2 \mathcal{R}(\zeta,0) &= \{\mu \in H^{-1}(\Omega) |\, \langle \mu, v\rangle_{H_0^1(\Omega)} \leq \mathcal{R}(\zeta,v) - \mathcal{R}(\zeta,0) \quad \forall\, v \in H_0^1(\Omega)\} \\ &= \{\mu \in H^{-1}(\Omega) |\, \langle \mu - \kappa(\zeta), v\rangle_{H_0^1(\Omega)} \leq 0 \quad \forall\, v \in H_0^1(\Omega) \text{ with } v \geq 0 \text{ a.e. in } \Omega\} \\ &= \partial I_{\mathcal{C}}(0) + \kappa(\zeta) \quad \forall\, \zeta \in L^2(\Omega), \end{aligned} \tag{24}$$

where the second identity in (24) is due to (20). Moreover, given a closed convex set $M \subset H^{-1}(\Omega)$, it holds

$$P_M \omega = P_{M-\psi}(\omega - \psi) + \psi \quad \forall\, \omega, \psi \in H^{-1}(\Omega),$$

which can be easily checked by employing the definition of the projection operator. The desired assertion now follows from (24). □

Proposition 1 (Control-to-state map)**.** *For every $\ell \in L^2(0,T;H^{-1}(\Omega))$, the viscous damage problem with fatigue* (21) *admits a unique solution $q \in H_0^1(0,T;H_0^1(\Omega))$, which is characterized by*

$$\dot{q}(t) = \frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I} - P_{\partial I_{\mathcal{C}}(0)})(\alpha \Delta q(t) + \ell(t) - (\kappa \circ H)(q)(t)) \quad \text{in } H_0^1(\Omega), \tag{25}$$

$$q(0) = 0 \tag{26}$$

a.e. in $(0,T)$.

Proof. In view of (23), (20) and the sum rule for convex subdifferentials, it holds

$$\partial_2 \mathcal{R}_\epsilon(H(q)(t), \dot{q}(t)) = \partial_2 \mathcal{R}(H(q)(t), \dot{q}(t)) - \epsilon\, \Delta \dot{q}(t)$$

a.e. in $(0,T)$. Thus, on account of (22), the evolution (21) is equivalent to

$$\begin{aligned}&\mathcal{R}(H(q)(t),v)-\mathcal{R}(H(q)(t),\dot q(t))+\epsilon\,(\nabla\dot q(t),\nabla(v-\dot q(t)))_{L^2(\Omega)}\\&\geq\langle\alpha\Delta q(t)+\ell(t),v-\dot q(t)\rangle_{H_0^1(\Omega)}\end{aligned}\tag{27}$$

for all $v\in H_0^1(\Omega)$. Now, with the notations from section 3, we see that if we set

$$X:=L^2(\Omega),\quad Y:=H_0^1(\Omega),\quad Z:=H^{-1}(\Omega),\tag{28a}$$

$$R:=\mathcal{R},\quad \mathcal{H}:=H,\quad \mathcal{V}:=-\epsilon\,\Delta,\quad g(\widetilde q,\widetilde\ell):=\alpha\Delta\widetilde q+\widetilde\ell,\tag{28b}$$

then (EVI) coincides with (27). Indeed, the quantities in (28) satisfy Assumption 7, as we will next see. While Assumption 7.1a can be easily checked, the condition in Assumption 7.1b is due to

$$\begin{aligned}R(\zeta_1,\eta_2)-&R(\zeta_1,\eta_1)+R(\zeta_2,\eta_1)-R(\zeta_2,\eta_2)\\&=\int_\Omega\kappa(\zeta_1)\,(\eta_2-\eta_1)\,dx-\int_\Omega\kappa(\zeta_2)\,(\eta_2-\eta_1)\,dx\\&\leq L_\kappa\,\|\zeta_1-\zeta_2\|_{L^2(\Omega)}\|\eta_1-\eta_2\|_{H_0^1(\Omega)}\\&\qquad\forall\,\zeta_1,\zeta_2\in L^2(\Omega),\ \forall\,\eta_1,\eta_2\in\mathcal{C}.\end{aligned}$$

Note that $\operatorname{dom}R(\zeta,\cdot)=\mathcal{C}$ for all $\zeta\in L^2(\Omega)$ and that the above inequality follows from Assumption 1.2. Further, in view of $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ and Assumption 1.1, we see that Assumption 7.2 is satisfied as well. Moreover, we observe that Assumption 7.3 (with $\vartheta=\epsilon$) and Assumption 7.4 also hold true in the setting (28).

Since the quantities in (28) satisfy the entire Assumption 7 and since the evolution in (21) is equivalent to (27) we immediately obtain from Theorem 8 the unique solvability of (21) and the desired regularity of the solution. Moreover, according to Theorem 8, (27) is equivalent to

$$\dot q(t)=\mathcal{F}\big(H(q)(t),g(q,\ell)(t)\big)\quad\text{a.e. in }(0,T),$$

where

$$\mathcal{F}(\zeta,\omega)=\frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I}-P_{\partial_2\mathcal{R}(\zeta,0)})\omega\quad\forall\,(\zeta,\omega)\in L^2(\Omega)\times H^{-1}(\Omega).\tag{29}$$

That is, (27), and thus, the evolution in (21), can be rewritten as

$$\dot q(t)=\frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I}-P_{\partial_2\mathcal{R}(H(q)(t),0)})\big(g(q,\ell)(t)\big)\quad\text{a.e. in }(0,T).\tag{30}$$

Applying Lemma 6 now yields that

$$P_{\partial_2 \mathcal{R}(H(q)(t),0)}\big(g(q,\ell)(t)\big) = P_{\partial I_{\mathcal{C}}(0)}\big[g(q,\ell)(t) - (\kappa\circ H)(q)(t)\big] + (\kappa\circ H)(q)(t),$$

which inserted in (30) gives in turn

$$\dot{q}(t) = \frac{1}{\epsilon}(-\Delta)^{-1}\big(g(q,\ell)(t) - P_{\partial I_{\mathcal{C}}(0)}\big[g(q,\ell)(t) - (\kappa\circ H)(q)(t)\big] - (\kappa\circ H)(q)(t)\big)$$

a.e. in $(0,T)$. In view of the definition of the mapping g in (28) we can finally deduce that (21) is equivalent to (25). □

Lemma 7. *The mapping $f : H^{-1}(\Omega) \to H_0^1(\Omega)$ defined as*

$$f(\omega) = \frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I} - P_{\partial I_{\mathcal{C}}(0)})(\omega)$$

is directionally differentiable with

$$f'(\omega;\delta\omega) = \frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I} - P_{T(\omega)})(\delta\omega) \quad \forall\, \omega, \delta\omega \in H^{-1}(\Omega),$$

where $T(\omega) := \overline{\mathbb{R}^+(\partial I_{\mathcal{C}}(0) - P_{\partial I_{\mathcal{C}}(0)}\omega)}^{H^{-1}(\Omega)} \cap [\psi(\omega)]^\perp$ denotes the critical cone of $\partial I_{\mathcal{C}}(0) \subset H^{-1}(\Omega)$ at $(\omega, \psi(\omega))$ and $\psi(\omega) := \frac{1}{\epsilon}(-\Delta)^{-1}(\omega - P_{\partial I_{\mathcal{C}}(0)}\omega)$.

Proof. Since C is polyhedric, see Lemma 5, the set $\partial I_{\mathcal{C}}(0) = \mathcal{C}^\circ \subset H^{-1}(\Omega)$ is polyhedric as well, cf. [40, Lem. 3.2]. Now, by [25, Thm. 2.1] we have that $P_{\partial I_{\mathcal{C}}(0)} : H^{-1}(\Omega) \to H^{-1}(\Omega)$ is directionally differentiable with

$$P'_{\partial I_{\mathcal{C}}(0)}(\omega;\delta\omega) = P_{T(\omega)}\delta\omega \quad \forall\, \omega, \delta\omega \in H^{-1}(\Omega).$$

□

Proposition 2 (Directional differentiability)**.** *The solution map associated to* (21)

$$S : L^2(0,T;H^{-1}(\Omega)) \ni \ell \mapsto q \in H_0^1(0,T;H_0^1(\Omega))$$

is directionally differentiable. Its directional derivative $\delta q := S'(\ell;\delta\ell)$ at the point $\ell \in L^2(0,T;H^{-1}(\Omega))$ in direction $\delta\ell \in L^2(0,T;H^{-1}(\Omega))$ is the unique solution of

$$\begin{aligned}&\dot{\delta q}(t) = \frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I} - P_{T(z(t))})\Big(\alpha\Delta\delta q(t) + \delta\ell(t) - (\kappa\circ H)'(q)(\delta q)(t)\Big) \ \text{ in } H_0^1(\Omega),\\ &\delta q(0) = 0,\end{aligned} \tag{31}$$

a.e. in $(0,T)$. *Here,*

$$T(z(t)) = \overline{\mathbb{R}^+(\partial I_{\mathcal{C}}(0) - P_{\partial I_{\mathcal{C}}(0)} z(t))}^{H^{-1}(\Omega)} \cap [\dot q(t)]^\perp$$

denotes the critical cone of $\partial I_{\mathcal{C}}(0) \subset H^{-1}(\Omega)$ *at* $(z(t), \dot q(t))$, *where we abbreviate* $z(t) := \alpha \Delta q(t) + \ell(t) - (\kappa \circ H)(q)(t)$.

Proof. According to Theorem 9,

$$S : L^2(0,T; H^{-1}(\Omega)) \ni \ell \mapsto q \in H^1_0(0,T; H^1_0(\Omega))$$

is directionally differentiable, if

$$B : L^2(\Omega) \times H^{-1}(\Omega) \ni (\zeta, \omega) \mapsto P_{\partial_2 \mathcal{R}(\zeta,0)} \omega \in H^{-1}(\Omega)$$

does so. Thus, in the light of Lemma 6 we need to check that

$$B : L^2(\Omega) \times H^{-1}(\Omega) \ni (\zeta, \omega) \mapsto P_{\partial I_{\mathcal{C}}(0)}(\omega - \kappa(\zeta)) + \kappa(\zeta) \in H^{-1}(\Omega) \tag{32}$$

is directionally differentiable. In view of Assumption 10.2 and by employing Lebesgue's dominated convergence theorem we obtain that $\kappa : L^2(\Omega) \to L^2(\Omega)$ is Gâteaux-differentiable. As $P_{\partial I_{\mathcal{C}}(0)} : H^{-1}(\Omega) \to H^{-1}(\Omega)$ is Lipschitz continuous and directionally differentiable, see Lemma 7, we can make use of the chain rule for directionally differentiable functions [33, Prop. 3.6(i)]. This implies that the mapping B from (32) is indeed directionally differentiable with

$$B'((\zeta, \omega); (\delta\zeta, \delta\omega)) = P_{T(\omega - \kappa(\zeta))}(\delta\omega - \kappa'(\zeta)(\delta\zeta)) + \kappa'(\zeta)(\delta\zeta) \tag{33}$$

for all $(\zeta, \omega), (\delta\zeta, \delta\omega) \in L^2(\Omega) \times H^{-1}(\Omega)$. By Theorem 9 we then obtain that $S : L^2(0,T; H^{-1}(\Omega)) \ni \ell \mapsto q \in H^1_0(0,T; H^1_0(\Omega))$ is directionally differentiable with directional derivative $\delta q = S'(\ell; \delta\ell)$ satisfying

$$\begin{aligned}
\dot{\delta q}(t) &= \mathcal{F}'\Big((H(q), g(q,\ell)); \big(H'(q)(\delta q), g'((q,\ell);(\delta q, \delta\ell))\big)\Big)(t) \\
&= \frac{1}{\epsilon}(-\Delta)^{-1}[g'((q,\ell);(\delta q, \delta\ell))(t) - B'((H(q), g(q,\ell)); \\
&\quad (H'(q)(\delta q), g'((q,\ell);(\delta q, \delta\ell)))(t)] \\
&= \frac{1}{\epsilon}(-\Delta)^{-1}[\mathbb{I} - P_{T(z(t))}][g'((q,\ell);(\delta q, \delta\ell))(t) - \kappa'(H(q))(H'(q)(\delta q))(t)],
\end{aligned} \tag{34}$$

a.e. in $(0,T)$, where $\mathcal{F}$ is given by (29) and g is the mapping from (28); recall that $z = \alpha\Delta q + \ell - (\kappa \circ H)(q)$ and that H is Gâteaux-differentiable, by

Assumption 10.1. Note that the second identity in (34) is due to Corollary 1, while the last identity follows from (33). Since $g'((q,\ell);(\delta q,\delta\ell)) = \alpha\Delta\delta q + \delta\ell$ and

$$\psi(z(t)) = \frac{1}{\epsilon}(-\Delta)^{-1}(z(t) - P_{\partial I_C(0)} z(t)) = \dot{q}(t) \quad \text{a.e. in } (0,T),$$

cf. (25), the proof is now complete. □

Next, we want to apply the strong stationarity result from section 2 to the following optimal control problem:

$$\left.\begin{aligned} \min_{\ell \in L^2(0,T;L^2(\Omega))} \quad & \mathcal{J}(q,\ell) \\ \text{s.t.} \quad & q \text{ solves (21) with r.h.s. } \ell. \end{aligned}\right\} \tag{P}$$

In the sequel, the objective $\mathcal{J}$ is supposed to fulfill

Assumption 11. *The functional* $\mathcal{J} : L^2(0,T;H^1(\Omega)) \times L^2(0,T;L^2(\Omega)) \to \mathbb{R}$ *is Fréchet-differentiable.*

Before stating the strong stationary optimality conditions we establish an estimate which will be useful in the proof of Theorem 12 below.

Lemma 8. *There exists a constant* $K > 0$ *dependent only on the given data such that for all* $\ell, \delta\ell_1, \delta\ell_2 \in L^2(0,T;H^{-1}(\Omega))$ *it holds*

$$\|S'(\ell;\delta\ell_1) - S'(\ell;\delta\ell_2)\|_{L^2(0,T;H_0^1(\Omega))} \le K\, \|\delta\ell_1 - \delta\ell_2\|_{L^2(0,T;H^{-1}(\Omega))}, \tag{35}$$

where $S : L^2(0,T;H^{-1}(\Omega)) \to H_0^1(0,T;H_0^1(\Omega))$ *is the solution operator to* (21).

Proof. In the proof of Proposition 1 we established that (21) fits in the setting of section 3 with the quantities from (28). This means that $S : L^2(0,T;H^{-1}(\Omega)) \ni \ell \mapsto q \in H_0^1(0,T;H_0^1(\Omega))$ is Lipschitz continuous according to Proposition 1. In view of Proposition 2, we can now conclude the desired estimate. □

The main result of this subsection reads as follows.

Theorem 12 (Strong stationarity for the optimal control of the viscous damage model with fatigue)**.** *Let* $\bar{\ell} \in L^2(0,T;L^2(\Omega))$ *be locally optimal for* (P) *with associated state* $\bar{q} \in H_0^1(0,T;H_0^1(\Omega))$*. Then, there exists a* *unique adjoint state* $\xi \in H^1_T(0,T;H^{-1}(\Omega))$ *and a unique multiplier* $\lambda \in L^2(0,T;H^1_0(\Omega))$ *such that the following system is satisfied*

$$-\dot{\xi} - \alpha\Delta\lambda + [(\kappa \circ H)'(\bar{q})]^\star \lambda = \partial_q \mathcal{J}(\bar{q},\bar{\ell}) \quad in\ L^2(0,T;H^{-1}(\Omega)), \quad \xi(T) = 0, \tag{36a}$$

$$\langle \xi(t), \frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I} - P_{T(\bar{z}(t))})\, v\rangle_{H^1_0(\Omega)} \geq \langle \lambda(t), v\rangle_{H^{-1}(\Omega)} \quad \forall\, v \in H^{-1}(\Omega),\ a.e.\ in\,(0,T), \tag{36b}$$

$$\lambda + \partial_\ell \mathcal{J}(\bar{q},\bar{\ell}) = 0 \quad in\ L^2(0,T;L^2(\Omega)), \tag{36c}$$

where we abbreviate $\bar{z} := \alpha\Delta\bar{q} + \bar{\ell} - (\kappa \circ H)(\bar{q})$*. Again,*

$$T(\bar{z}(t)) = \overline{\mathbb{R}^+(\partial I_{\mathcal{C}}(0) - P_{\partial I_{\mathcal{C}}(0)}\bar{z}(t))}^{H^{-1}(\Omega)} \cap [\dot{\bar{q}}(t)]^\perp$$

denotes the critical cone of $\partial I_{\mathcal{C}}(0) \subset H^{-1}(\Omega)$ *at* $(\bar{z}(t), \dot{\bar{q}}(t))$*.*

Proof. We aim to apply the strong stationarity result given by Theorem 5 for the optimal control problem (P). To this end, we have to check if (P) fits in the general setting from section 2, see Assumption 1. After that, we verify Assumptions 2, 3 and 4. Indeed, with the notations from section 2, we see that if we set

$$V := L^2(\Omega),\ Y := H^1_0(\Omega),\ U := L^2(\Omega),\ J := \mathcal{J}, \tag{37a}$$

$$f : Y^* \to Y, \quad f(\omega) = \frac{1}{\epsilon}(-\Delta)^{-1}(\mathbb{I} - P_{\partial I_{\mathcal{C}}(0)})(\omega), \tag{37b}$$

$$\Phi : L^2(0,T;Y \times U) \ni (q,u) \mapsto \alpha\Delta q + u - (\kappa \circ H)(q) \in L^2(0,T;Y^*), \tag{37c}$$

$$\Psi : L^2(0,T;Y \times U) \ni (q,u) \mapsto u \in L^2(0,T;U^*), \tag{37d}$$

then (1) coincides with (P), thanks to Proposition 1. Clearly, Assumption 1.1 is satisfied. Since $(\kappa \circ H) : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ is Gâteaux-differentiable, see Assumption 10, the requirement in Assumption 1.2 is fulfilled as well. Assumption 1.3 is satisfied by the mapping introduced in (37b), see Lemma 7. Thus, the entire Assumption 1 holds true in the setting (37), cf. also Assumption 14.

Moreover, since for the setting considered in (37), (2) is equivalent to (25), we see that, in view of Proposition 1, Assumption 2.1 holds. The solution operator of (2) is given by $L^2(0,T;L^2(\Omega)) \ni \ell \mapsto (S(\ell),\ell) \in H^1_0(0,T;H^1_0(\Omega)) \times L^2(0,T;L^2(\Omega))$, where $S : L^2(0,T;H^{-1}(\Omega)) \ni \ell \mapsto q \in H^1_0(0,T;H^1_0(\Omega))$ is the solution operator of (25). According to Proposition 2, this is directionally differentiable and its directional derivative $S'(\ell;\delta\ell)$

at ℓ in direction $\delta\ell$ is the unique solution of (31). In light of Lemma 7, this means that the pair $(\delta q, \delta u) = (S'(\ell; \delta\ell), \delta\ell)$ satisfies

$$\begin{aligned}\dot{\delta q}(t) = f'(\Phi(q,u)(t); \Phi'(q,u)(\delta q, \delta u)(t)), \quad \delta q(0) = 0,\\ \Psi'(q,u)(\delta q, \delta u)(t) = \delta\ell(t) \quad \text{a.e. in } (0,T).\end{aligned}$$

that is, (3). Thus, Assumption 2.2 holds true. Further, (35) in combination with the embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ implies that Assumption 2.3 is verified as well; note that the second statement in Assumption 2.3 is true in our setting, since $S'(\ell;\cdot) : L^2(0,T;L^2(\Omega)) \to L^2(0,T;H_0^1(\Omega))$ is continuous for any $\ell \in L^2(0,T;L^2(\Omega))$. Hence, the entire Assumption 2 is true for the quantities in (46).

It remains to check that Assumptions 3 and 4 are guaranteed. To this end, we observe that

$$\partial_u \Phi(\bar q, \bar u) = \mathbb{I} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;H^{-1}(\Omega)).$$

As a result of $L^2(0,T;L^2(\Omega)) \overset{d}{\hookrightarrow} L^2(0,T;H^{-1}(\Omega))$, the 'constraint qualification' in Assumption 3 is fulfilled. In the light of (37c)-(37d), the adjoints of the partial derivatives of Φ and Ψ are given by

$$\begin{aligned}&\partial_q \Phi(\bar q, \bar u)^* = \alpha\Delta - [(\kappa \circ H)'(\bar q)]^\star : L^2(0,T;H_0^1(\Omega)) \to L^2(0,T;H^{-1}(\Omega)),\\ &\partial_u \Phi(\bar q, \bar u)^* = \mathbb{I} : L^2(0,T;H_0^1(\Omega)) \to L^2(0,T;L^2(\Omega)),\\ &\partial_q \Psi(\bar q, \bar u)^* = 0 : L^2(0,T;L^2(\Omega)) \to L^2(0,T;H^{-1}(\Omega)),\\ &\partial_u \Psi(\bar q, \bar u)^* = \mathbb{I} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega)).\end{aligned} \tag{38}$$

To see that Assumption 4 is true, we only need to check if $\partial_\ell \mathcal{J}(\bar q, \bar\ell) \in L^2(0,T;H_0^1(\Omega))$ (cf. (38) and note that $\partial_u \mathcal{J} = 0$). To this end, we make use of (4), which in the setting (37) reads

$$\partial_q \mathcal{J}(\bar q, \bar\ell) S'(\bar\ell; \delta\ell) + \partial_\ell \mathcal{J}(\bar q, \bar\ell)\delta\ell \geq 0 \quad \forall\, \delta\ell \in L^2(0,T;L^2(\Omega)).$$

Since

$$\|S'(\bar\ell;\delta\ell)\|_{L^2(0,T;H_0^1(\Omega))} \leq K\, \|\delta\ell\|_{L^2(0,T;H^{-1}(\Omega))} \quad \forall\, \delta\ell \in L^2(0,T;H^{-1}(\Omega)),$$

see (35), Hahn-Banach theorem now gives in turn that

$$\partial_\ell \mathcal{J}(\bar q, \bar\ell) \in L^2(0,T;H_0^1(\Omega)).$$

Thus, we can apply Theorem 5, which in combination with (38) tells us that there exist unique adjoint "states" $\xi \in H_T^1(0,T;H^{-1}(\Omega))$ and $w \in L^2(0,T;L^2(\Omega))$ and a unique multiplier $\lambda \in L^2(0,T;H_0^1(\Omega))$ such that

$$-\dot{\xi} - \alpha\Delta\lambda + [(\kappa \circ H)'(\bar{q})]^\star\lambda = \partial_q \mathcal{J}(\bar{q},\bar{\ell}) \text{ in } L^2(0,T;H^{-1}(\Omega)), \quad \xi(T) = 0, \tag{39a}$$

$$-\lambda + w = 0 \text{ in } L^2(0,T;L^2(\Omega)), \tag{39b}$$

$$\langle \xi(t), f'(\Phi(\bar{q},\bar{u})(t);v)\rangle_{H_0^1(\Omega)} \geq \langle \lambda(t), v\rangle_{H^{-1}(\Omega)} \quad \forall\, v \in H^{-1}(\Omega), \text{ a.e. in } (0,T), \tag{39c}$$

$$w + \partial_\ell \mathcal{J}(\bar{q},\bar{\ell}) = 0 \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{39d}$$

Inserting (39b) in (39d) and employing Lemma 7 to compute the derivative in (39c) finally yields (36); recall that $\psi(z(t)) = \dot{q}(t)$, see the end of the proof of Proposition 2 and Lemma 7. The proof is now complete. □

The optimality system in Theorem 12 is indeed of strong stationary type, as the next result shows:

Theorem 13 (Equivalence between B- and strong stationarity)**.** *Assume that $\bar{\ell} \in L^2(0,T;L^2(\Omega))$ together with its state $\bar{q} \in H_0^1(0,T;H_0^1(\Omega))$, some adjoint state $\xi \in H_T^1(0,T;H^{-1}(\Omega))$ and a multiplier $\lambda \in L^2(0,T;H_0^1(\Omega))$ satisfy the optimality system* (36)*. Then, it also satisfies the variational inequality*

$$\partial_q \mathcal{J}(\bar{q},\bar{\ell})S'(\bar{\ell};\delta\ell) + \partial_\ell \mathcal{J}(\bar{q},\bar{\ell})\delta\ell \geq 0 \quad \forall\, \delta\ell \in L^2(0,T;L^2(\Omega)), \tag{40}$$

where S is the solution mapping associated to (21)*, see Proposition 1.*

Proof. We show the result by means of Theorem 6. In the proof of Theorem 12, we have seen that the problem (P) fits in the setting from section 2, i.e., Assumptions 1 and 2 are satisfied for the quantities in (37). According to the end of the proof of Theorem 12, the system (36) coincides with (39), which is the same as (5) in this particular setting, see (38). We also note that (4) is just (40). Hence, the desired statement is true. Note that, since Assumptions 3 and 4 are fulfilled, cf. the proof of Theorem 12, we have the equivalence (40) $\Longleftrightarrow$ (36). □

4.2 Penalization (L^2-viscosity)

In this subsection we apply the result from section 2 to obtain strong stationary optimality conditions for the control of a *two-field* gradient damage model with fatigue. The problem we consider is a penalized version of the viscous fatigue damage model addressed in [1]. This kind of penalization has already been proven to be successful in the context of classical damage models (without fatigue). Firstly, it approximates the single-field damage model, cf. [23], and secondly, it is frequently employed in computational mechanics (see e.g. [16] and the references therein).

The model we intend to examine describes the evolution of damage under the influence of a time-dependent load $\ell : [0,T] \to H^1(\Omega)^*$ (control) acting on a body occupying the bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$. The induced 'local' and 'nonlocal' damage are expressed in terms of $\varphi : [0,T] \to H^1(\Omega)$ and $d : [0,T] \to L^2(\Omega)$, respectively (states). For more details, see [10, Sec. 1].

The viscous two-field gradient damage problem with fatigue reads as follows:

$$\left.\begin{aligned} \varphi(t) \in \operatorname*{arg\,min}_{\varphi \in H^1(\Omega)} \mathcal{E}(t, \varphi, d(t)), \\ -\partial_d \mathcal{E}(t, \varphi(t), d(t)) \in \partial_2 \mathcal{R}_\epsilon(H(d)(t), \dot{d}(t)) \quad \text{in } L^2(\Omega), \quad d(0) = 0 \end{aligned}\right\} \tag{41}$$

a.e. in $(0,T)$. In (41), the stored energy $\mathcal{E} : [0,T] \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{E}(t, \varphi, d) := \frac{\alpha}{2}\|\nabla \varphi\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\varphi - d\|^2_{L^2(\Omega)} - \langle \ell(t), \varphi \rangle_{H^1(\Omega)},$$

where $\alpha > 0$ is the gradient regularization and $\beta > 0$ denotes the penalization parameter. The viscous dissipation $\mathcal{R}_\epsilon : L^2(\Omega) \times L^2(\Omega) \to (-\infty, \infty]$ is defined as

$$\mathcal{R}_\epsilon(\zeta, \eta) := \begin{cases} \displaystyle\int_\Omega \kappa(\zeta)\, \eta \, dx + \frac{\epsilon}{2}\|\eta\|^2_{L^2(\Omega)}, & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$ is the viscosity parameter.

Remark 6. *Optimality conditions for the control of* (41) *have been recently established in [10], however in a more general setting. Therein, the fatigue degradation function* κ *is assumed to be only directionally differentiable, so that conditions of strong stationary type are not to be expected [10, Remark 3.22]. Moreover, in the contribution [10], the control space is* $H^1(0,T;L^2(\Omega))$, *whereas we will work with* $L^2(0,T;L^2(\Omega))$. *Note that there is no need to apply the findings in section 3 to see that* (41) *is a system of the type* (2)*; this is already stated in [10, Prop. 2.3], see below.*

Lemma 9 (Control-to-state map, directional differentiability [10, Prop. 2.3, 2.6])**.** *The following assertions are true:*

1. *For every $\ell \in L^2(0,T;H^1(\Omega)^*)$, the viscous damage problem with fatigue* (41) *admits a unique solution*

$$(d,\varphi) \in H_0^1(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega)),$$

which is characterized by the following PDE system

$$\dot{d}(t) = \frac{1}{\epsilon}\max(-\beta(d(t)-\varphi(t)) - (\kappa \circ H)(d)(t), 0) \quad \text{in } L^2(\Omega), \quad d(0)=0, \tag{42a}$$

$$-\alpha\Delta\varphi(t) + \beta\,\varphi(t) = \beta d(t) + \ell(t) \quad \text{in } H^1(\Omega)^* \tag{42b}$$

a.e. in $(0,T)$.

2. *The solution map associated to* (41)

$$S : L^2(0,T;H^1(\Omega)^*) \ni \ell \mapsto (d,\varphi) \in H_0^1(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega))$$

is directionally differentiable. Its directional derivative $(\delta d, \delta\varphi) := S'(\ell;\delta\ell)$ at $\ell \in L^2(0,T;H^1(\Omega)^*)$ in direction $\delta\ell \in L^2(0,T;H^1(\Omega)^*)$ is the unique solution of the system

$$\begin{aligned} \dot{\delta d}(t) &= \frac{1}{\epsilon}\max{}'\big(z(t); -\beta(\delta d(t) - \delta\varphi(t)) - (\kappa\circ H)'(d)(\delta d)(t)\big) \quad \text{in } L^2(\Omega), \\ -\alpha\Delta\delta\varphi(t) &+ \beta\,\delta\varphi(t) = \beta\delta d(t) + \delta\ell(t) \quad \text{in } H^1(\Omega)^* \quad \text{a.e. in } (0,T), \end{aligned} \tag{43}$$

with initial condition $\delta d(0)=0$, where we abbreviate $z(t) := -\beta(d(t) - \varphi(t)) - (\kappa\circ H)(d)(t)$.

Next, we want to apply the strong stationarity result from section 2 to the following optimal control problem:

$$\left.\begin{aligned} \min_{\ell \in L^2(0,T;L^2(\Omega))} \quad & \mathcal{J}(d,\varphi,\ell) \\ \text{s.t.} \quad & (d,\varphi) \text{ solves (41) with r.h.s. } \ell. \end{aligned}\right\} \tag{Q}$$

In the sequel, the objective $\mathcal{J}$ is supposed to fulfill

Assumption 14. *The objective functional*

$$\mathcal{J} : L^2(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega)) \times L^2(0,T;L^2(\Omega)) \to \mathbb{R}$$

is continuously Fréchet-differentiable and Lipschitz continuous on bounded sets, i.e., for all $M > 0$ *there exists* $L_M > 0$ *so that*

$$|\mathcal{J}(v_1) - \mathcal{J}(v_2)| \leq L_M \, \|v_1 - v_2\|_X \quad \forall \, v_1, v_2 \in B_X(0, M),$$

where we abbreviate $X := L^2(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega)) \times L^2(0,T;L^2(\Omega))$.

Note that Assumption 14 is satisfied by classical objectives of tracking type such as

$$\mathcal{J}_{ex}(d, \varphi, \ell) := \frac{1}{2} \, \|\varphi - \varphi_d\|^2_{L^2(0,T;H^1(\Omega))} + \frac{\alpha_1}{2} \|d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha_2}{2} \|\ell - \ell_d\|^2_{L^2(0,T;L^2(\Omega))},$$

where $\varphi_d \in L^2(0,T;H^1(\Omega))$, $\ell_d \in L^2(0,T;L^2(\Omega))$ and $\alpha_1, \alpha_2 > 0$.

Before stating the strong stationary optimality conditions, we check that Assumption 4 is satisfied in our setting. As it will turn out in the proof of Theorem 15 below, this is indeed the case, as a result of the following

Lemma 10. *For any local optimum* $\bar{\ell}$ *of* (Q)*, there exists*

$$(\lambda, w) \in L^2(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega))$$

so that

$$-\alpha \Delta w(t) + \beta(w(t) - \lambda(t)) = \partial_\varphi \mathcal{J}(S(\bar{\ell}), \bar{\ell})(t) \quad \text{in } H^1(\Omega)^*, \tag{44a}$$

$$w(t) + \partial_\ell \mathcal{J}(S(\bar{\ell}), \bar{\ell})(t) = 0 \quad \text{in } H^1(\Omega), \quad \text{a.e. in } (0,T), \tag{44b}$$

where S *is the solution operator associated to* (41)*, see Lemma 9.2.*

Proof. We refer to the proof of [8, Lem. 4.3], where a very similar setting is considered. The result is shown by a classical regularization approach, see [39, 4] for instance. One defines a smooth approximation of the function $\max(\cdot, 0)$, to which one associates a state equation where the solution mapping is Gâteaux-differentiable. Then, it is shown that $\bar{\ell}$ can be approximated by a sequence of local minimizers of an optimal control problem governed by the regularized state equation. Passing to the limit in the adjoint system associated to the regularized optimal control problem finally yields the desired assertion. The only difference to the proof of [8, Lem. 4.3] consists in having to show the existence and uniformly boundedness of the regularized adjoint state. To this end, one can follow arguments employed in the proof of [10, Prop. 3.10]. □

The main result of this subsection reads as follows.

Theorem 15 (Strong stationarity for the optimal control of the viscous two-field damage model with fatigue)**.** *Let $\bar{\ell} \in L^2(0,T;L^2(\Omega))$ be locally optimal for* (Q) *with associated states*

$$\bar{d} \in H_0^1(0,T;L^2(\Omega)) \quad \textit{and} \quad \bar{\varphi} \in L^2(0,T;H^1(\Omega)).$$

Then, there exist unique adjoint states

$$\xi \in H_T^1(0,T;L^2(\Omega)) \quad \textit{and} \quad w \in L^2(0,T;H^1(\Omega)),$$

and a unique multiplier $\lambda \in L^2(0,T;L^2(\Omega))$ such that the following system is satisfied

$$-\dot{\xi} - \beta(w-\lambda) + [(\kappa \circ H)'(\bar{d})]^* \lambda = \partial_d \mathcal{J}(\bar{d},\bar{\varphi},\bar{\ell}) \quad \textit{in } L^2(0,T;L^2(\Omega)),$$
$$\xi(T) = 0, \tag{45a}$$

$$-\alpha\Delta w + \beta(w-\lambda) = \partial_\varphi \mathcal{J}(\bar{d},\bar{\varphi},\bar{\ell}) \quad \textit{in } L^2(0,T;H^1(\Omega)^*), \tag{45b}$$

$$\left.\begin{aligned} \lambda(t,x) &= \frac{1}{\epsilon}\chi_{\{\bar{z}>0\}}(t,x)\xi(t,x) \quad \textit{a.e. where } \bar{z}(t,x) \neq 0, \\ \lambda(t,x) &\in \big[0, \frac{1}{\epsilon}\xi(t,x)\big] \quad \textit{a.e. where } \bar{z}(t,x) = 0, \end{aligned}\right\} \tag{45c}$$

$$w + \partial_\ell \mathcal{J}(\bar{d},\bar{\varphi},\bar{\ell}) = 0 \quad \textit{in } L^2(0,T;H^1(\Omega)), \tag{45d}$$

where we abbreviate $\bar{z} := -\beta(\bar{d}-\bar{\varphi}) - (\kappa \circ H)(\bar{d})$.

Proof. We aim to apply the strong stationarity result given by Theorem 5 for the optimal control problem (Q). To this end, we have to check if (Q) fits in the general setting from section 2. After that, we verify Assumptions 2, 3 and 4. Indeed, with the notations from section 2, we see that if we set

$$V := L^2(\Omega),\ Y := L^2(\Omega),\ U := H^1(\Omega),\ J := \mathcal{J}, \tag{46a}$$

$$f : Y^* \to Y, \quad f(\omega) = \frac{1}{\epsilon}\max(\omega, 0), \tag{46b}$$

$$\Phi : L^2(0,T;Y \times U) \ni (d,\varphi) \mapsto -\beta(d-\varphi) - (\kappa \circ H)(d) \in L^2(0,T;Y^*), \tag{46c}$$

$$\Psi : L^2(0,T;Y \times U) \ni (d,\varphi) \mapsto -\alpha\Delta\varphi + \beta\varphi - \beta d \in L^2(0,T;U^*), \tag{46d}$$

then (1) coincides with (Q), thanks to Lemma 9.1. Notice that $V \overset{d}{\hookrightarrow} U^*$ so that Assumption 1.1 is satisfied. Since $(\kappa \circ H) : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ is Gâteaux-differentiable, see Assumption 10, the requirement in Assumption 1.2 is fulfilled as well. Since $\max(\cdot,0) : L^2(\Omega) \to L^2(\Omega)$

is Lipschitz continuous and directionally differentiable, Assumption 1.3 is satisfied by (46b). Thus, the entire Assumption 1 holds true in the setting (46), cf. also Assumption 14.

Moreover, by employing Lemma 9.1, we see that Assumption 2.1 holds. The resulting solution operator of (2), i.e.,

$$S : L^2(0,T;H^1(\Omega)^*) \ni \ell \mapsto (d,\varphi) \in H_0^1(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega))$$

is directionally differentiable, cf. Lemma 9.2. According to the latter, its directional derivative $S'(\ell;\delta\ell)$ at ℓ in direction $\delta\ell$ is the unique solution of (43), and thus, of (3), whence Assumption 2.2 follows. According to [10, Lemma 2.4], $S : L^2(0,T;H^1(\Omega)^*) \to H_0^1(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega))$ is Lipschitz continuous, which implies that Assumption 2.3 is verified as well. Hence, the entire Assumption 2 is true for the setting (46).

It remains to check that Assumptions 3 and 4 are guaranteed. To this end, we observe that

$$\partial_\varphi \Phi(\bar d,\bar\varphi) = \beta\,\mathbb{I} : L^2(0,T;H^1(\Omega)) \to L^2(0,T;L^2(\Omega)).$$

As a result of $L^2(0,T;H^1(\Omega)) \overset{d}{\hookrightarrow} L^2(0,T;L^2(\Omega))$, the 'constraint qualification' in Assumption 3 is fulfilled. In the light of (46c)-(46d), the adjoints of the partial derivatives of Φ and Ψ are given by

$$\begin{aligned}
&\partial_d \Phi(\bar d,\bar\varphi)^* = -\beta\,\mathbb{I} - [(\kappa\circ H)'(\bar d)]^* : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega)),\\
&\partial_\varphi \Phi(\bar d,\bar\varphi)^* = \beta\,\mathbb{I} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;H^1(\Omega)^*),\\
&\partial_d \Psi(\bar d,\bar\varphi)^* = -\beta\,\mathbb{I} : L^2(0,T;H^1(\Omega)) \to L^2(0,T;L^2(\Omega)),\\
&\partial_\varphi \Psi(\bar d,\bar\varphi)^* = -\alpha\Delta + \beta\,\mathbb{I} : L^2(0,T;H^1(\Omega)) \to L^2(0,T;H^1(\Omega)^*).
\end{aligned} \tag{47}$$

Now Lemma 10 gives in turn that Assumption 4 is true for the setting (46). Thus, we can apply Theorem 5, which in combination with (47) tells us that there exist unique adjoint states $\xi \in H_T^1(0,T;L^2(\Omega))$ and $w \in L^2(0,T;H^1(\Omega))$ and a unique multiplier $\lambda \in L^2(0,T;L^2(\Omega))$ such that

$$\begin{aligned}
-\dot\xi + \beta\lambda + [(\kappa\circ H)'(\bar d)]^*\lambda - \beta w = \partial_d \mathcal{J}(\bar d,\bar\varphi,\bar\ell) \quad \text{in } L^2(0,T;L^2(\Omega)),\\
\xi(T) = 0,
\end{aligned} \tag{48a}$$

$$-\beta\lambda - \alpha\Delta w + \beta w = \partial_\varphi \mathcal{J}(\bar d,\bar\varphi,\bar\ell) \quad \text{in } L^2(0,T;H^1(\Omega)^*), \tag{48b}$$

$$\big(\xi(t), f'(\Phi(\bar d,\bar\varphi)(t);v)\big)_{L^2(\Omega)} \ge (\lambda(t),v)_{L^2(\Omega)} \quad \forall\, v \in L^2(\Omega), \text{ a.e. in } (0,T), \tag{48c}$$

$$w + \partial_\ell \mathcal{J}(\bar d,\bar\varphi,\bar\ell) = 0 \quad \text{in } L^2(0,T;H^1(\Omega)). \tag{48d}$$

It remains to show that (48c) implies (45c). Here, we recall the abbreviation $\bar z := -\beta(\bar d - \bar\varphi) - (\kappa \circ H)(\bar d)$ and (46c), i.e., $\bar z = \Phi(\bar d, \bar\varphi)$. An argument based on the fundamental lemma of calculus of variations and the positive homogeneity of the directional derivative w.r.t. direction yields

$$\frac{1}{\epsilon}\xi(t,x)\max{}'(\bar z(t,x);1) \geq \lambda(t,x) \geq -\frac{1}{\epsilon}\xi(t,x)\max{}'(\bar z(t,x);-1)$$
$$\text{a.e. in } (0,T)\times\Omega,$$

in view of (46b). The desired assertion now follows by distinguishing between the sets $\{(t,x) : \bar z(t,x) > 0\}$, $\{(t,x) : \bar z(t,x) < 0\}$ and $\{(t,x) : \bar z(t,x) = 0\}$. □

Remark 7. *If $\bar z(t,x) \neq 0$ a.e. in $(0,T)\times\Omega$, then (45) reduces to the standard KKT-conditions, since in this case, (45c) is equivalent to*

$$\lambda = \frac{1}{\epsilon}\max{}'(\bar z)\xi \quad \text{a.e. in } (0,T)\times\Omega.$$

The optimality system in Theorem 15 is indeed of strong stationary type, as the next result shows:

Theorem 16 (Equivalence between B- and strong stationarity)**.** *Assume that $\bar\ell \in L^2(0,T;L^2(\Omega))$ together with its states*

$$(\bar d, \bar\varphi) \in H_0^1(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega)),$$

some adjoint states

$$(\xi, w) \in H_T^1(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega)),$$

and a multiplier $\lambda \in L^2(0,T;L^2(\Omega))$ satisfy the optimality system (45a)–(45d). Then, it also satisfies the variational inequality

$$\partial_{(d,\varphi)}\mathcal{J}(\bar d,\bar\varphi,\bar\ell)S'(\bar\ell;\delta\ell) + \partial_\ell\mathcal{J}(\bar d,\bar\varphi,\bar\ell)\delta\ell \geq 0 \quad \forall\,\delta\ell \in L^2(0,T;L^2(\Omega)), \quad (49)$$

where $S : L^2(0,T;H^1(\Omega)^*) \to H_0^1(0,T;L^2(\Omega)) \times L^2(0,T;H^1(\Omega))$ is the solution mapping associated to (41), see Lemma 9.

Proof. We show the result by means of Theorem 6. In the proof of Theorem 15, we have seen that the problem (Q) fits in the setting from Section 2, i.e., Assumptions 1 and 2 are satisfied for the quantities in (46). According to the proof of Theorem 15, the system (5) coincides with (48) in this particular setting, see (47). We also note that (4) is just (49). Thus, in view of Theorem

6, we only need to show that (45c) implies (48c), which, in view of (46b) and (46c), reads

$$\big(\xi(t), \frac{1}{\epsilon} \max{}'(\bar{z}(t); v)\big)_{L^2(\Omega)} \geq (\lambda(t), v)_{L^2(\Omega)} \quad \forall\, v \in L^2(\Omega), \text{ a.e. in } (0,T), \tag{50}$$

where $\bar{z} := -\beta(\bar{d} - \bar{\varphi}) - (\kappa \circ H)(\bar{d})$.

To this end, let $v \in L^2(\Omega)$ be arbitrary, but fixed. From the first identity in (45c), we know that

$$\begin{aligned} \lambda(t,x)v(x) &= \frac{1}{\epsilon}\chi_{\{\bar{z}>0\}}(t,x)v(x)\xi(t,x) \\ &= \frac{1}{\epsilon} \max{}'(\bar{z}(t,x))v(x)\xi(t,x) \quad \text{a.e. where } \bar{z}(t,x) \neq 0. \end{aligned} \tag{51}$$

Further, we define $M^+ := \{(t,x) \in (0,T) \times \Omega : \bar{z}(t,x) = 0 \text{ and } v(x) > 0\}$ and $M^- := \{(t,x) \in (0,T) \times \Omega : \bar{z}(t,x) = 0 \text{ and } v(x) \leq 0\}$ (up to sets of measure zero). Then, the second identity in (45c) yields

$$\begin{aligned} \lambda(t,x)v(x) &\leq \begin{cases} \frac{1}{\epsilon}\xi(t,x)v(x) & \text{a.e. in } M^+ \\ 0 & \text{a.e. in } M^- \end{cases} \\ &= \frac{1}{\epsilon} \max{}'(\bar{z}(t,x); v(x))\xi(t,x) \quad \text{a.e. where } \bar{z}(t,x) = 0. \end{aligned} \tag{52}$$

Now, (50) follows from (51) and (52). Note that, since Assumptions 3 and 4 are fulfilled, cf. the proof of Theorem 15, we have the equivalence (49) $\Longleftrightarrow$ (45). □

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