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RESULTS OF EXISTENCE OF SOLUTIONS FOR SOME VARIATIONAL CONTROL INEQUALITIES*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

The paper deals with the study of solutions for some weak variational control inequalities of vector type, and the efficient solutions to the corresponding optimization problem. More exactly, to formulate and prove the principal results, we consider the Fréchet differentiability, the concept of invex set, and invexity & pseudoinvexity of the curvilinear integral type functionals which are involved in the study.

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1 Introduction

The concept of *efficient solution* is fundamental to investigate multi-objective optimization problems. Geoffrion [1] introduced the notion *proper efficiency*. Klinger [2] defined *improper solutions* for multi-objective optimization. Also, by considering variational inequalities of vector type, Kazmi [3] proved several existence results of a *weak minimum* in a class of problems with multiple objectives. *Approximate solutions* have been formulated by Ghaznavi-Ghosoni and Khorram [4] in order to establish some efficiency conditions for general multi-objective variational problems.

At the same time, the term of *convexity*, which plays a crucial role in variational analysis and optimization, does not cover all the real-life problems arising in science. Therefore, a generalization for this notion was a necessary requirement. Thus, Hanson [5] defined the new notion of *invex functions*. During this time, other extensions have been considered (see, for example, preinvexity, quasiinvexity, pseudoinvexity, approximate convexity, univexity) by authors like Mishra et al. [8], Antczak [6], Ahmad et al. [7]). In addition, a part of these concepts were transferred in the multidimensional framework (Treanţă [9], Mititelu et al. [10]).

Giannessi [11] established notable theorems for variational-type inequalities. Tiba [16, 17] provided important aspects on the optimal control of elliptic equations and nonsmooth distributed parameter systems. Vector variational inequalities help us to formulate existence results in multi-objective optimization problems (the reader can consult Ruiz-Garzón et al. [12]). As a natural continuation of the above-mentioned research papers, Treanţă [13] investigated a class of variational control inequalities driven by curvilinear integral type functionals. Also, Kim [14] stated important relations between multiple objective variational problems and vector variational inequalities. As a generalization of variational problems in continuous time, the controlled variational problems were considered in studying many processes in operations research, economics, and game theory. Thus, Treanţă [18, 19], Jha et al. [15] formulated conditions of efficiency, modified objective function method, saddle-point optimality criteria, and well-posedness of multi-time variational problems generated by multiple/curvilinear integral functionals. Recently, Treanţă [20] stated some relations of solutions for some variational control inequalities of vector type, and (proper and efficient) solutions of the attached multiple objective variational problems. In the current paper, we introduce weak vector-controlled variational inequalities governed by partial derivatives of second order and the associated multi-objective variational control problem driven by path-independent curvilinear integral-type functionals. By considering invex sets, Fréchet differentiability, invexity and pseudoinvexity attached with the involved integral functionals (which calculate, in physical terms, the mechanical work), we formulate and prove some connections between solutions of the studied variational control problems. An application and an example to show the aforementioned class of vector variational control inequalities is solvable at a point can be consulted in Treanţă [22].

This study is structured as follows. The preliminaries/auxiliary results and problem description are given in Section 2; in Section 3, we formulate and prove some characterization theorems of solutions for the studied variational control problems; in Section 4, we give the study's conclusions.

2 Preliminaries

Let $\mathcal{U}$ be a compact set in $\mathbb{R}^b$ and $\mathcal{U} \ni u = (u^m)$, $m = \overline{1,b}$, be a multiple variable of evolution. Consider $\mathcal{U} \supset \Gamma : u = u(\varsigma), \varsigma \in [t_0, t_1]$, is a piece-wise differentiable curve that links the fixed points $u_1 = (u_1^1, \dots, u_1^b)$, $u_2 = (u_2^1, \dots, u_2^b)$ in $\mathcal{U}$. Also, let B be the space of piece-wise differentiable functions $h : \mathcal{U} \to \mathbb{R}^v$ (*state* variables) and C be the space of piece-wise continuous functions $\lambda : \mathcal{U} \to \mathbb{R}^s$ (*control* variables).

Next, let us define a p-dimensional functional driven by curvilinear integrals $G : \mathsf{B} \times \mathsf{C} \to \mathbb{R}^p$,

$$G(h, \lambda) = \int_\Gamma g_m\left(u, h(u), h_n(u), h_{\gamma\delta}(u), \lambda(u)\right) du^m$$

$$= \Big(\int_\Gamma g_m^1\left(u, h(u), h_n(u), h_{\gamma\delta}(u), \lambda(u)\right) du^m,$$

$$\dots, \int_\Gamma g_m^p\left(u, h(u), h_n(u), h_{\gamma\delta}(u), \lambda(u)\right) du^m \Big),$$

where the vector-valued functions $g_m = (g_m^\zeta) : \mathcal{U} \times \mathbb{R}^v \times \mathbb{R}^{vb} \times \mathbb{R}^{vb^2} \times \mathbb{R}^s \to \mathbb{R}^p$, $m = \overline{1,b}$, $\zeta = \overline{1,p}$, are of C^3-class, D_n, $n \in \{1, \dots, b\}$, is the total derivative operator, $D^2_{\gamma\delta} := D_\gamma(D_\delta)$, $h_n(u) := \dfrac{\partial h}{\partial u^n}(u)$, $h_{\gamma\delta}(u) := \dfrac{\partial^2 h}{\partial u^\gamma \partial u^\delta}(u)$, and assume the Lagrange 1-form densities

$$g_m = \left(g_m^1, \dots, g_m^p\right) : \mathcal{U} \times \mathbb{R}^v \times \mathbb{R}^{vb} \times \mathbb{R}^{vb^2} \times \mathbb{R}^s \to \mathbb{R}^p, \quad m = \overline{1,b},$$

satisfy $D_n g_m^\zeta = D_m g_n^\zeta,\ m,n=\overline{1,b},\ m\neq n,\ \zeta=\overline{1,p}$. Also, we consider the following hypotheses:

$$\sigma=\epsilon \Leftrightarrow \sigma^\zeta=\epsilon^\zeta,\quad \sigma\le\epsilon \Leftrightarrow \sigma^\zeta\le\epsilon^\zeta,$$

$$\sigma<\epsilon \Leftrightarrow \sigma^\zeta<\epsilon^\zeta,\quad \sigma\preceq\epsilon \Leftrightarrow \sigma\le\epsilon,\ \sigma\neq\epsilon,\quad \zeta=\overline{1,p},$$

for all $\sigma=\left(\sigma^1,\cdots,\sigma^p\right),\ \epsilon=\left(\epsilon^1,\cdots,\epsilon^p\right)$ in $\mathbb{R}^p$.

Now, let us define the following variational control problem with mixed constraints

$$(\mathcal{P})\qquad \min_{(h,\lambda)}\left\{G(h,\lambda)=\int_\Gamma g_m\left(u,h(u),h_n(u),h_{\gamma\delta}(u),\lambda(u)\right)du^m\right\}$$

$$\text{subject to } (h,\lambda)\in\mathcal{F},$$

where

$$\begin{aligned}G(h,\lambda)&=\int_\Gamma g_m\left(u,h(u),h_n(u),h_{\gamma\delta}(u),\lambda(u)\right)du^m\\&=\Big(\int_\Gamma g_m^1\left(u,h(u),h_n(u),h_{\gamma\delta}(u),\lambda(u)\right)du^m,\\&\dots,\int_\Gamma g_m^p\left(u,h(u),h_n(u),h_{\gamma\delta}(u),\lambda(u)\right)du^m\Big)\\&=\left(G^1(h,\lambda),\dots,G^p(h,\lambda)\right)\end{aligned}$$

and

$$\mathcal{F}=\Big\{(h,\lambda)\in\mathsf{B}\times\mathsf{C}\mid Y\left(u,h(u),h_n(u),h_{\gamma\delta}(u),\lambda(u)\right)\le 0,$$

$$A\left(u,h(u),h_n(u),h_{\gamma\delta}(u),\lambda(u)\right):=h_{\gamma\delta}(u)-T\left(u,h(u),h_n(u),\lambda(u)\right)=0,$$

$$h|_{u=u_1,u_2}=\text{given},\ h_n|_{u=u_1,u_2}=\text{given}\Big\}.$$

Previous, we have considered $A=(A^\iota):\mathcal{U}\times\mathbb{R}^v\times\mathbb{R}^{vb}\times\mathbb{R}^{vb^2}\times\mathbb{R}^s\to\mathbb{R}^t,\ \iota=\overline{1,t},\ Y=(Y^r):\mathcal{U}\times\mathbb{R}^v\times\mathbb{R}^{vb}\times\mathbb{R}^{vb^2}\times\mathbb{R}^s\to\mathbb{R}^a,\ r=\overline{1,a}$, are some C^2-class functions.

Definition 1. *The pair $(h^0,\lambda^0)\in\mathcal{F}$ is said to be efficient solution of $(\mathcal{P})$ if there exists no other $(h,\lambda)\in\mathcal{F}$ satisfying $G(h,\lambda)\preceq G(h^0,\lambda^0)$, or, for at least one ζ, we have $G^\zeta(h,\lambda)-G^\zeta(h^0,\lambda^0)\le 0$ with strict inequality.*

Definition 2. *The pair $(h^0,\lambda^0)\in\mathcal{F}$ is said to be weak efficient solution in $(\mathcal{P})$ if there exists no other $(h,\lambda)\in\mathcal{F}$ satisfying $G(h,\lambda)<G(h^0,\lambda^0)$, or $G^\zeta(h,\lambda)-G^\zeta(h^0,\lambda^0)<0,\ (\forall)\zeta=\overline{1,p}$.*

In the following, in accordance with Treanţă [20, 21] and by using Saunders's multi-index notation (Saunders [23]), we consider a vector-valued functional of curvilinear integral type (assumed to be path-independent)

$$K : \mathsf{B} \times \mathsf{C} \to \mathbb{R}^p, \quad K(h, \lambda) = \int_\Gamma \kappa_m \left(u, h(u), h_n(u), h_{\gamma\delta}(u), \lambda(u)\right) du^m.$$

Definition 3. *The functional K is called invex at $\left(h^0, \lambda^0\right) \in \mathsf{B} \times \mathsf{C}$ with respect to ρ and μ if there exist*

$$\rho : \mathcal{U} \times (\mathbb{R}^v \times \mathbb{R}^s)^2 \to \mathbb{R}^v,$$

$$\rho = \rho\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right) = \left(\rho^i\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right)\right),\ i = \overline{1, v},$$

of C^2-class with $\rho\left(u, h^0(u), \lambda^0(u), h^0(u), \lambda^0(u)\right) = 0$, $(\forall) u \in \mathcal{U}$, $\rho(u_1) = \rho(u_2) = 0$ and

$$\mu : \mathcal{U} \times (\mathbb{R}^v \times \mathbb{R}^s)^2 \to \mathbb{R}^s,$$

$$\mu = \mu\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right) = \left(\mu^j\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right)\right),\ j = \overline{1, s},$$

of C^0-class with $\mu\left(u, h^0(u), \lambda^0(u), h^0(u), \lambda^0(u)\right) = 0$, $(\forall) u \in \mathcal{U}$, $\mu(u_1) = \mu(u_2) = 0$, such that

$$\begin{aligned}
&K(h, \lambda) - K\left(h^0, \lambda^0\right) \\
&\geq \int_\Gamma \Big[\frac{\partial \kappa_m}{\partial h}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) \rho \\
&+ \frac{\partial \kappa_m}{\partial h_n}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) D_n \rho\Big] du^m \\
&+ \frac{1}{x(\gamma, \delta)} \int_\Gamma \left[\frac{\partial \kappa_m}{\partial h_{\gamma\delta}}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) D_{\gamma\delta} \rho\right] du^m \\
&+ \int_\Gamma \left[\frac{\partial \kappa_m}{\partial \lambda}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) \mu\right] du^m,
\end{aligned}$$

for $(h, \lambda) \in \mathsf{B} \times \mathsf{C}$.

Definition 4. *In Definition 3, if we replace $\geq$ with $>$, and consider $(h, \lambda) \neq \left(h^0, \lambda^0\right)$, we obtain the strictly invexity of K at $\left(h^0, \lambda^0\right) \in \mathsf{B} \times \mathsf{C}$ with respect to ρ and μ.*

Definition 5. *The functional K is called pseudoinvex at $\left(h^0, \lambda^0\right) \in \mathsf{B} \times \mathsf{C}$ with respect to ρ and μ if there exist*

$$\rho : \mathcal{U} \times (\mathbb{R}^v \times \mathbb{R}^s)^2 \to \mathbb{R}^v,$$

$\rho = \rho\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right) = \left(\rho^i\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right)\right),\ i = \overline{1, v}$, *of* C^2*-class with* $\rho\left(u, h^0(u), \lambda^0(u), h^0(u), \lambda^0(u)\right) = 0,\ (\forall) u \in \mathcal{U},\ \rho(u_1) = \rho(u_2) = 0$ *and*

$$\mu : \mathcal{U} \times (\mathbb{R}^v \times \mathbb{R}^s)^2 \to \mathbb{R}^s,$$

$\mu = \mu\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right) = \left(\mu^j\left(u, h(u), \lambda(u), h^0(u), \lambda^0(u)\right)\right),\ j = \overline{1, s}$, *of* C^0*-class with* $\mu\left(u, h^0(u), \lambda^0(u), h^0(u), \lambda^0(u)\right) = 0,\ (\forall) u \in \mathcal{U},\ \mu(u_1) = \mu(u_2) = 0$, *such that*

$$K(h, \lambda) - K\left(h^0, \lambda^0\right) < 0$$

involves

$$\begin{aligned}
&\int_\Gamma \Big[\frac{\partial \kappa_m}{\partial h}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right)\rho \\
&\quad + \frac{\partial \kappa_m}{\partial h_n}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) D_n\rho\Big] du^m \\
&\quad + \frac{1}{x(\gamma,\delta)}\int_\Gamma \left[\frac{\partial \kappa_m}{\partial h_{\gamma\delta}}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) D_{\gamma\delta}\rho\right] du^m \\
&\quad + \int_\Gamma \left[\frac{\partial \kappa_m}{\partial \lambda}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right)\mu\right] du^m < 0,
\end{aligned}$$

or, in an equivalent manner,

$$\begin{aligned}
&\int_\Gamma \Big[\frac{\partial \kappa_m}{\partial h}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right)\rho \\
&\quad + \frac{\partial \kappa_m}{\partial h_n}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) D_n\rho\Big] du^m \\
&\quad + \frac{1}{x(\gamma,\delta)}\int_\Gamma \left[\frac{\partial \kappa_m}{\partial h_{\gamma\delta}}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right) D_{\gamma\delta}\rho\right] du^m \\
&\quad + \int_\Gamma \left[\frac{\partial \kappa_m}{\partial \lambda}\left(u, h^0(u), h^0_n(u), h^0_{\gamma\delta}(u), \lambda^0(u)\right)\mu\right] du^m \geq 0 \\
&\quad \Rightarrow K(h, \lambda) - K\left(h^0, \lambda^0\right) \geq 0,
\end{aligned}$$

for $(h, \lambda) \in \mathsf{B} \times \mathsf{C}$.

Invex and/or pseudoinvex functionals of curvilinear integral type can be seen in Treanţă [21].

Definition 6. (Treanţă [20]) *The subset* $\emptyset \neq \mathsf{X} \times \mathsf{Q} \subset \mathsf{B} \times \mathsf{C}$ *is named invex with respect to* ρ *and* μ *if*

$$(h^0, \lambda^0) + \nu\left(\rho\left(u, h, \lambda, h^0, \lambda^0\right), \mu\left(u, h, \lambda, h^0, \lambda^0\right)\right) \in \mathsf{X} \times \mathsf{Q}$$

for any $\nu \in [0, 1]$ *and* $(h, \lambda),\ (h^0, \lambda^0) \in \mathsf{X} \times \mathsf{Q}$.

For establishing some characterization results of $(\mathcal{P})$, we formulate the next *weak variational control inequalities of vector type*: *find* $(h^0,\lambda^0)\in\mathcal{F}$ *so that there exists no* $(h,\lambda)\in\mathcal{F}$ *satisfying*

$$
\begin{aligned}
(WVI)\quad & \Bigg(\int_\Gamma \left[\frac{\partial g_m^1}{\partial h}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)\rho\right]du^m\\
&+\int_\Gamma \left[\frac{\partial g_m^1}{\partial \lambda}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)\mu\right]du^m\\
&+\int_\Gamma \left[\frac{\partial g_m^1}{\partial h_n}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)D_n\rho\right]du^m\\
&+\frac{1}{x(\gamma,\delta)}\int_\Gamma \left[\frac{\partial g_m^1}{\partial h_{\gamma\delta}}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)D_{\gamma\delta}\rho\right]du^m,\\
&\dots,\int_\Gamma \left[\frac{\partial g_m^p}{\partial h}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)\rho\right]du^m\\
&+\int_\Gamma \left[\frac{\partial g_m^p}{\partial \lambda}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)\mu\right]du^m\\
&+\int_\Gamma \left[\frac{\partial g_m^p}{\partial h_n}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)D_n\rho\right]du^m\\
&+\frac{1}{x(\gamma,\delta)}\int_\Gamma \left[\frac{\partial g_m^p}{\partial h_{\gamma\delta}}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)D_{\gamma\delta}\rho\right]du^m\Bigg)\\
&<0.
\end{aligned}
$$

An application and an example to illustrate the aforementioned class of variational control inequalities of vector type is solvable at a point can be consulted in Treanţă [22].

3 Main results

This part of our study establishes the characterization theorems for the variational control problem $(\mathcal{P})$ and weak variational control inequalities of vector type (WVI).

A sufficient condition for $(h^0,\lambda^0)\in\mathcal{F}$ to become a solution to (WVI) is provided by the next theorem.

Theorem 1. *Let* $\mathcal{F}$ *be an invex set, assume that* $(h^0,\lambda^0)\in\mathcal{F}$ *is a weak efficient solution in* $(\mathcal{P})$*, and each* $G^\zeta(h,\lambda)$*,* $\zeta=\overline{1,p}$*, is Fréchet differentiable at the point* $(h^0,\lambda^0)\in\mathcal{F}$*. Then the pair* (h^0,λ^0) *is solution to* (WVI)*.*

Proof. Since, by hypothesis, $(h^0, \lambda^0) \in \mathcal{F}$ is a weak efficient solution to $(\mathcal{P})$, thus, there exists no other $(h, \lambda) \in \mathcal{F}$ with $G(h, \lambda) < G(h^0, \lambda^0)$, or, in an equivalent way,

$$G^\zeta(h, \lambda) - G^\zeta(h^0, \lambda^0) < 0, \quad (\forall)\zeta = \overline{1, p}. \tag{1}$$

Also, since $\mathcal{F} \subset \mathsf{B} \times \mathsf{C}$ is an invex set, for $\nu \in [0, 1]$, it results $(z, w) = (h^0, \lambda^0) + \nu \left(\rho \left(u, h, \lambda, h^0, \lambda^0\right), \mu \left(u, h, \lambda, h^0, \lambda^0\right)\right) \in \mathcal{F}$. In consequence, by using (1), we get that there exists no other feasible solution $(h, \lambda) \in \mathcal{F}$ such that $G(z, w) < G(h^0, \lambda^0)$, or, in an equivalent way,

$$G^\zeta(z, w) - G^\zeta(h^0, \lambda^0) < 0, \quad (\forall)\zeta = \overline{1, p}. \tag{2}$$

Furthermore, by (2) and taking into account that each $G^\zeta(h, \lambda)$, $\zeta = \overline{1, p}$, is Fréchet differentiable at $(h^0, \lambda^0) \in \mathcal{F}$, we get there exists no other $(h, \lambda) \in \mathcal{F}$ with

$$\begin{aligned}
&\int_\Gamma \Big[\frac{\partial g_m^\zeta}{\partial h} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) \rho \\
&\quad + \frac{\partial g_m^\zeta}{\partial \lambda} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) \mu \Big] du^m \\
&\quad + \int_\Gamma \left[\frac{\partial g_m^\zeta}{\partial h_n} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) D_n \rho \right] du^m \\
&\quad + \frac{1}{x(\gamma, \delta)} \int_\Gamma \left[\frac{\partial g_m^\zeta}{\partial h_{\gamma\delta}} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) D_{\gamma\delta} \rho \right] du^m < 0
\end{aligned}$$

for all $\zeta = \overline{1, p}$, and this completes the proof. □

The next result, by using the weak variational control inequality of vector type (WVI), gives us a characterization of weak efficient solutions for $(\mathcal{P})$.

Theorem 2. *Let $(h^0, \lambda^0) \in \mathcal{F}$ be a solution to (WVI), each $G^\zeta(h, \lambda)$, $\zeta = \overline{1, p}$, is pseudoinvex and Fréchet differentiable at $(h^0, \lambda^0) \in \mathcal{F}$. Then (h^0, λ^0) is a weak efficient solution to $(\mathcal{P})$.*

Proof. Let us consider, by contrary, that $(h^0, \lambda^0) \in \mathcal{F}$ is a solution of (WVI) but it is not a weak efficient solution to $(\mathcal{P})$. Therefore, there exists $(h, \lambda) \in \mathcal{F}$ satisfying, for all $\zeta = \overline{1, p}$, the following inequality

$$G^\zeta(h, \lambda) - G^\zeta(h^0, \lambda^0) < 0.$$

Since each $G^{\zeta}(h,\lambda)$, $\zeta=\overline{1,p}$, is pseudoinvex and Fréchet differentiable at $(h^0,\lambda^0)\in\mathcal{F}$, we obtain

$$\int_{\Gamma}\Big[\frac{\partial g_m^{\zeta}}{\partial h}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)\rho$$
$$+\frac{\partial g_m^{\zeta}}{\partial \lambda}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)\mu\Big]du^m$$
$$+\int_{\Gamma}\left[\frac{\partial g_m^{\zeta}}{\partial h_n}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)D_n\rho\right]du^m$$
$$+\frac{1}{x(\gamma,\delta)}\int_{\Gamma}\left[\frac{\partial g_m^{\zeta}}{\partial h_{\gamma\delta}}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)D_{\gamma\delta}\rho\right]du^m<0,$$

for $(h,\lambda)\in\mathcal{F}$ and $\zeta=\overline{1,p}$. This is in contradiction with $(h^0,\lambda^0)\in\mathcal{F}$ is a solution of (WVI). □

Finally, we formulate a sufficient condition such that a weak efficient solution of $(\mathcal{P})$ becomes an efficient solution of $(\mathcal{P})$.

Theorem 3. *Consider $(h^0,\lambda^0)\in\mathcal{F}$ is a weak efficient solution in $(\mathcal{P})$, each integral $G^{\zeta}(h,\lambda)$, $\zeta=\overline{1,p}$, is strictly invex and Fréchet differentiable at $(h^0,\lambda^0)\in\mathcal{F}$, and $\mathcal{F}$ is an invex set. Then (h^0,λ^0) is an efficient solution to $(\mathcal{P})$.*

Proof. Let assume, by contrary, that $(h^0,\lambda^0)\in\mathcal{F}$ is a weak efficient solution to $(\mathcal{P})$ but it is not an efficient solution to $(\mathcal{P})$. Thus, there exists $(h,\lambda)\in\mathcal{F}$ with $G(h,\lambda)\preceq G(h^0,\lambda^0)$, or, for at least one ζ, the relation

$$G^{\zeta}(h,\lambda)-G^{\zeta}(h^0,\lambda^0)\le 0,\ (\forall)\zeta=\overline{1,p}, \tag{3}$$

is with strict inequality.

Since each integral $G^{\zeta}(h,\lambda)$, $\zeta=\overline{1,p}$, is Fréchet differentiable and strictly invex at $(h^0,\lambda^0)\in\mathcal{F}$ with respect to ρ and μ, we can write

$$G^{\zeta}(h,\lambda)-G^{\zeta}(h^0,\lambda^0)>\int_{\Gamma}\left[\frac{\partial g_m^{\zeta}}{\partial h}\left(u,h^0(u),h_n^0(u),h_{\gamma\delta}^0(u),\lambda^0(u)\right)\rho\right]du^m \tag{4}$$

$$+ \int_{\Gamma} \left[\frac{\partial g_m^{\zeta}}{\partial \lambda} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) \mu \right] du^m$$
$$+ \int_{\Gamma} \left[\frac{\partial g_m^{\zeta}}{\partial h_n} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) D_n\rho \right] du^m$$
$$+ \frac{1}{x(\gamma,\delta)} \int_{\Gamma} \left[\frac{\partial g_m^{\zeta}}{\partial h_{\gamma\delta}} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) D_{\gamma\delta}\rho \right] du^m$$

for any $(h,\lambda) \neq (h^0,\lambda^0) \in \mathcal{F}$ and $\zeta = \overline{1,p}$. By (3) and (4), it results there exists $(h,\lambda) \in \mathcal{F}$ satisfying

$$\int_{\Gamma} \Big[\frac{\partial g_m^{\zeta}}{\partial h} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) \rho$$
$$+ \frac{\partial g_m^{\zeta}}{\partial \lambda} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) \mu \Big] du^m$$
$$+ \int_{\Gamma} \left[\frac{\partial g_m^{\zeta}}{\partial h_n} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) D_n\rho \right] du^m$$
$$+ \frac{1}{x(\gamma,\delta)} \int_{\Gamma} \left[\frac{\partial g_m^{\zeta}}{\partial h_{\gamma\delta}} \left(u, h^0(u), h_n^0(u), h_{\gamma\delta}^0(u), \lambda^0(u)\right) D_{\gamma\delta}\rho \right] du^m < 0,$$

for all $\zeta = \overline{1,p}$. Therefore, $(h^0,\lambda^0) \in \mathcal{F}$ is not a solution in (WVI). Now, according to Theorem 3.1, we obtain $(h^0,\lambda^0) \in \mathcal{F}$ isn't a weak efficient solution to $(\mathcal{P})$. The proof is now complete. □

4 Conclusions

In this paper, motivated by its physical significance (mechanical work), we investigated the solutions for some weak variational control inequalities of vector type, and the efficient solutions to the corresponding optimization problem. More exactly, to formulate and prove the principal results, we have considered the Fréchet differentiability, the concept of invex set, and invexity & pseudoinvexity of the curvilinear integral type functionals which are involved in the study.

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