Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 15, No. 1-2/2023

OPTIMAL TEMPERATURE DISTRIBUTION FOR A NONISOTHERMAL CAHN–HILLIARD SYSTEM IN TWO DIMENSIONS WITH SOURCE TERM AND DOUBLE OBSTACLE POTENTIAL*

P. Colli† G. Gilardi‡ A. Signori § J. Sprekels¶

DOI https://doi.org/10.56082/annalsarscimath.2023.1-2.175

Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

In this note, we study the optimal control of a nonisothermal phase field system of Cahn–Hilliard type that constitutes an extension of the classical Caginalp model for nonisothermal phase transitions with a conserved order parameter. It couples a Cahn–Hilliard type equation with source term for the order parameter with the universal balance

*Accepted for publication on March 27-th, 2023

†`pierluigi.colli@unipv.it` Dipartimento di Matematica, Università Degli Studi di Pavia "F. Casorati", and Research Associate at the IMATI – C.N.R. Pavia, via Ferrata 5, I–27100 Pavia, Italy;

‡`gianni.gilardi@unipv.it` Dipartimento di Matematica, Università Degli Studi di Pavia "F. Casorati", and Research Associate at the IMATI – C.N.R. Pavia, via Ferrata 5, I–27100 Pavia, Italy;

§`andrea.signori@polimi.it` Dipartimento di Matematica, Politecnico di Milano, via E. Bonardi 9, I-20133 Milano, Italy;

¶`juergen.sprekels@wias-berlin.de` Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, D–10099 Berlin, Germany, and Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D–10117 Berlin, Germany

law of internal energy. In place of the standard Fourier form, the constitutive law of the heat flux is assumed in the form given by the theory developed by Green and Naghdi, which accounts for a possible thermal memory of the evolution. This has the consequence that the balance law of internal energy becomes a second-order in time equation for the *thermal displacement* or *freezing index*, that is, a primitive with respect to time of the temperature. Another particular feature of our system is the presence of the source term in the equation for the order parameter, which entails further mathematical difficulties because the mass conservation of the order parameter is no longer satisfied. In this paper, we study the case that the double-well potential driving the evolution of the phase transition is given by the nondifferentiable double obstacle potential, thereby complementing recent results obtained for the differentiable cases of regular and logarithmic potentials. Besides existence results, we derive first-order necessary optimality conditions for the control problem. The analysis is carried out by employing the so-called *deep quench approximation* in which the nondifferentiable double obstacle potential is approximated by a family of potentials of logarithmic structure for which meaningful first-order necessary optimality conditions in terms of suitable adjoint systems and variational inequalities are available. Since the results for the logarithmic potentials crucially depend on the validity of the so-called *strict separation property* which is only available in the spatially two-dimensional situation, our whole analysis is restricted to the two-dimensional case.

MSC: 35K20, 35K55, 49J50, 49J52, 49K20.

keywords: Optimal control, nonisothermal Cahn–Hilliard equation, thermal memory, Cahn–Hilliard equation with source term, Cahn–Hilliard–Oono equation.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be some open, bounded, and connected set having a smooth boundary $\Gamma := \partial\Omega$ and the outward unit normal field $\mathbf{n}$. Denoting by $\partial_{\mathbf{n}}$ the directional derivative in the direction of $\mathbf{n}$, and putting, with a fixed final time $T > 0$,

$$Q_t := \Omega \times (0,t) \text{ and } \Sigma_t := \Gamma \times (0,t) \text{ for } t \in (0,T], \quad Q := Q_T, \quad \Sigma := \Sigma_T,$$