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A NOTE ON A CLASSICAL CONNECTION BETWEEN PARTITIONS AND DIVISORS*

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Dedicated to Dr. Dan Tiba on the occasion of his 70^{th} anniversary

Abstract

In this note, we consider the number of k's in all the partitions of n in order to provide a new proof of a classical identity involving Euler's partition function $p(n)$ and the sum of the positive divisors function $\sigma(n)$. New relations connecting classical functions of multiplicative number theory with the partition function $p(n)$ from additive number theory are introduced in this context. The fascinating feature of these relations is their common nature. A new identity for the number of 1's in all the partitions of n is derived in this context.

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1 Introduction

Let A be a given set of positive integers, and let $f(n)$ be a given arithmetical function. By Apostol [3, Theorem 14.8], we know that the numbers $p_{A,f}(n)$

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defined by the equation

$$\prod_{n\in A}(1-q^n)^{-f(n)/n} = 1+\sum_{n=1}^{\infty} p_{A,f}(n)\,q^n \tag{1}$$

satisfy the recurrence relation

$$n\,p_{A,f}(n) = \sum_{k=1}^{n} f_A(k)\,p_{A,f}(n-k), \tag{2}$$

where $p_{A,f}(0)=1$ and

$$f_A(n) = \sum_{\substack{d|n\\ d\in A}} f(d).$$

The formula (2) was derived by logarithmic differentiation of generating functions

$$F_A(q) = \prod_{n\in A}(1-q^n)^{-f(n)/n}$$

and

$$G_A(q) = \sum_{n\in A}\frac{f(n)}{n}\,q^n.$$

If A is the set of all positive integers, then for $f(n)=n$ we have

$$p_{A,f}(n) = p(n),$$

the unrestricted partition function, and

$$f_A(n) = \sigma(n),$$

the sum of the positive divisors of n.

Recall that a partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n [1]. For example, the following are the partitions of 6:

$$\begin{aligned}&(6),\ (5,1),\ (4,2),\ (4,1,1),\ (3,3),\ (3,2,1),\ (3,1,1,1),\\ &(2,2,2),\ (2,2,1,1),\ (2,1,1,1,1),\ (1,1,1,1,1,1).\end{aligned} \tag{3}$$

In this context, the equation (2) provides a remarkable relation connecting a function of multiplicative number theory with one of additive number theory, namely

$$n\,p(n) = \sum_{k=1}^{n}\sigma(k)\,p(n-k). \tag{4}$$

In this paper, we provide a new proof for the relation (4) considering the number of k's in all partitions of n. We denoted this number by $S_{n,k}$. By (3), we see that $S_{6,1} = 19$, $S_{6,2} = 8$, $S_{6,3} = 4$, $S_{6,4} = 2$, $S_{6,5} = 1$ and $S_{6,6} = 1$.

Theorem 1. *Let n be a positive integer. If $g(n) = \sum_{d|n} f(d)$, then*

$$\sum_{k=1}^{n} f(k)\, S_{n,k} = \sum_{k=1}^{n} g(k)\, p(n-k). \tag{5}$$

The general nature of the function $f(n)$ allows for applications of Theorem 1 to classical functions from multiplicative number theory: the divisor function $\sigma_x(n)$, the Möbius function $\mu(n)$, Euler's totient $\varphi(n)$, Jordan's totient $J_k(n)$, Liouville's function $\lambda(n)$, and others. The fascinating feature of these identities is their common nature.

2 Proof of Theorem 1

We first sketch the proof of the generating function for $S_{n,k}$. Note that the generating function for partitions where z keeps track of parts equal to k is given by

$$(1 + z\, q^k + z^2\, q^{k+k} + z^3\, q^{k+k+k} + \cdots) \prod_{\substack{n=1\\ n\neq k}}^{\infty} (1 + q^n + q^{n+n} + q^{n+n+n} + \cdots)$$
$$= \frac{1-q^k}{1-z\, q^k} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1-q^k}{(1-z\, q^k)(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. In this note, all identities involving infinite products of the form $(a;q)_\infty$ may be understood in the sense of formal power series in q.

Taking the derivative with respect to z, and setting z equal to 1, we obtain the expression of the generating function for $S_{k,n}$:

$$\begin{aligned}\sum_{n=k}^{\infty} S_{n,k}\, q^n &= \frac{d}{dz} \frac{1-q^k}{(1-z\,q^k)(q;q)_\infty}\bigg|_{z=1} \\ &= \frac{q^k}{1-q^k} \cdot \frac{1}{(q;q)_\infty}. \end{aligned} \tag{6}$$

Multiplying both sides of (6) by $f(k)$, we derive the relation

$$\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} S_{n,k}\, q^n \right) f(k) = \frac{1}{(q;q)_\infty} \sum_{k=1}^{\infty} f(k) \frac{q^k}{1-q^k},$$

that can be rewritten as

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} f(k)\, S_{n,k} \right) q^n = \left(\sum_{n=0}^{\infty} p(n)\, q^n \right) \left(\sum_{k=1}^{\infty} g(k)\, q^k \right) \tag{7}$$

where we have invoked the well-known generating function of $p(n)$

$$\sum_{n=0}^{\infty} p(n)\, q^n = \frac{1}{(q;q)_\infty},$$

and the well-known Lambert series

$$\sum_{k=1}^{\infty} \frac{f(k)\, q^k}{1-q^k} = \sum_{k=1}^{\infty} \left(\sum_{d|k} f(k) \right) q^k.$$

Equating the coefficient of q^n in (7) concludes the proof.

3 Some applications

Theorem 1 can be used to provide new connections between the partitions and many classical special arithmetic functions often studied in multiplicative number theory: the divisor function $\sigma_x(n)$, the Möbius function $\mu(n)$, Euler's totient $\varphi(n)$, Jordan's totient $J_k(n)$, and Liouville's function $\lambda(n)$.

3.1 Divisor functions

Firstly, we remark that

$$\sum_{k=1}^{n} k\, S_{n,k} = n\, p(n).$$

So it is clear that the relation (4) is the case $f(n) = n$ of our theorem.

By transposing the Ferrers graph of each partition of n it follows that the total number of parts in all the partitions of n equals the sum of largest parts of all the partitions of n. Considering $f(n) = 1$ in Theorem 1, we derive the following result.

Corollary 1. *For $n > 0$, the sum of largest parts of all partitions of n can be expressed as*

$$\sum_{k=1}^{n} \tau(k)\, p(n-k),$$

where $\tau(n)$ counts the positive divisors of n.

Example 1. *According to* (3) *and Corollary 1, the sum of largest parts of all the partitions of* 6 *can be expressed as*

$$\begin{aligned}
&6+5+4+4+3+3+3+2+2+2+1\\
&= p(5)+2\,p(4)+2\,p(3)+3\,p(2)+2\,p(1)+4\,p(0)\\
&= 7+10+6+6+2+4 = 35.
\end{aligned}$$

The cases $f(n) = (-1)^n$ and $f(n) = (-1)^n \cdot n$ of Theorem 1 read as follows.

Corollary 2. *For $n > 0$, the difference between the number of odd parts and the number of even parts in all the partitions of n can be expressed as*

$$\sum_{k=1}^{n} \tau_{o,e}(k)\, p(n-k),$$

where $\tau_{o,e}(n)$ is the difference between the number of odd divisors and the number of even divisors of n.

Example 2. *According to* (3) *and Corollary 2, the difference between the number of odd parts and the number of even parts in all the partitions of* 6 *can be expressed as*

$$\begin{aligned}
&(0+2+0+2+2+2+4+0+2+4+6)\\
&\qquad\qquad -(1+0+2+1+0+1+0+3+2+1+0)\\
&= p(5)+2\,p(3)-\,p(2)+2\,p(1) = 7+6-2+2 = 13.
\end{aligned}$$

Corollary 3. *For $n > 0$, the difference between the sum of odd parts and the sum of even parts in all the partitions of n can be expressed as*

$$\sum_{k=1}^{n} \sigma_{o,e}(k)\, p(n-k),$$

where $\sigma_{o,e}(n)$ is the difference between the sum of odd divisors and the sum of even divisors of n.

Example 3. *According to* (3) *and Corollary 3, the difference between the sum of odd parts and the sum of even parts in all the partitions of* 6 *can be expressed as:*

$$\begin{aligned}&(0+6+0+2+6+4+6+0+2+4+6)\\&\qquad\qquad -(6+0+6+4+0+2+0+6+4+2+0)\\&=p(5)-p(4)+4\,p(3)-5\,p(2)+6\,p(1)-4\,p(0)\\&=7-5+12-10+6-4=6.\end{aligned}$$

3.2 Möbius function

The classical Möbius function $\mu(n)$ is defined for all positive integers n and has its values in $\{-1,0,1\}$ depending on the factorization of n into prime factors:

$$\mu(n)=\begin{cases}0, & \text{if } n \text{ has a squared prime factor,}\\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes.}\end{cases}$$

The sum over all positive divisors of n of the Möbius function is zero except when $n = 1$. By Theorem 1, with $f(n) = \mu(n)$ we obtain the following decomposition for Euler's partition function.

Corollary 4. *For $n \geqslant 0$,*

$$p(n)=\sum_{k=1}^{n+1} \mu(k)\, S_{n+1,k}.$$

Example 4. *The case $n = 5$ of Corollary 4 reads as follows*

$$p(5)=S_{6,1}-S_{6,2}-S_{6,3}-S_{6,5}+S_{6,6}=19-8-4-1+1=7.$$

Recall that a natural number d is a *unitary divisor* of a number n if d is a divisor of n and if d and n/d are coprime. The sum over all positive divisors of n of the absolute value of the Möbius function is equal to the number of unitary divisors of n [5, Theorem 264],

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)},$$

where $\omega(n)$ is an additive function defined as the number of distinct primes dividing n. On the other hand, $2^{\omega(n)}$ counts the squarefree divisors of n. We remark that the set of unitary divisors of n is not the set of squarefree divisors, e.g., the set of unitary divisors of number 20 is $\{1, 4, 5, 20\}$, the set of squarefree divisors of number 20 is $\{1, 2, 5, 10\}$. By Theorem 1, we get the following result.

Corollary 5. *For $n > 0$, the total number of squarefree parts in all partitions of n can be expressed in terms of the number of squarefree divisors of n as follows*

$$\sum_{k=1}^{n} 2^{\omega(k)}\, p(n-k).$$

Example 5. *According to* (3) *and Corollary 5, the total number of squarefree parts in all partitions of* 6 *can be expressed as*

$$\begin{aligned} &1+2+1+2+2+3+4+3+4+5+6 \\ &= p(5) + 2p(4) + 2p(3) + 2p(2) + 2p(1) + 4p(0) \\ &= 7 + 10 + 6 + 4 + 2 + 4 = 33. \end{aligned}$$

In addition, considering the relation [6, Exercise 1.52]

$$\sum_{d|n} 2^{\omega(d)} = \tau(n^2),$$

the case $f(n) = 2^{\omega(n)}$ of Theorem 1 can be written as follows.

Corollary 6. *For $n > 0$,*

$$\sum_{k=1}^{n} 2^{\omega(k)}\, S_{n,k} = \sum_{k=1}^{n} \tau(k^2)\, p(n-k).$$

Example 6. *By Corollary 6, for $n = 6$ we have*

$$S_{6,1} + 2\,S_{6,2} + 2\,S_{6,3} + 2\,S_{6,4} + 2\,S_{6,5} + 2\,S_{6,4} + 2\,S_{6,5} + 4\,S_{6,6}$$
$$= 19 + 16 + 8 + 4 + 2 + 4 = 53$$

and

$$p(5) + 3\,p(4) + 3\,p(3) + 5\,p(2) + 3\,p(1) + 9\,p(0)$$
$$= 7 + 15 + 9 + 10 + 3 + 9 = 53.$$

3.3 Euler's totient function

Euler's totient or phi function, $\varphi(n)$, is a multiplicative function that counts the totatives of n, that is the positive integers less than or equal to n that are relatively prime to n. According to Euler's classical formula [5, Theorem 63],

$$\sum_{d|n} \varphi(d) = n$$

by Theorem 1, we obtain the following identity.

Corollary 7. *For $n > 0$,*

$$\sum_{k=1}^{n} \varphi(k)\, S_{n,k} = \sum_{k=1}^{n} k\, p(n-k).$$

Example 7. *By Corollary 7, for $n = 6$ we have*

$$S_{6,1} + S_{6,2} + 2\,S_{6,3} + 2\,S_{6,4} + 4\,S_{6,5} + 2\,S_{6,6} = 19 + 8 + 8 + 4 + 4 + 2 = 45$$

and

$$p(5) + 2\,p(4) + 3\,p(3) + 4\,p(2) + 5\,p(1) + 6\,p(0) = 7 + 10 + 9 + 8 + 5 + 6 = 45.$$

In a similar way, considering the relations

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n} \qquad \text{and} \qquad \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)} = \frac{n}{\varphi(n)},$$

we obtain new identities which combine Euler's totient with Euler's partition function.

Corollary 8. *For $n > 0$,*

(i) $\displaystyle\sum_{k=1}^{n} \frac{\mu(k)}{k} S_{n,k} = \sum_{k=1}^{n} \frac{\varphi(k)}{k} p(n-k);$

(ii) $\displaystyle\sum_{k=1}^{n} \frac{\mu^2(k)}{\varphi(k)} S_{n,k} = \sum_{k=1}^{n} \frac{k}{\varphi(k)} p(n-k).$

Example 8. *By Corollary 8.(i), for $n = 6$ we have*

$$S_{6,1} - \frac{S_{6,2}}{2} - \frac{S_{6,3}}{3} - \frac{S_{6,5}}{5} + \frac{S_{6,6}}{6} = 19 - 4 - \frac{4}{3} - \frac{1}{5} + \frac{1}{6} = \frac{409}{30}$$

and

$$p(5) + \frac{p(4)}{2} + \frac{2\,p(3)}{3} + \frac{p(2)}{2} + \frac{4\,p(1)}{5} + \frac{p(0)}{3} = 7 + \frac{5}{2} + 2 + 1 + \frac{4}{5} + \frac{1}{3} = \frac{409}{30}.$$

On the other hand, by Corollary 8.(ii), with n replaced by 6, we obtain

$$S_{6,1} + S_{6,2} + \frac{S_{6,3}}{2} + \frac{S_{6,5}}{4} + \frac{S_{6,6}}{2} = 19 + 8 + 2 + \frac{1}{4} + \frac{1}{2} = \frac{119}{4}$$

and

$$p(5) + 2\,p(4) + \frac{3\,p(3)}{2} + 2\,p(2) + \frac{5\,p(1)}{4} + 3\,p(0) = 7 + 10 + \frac{9}{2} + 4 + \frac{5}{4} + 3 = \frac{119}{4}.$$

In number theory, Jordan's totient function of a positive integer n, $J_t(n)$, is the number of t-tuples of positive integers all less than or equal to n that form a coprime $(t+1)$-tuple together with n. This is a generalisation of Euler's totient function, which is J_1. Considering the identity [8, eq. 27.6.8, p. 641]

$$\sum_{d|n} J_t(d) = n^t,$$

by Theorem 1, we obtain the following generalization of Corollary 7.

Corollary 9. *For $n > 0$, $t > 0$,*

$$\sum_{k=1}^{n} J_t(k)\, S_{n,k} = \sum_{k=1}^{n} k^t\, p(n-k).$$

Example 9. *By Corollary 9, for $n = 6$ and $t = 2$ we have*

$$S_{6,1} + 3\,S_{6,2} + 8\,S_{6,3} + 12\,S_{6,4} + 24\,S_{6,5} + 24\,S_{6,6} = 19 + 24 + 32 + 24 + 24 + 24 = 147$$

and

$$p(5) + 4\,p(4) + 9\,p(3) + 16\,p(2) + 25\,p(1) + 36\,p(0) = 7 + 20 + 27 + 32 + 25 + 36 = 147.$$

3.4 Liouville's function

For a positive integer n, the Liouville function $\lambda(n)$ is a completely multiplicative function defined as:

$$\lambda(n) = (-1)^{\Omega(n)}$$

where $\Omega(n)$ is the number of not necessarily distinct prime factors of n, with $\Omega(1) = 0$. We remark that $\Omega(n)$ is a completely additive function. Considering Theorem 1 and the relation [8, eq. 27.7.6, p. 641]

$$\sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise,} \end{cases}$$

we derive the following identity.

Corollary 10. *For $n > 0$,*

$$\sum_{k=1}^{n} \lambda(k)\, S_{n,k} = \sum_{k=1}^{n} p(n-k^2).$$

Example 10. *By Corollary 10, for $n = 6$ we have*

$$S_{6,1} - S_{6,2} - S_{6,3} + S_{6,4} - S_{6,5} + S_{6,6} = 19 - 8 - 4 + 2 - 1 + 1 = 9$$

and

$$p(5) + p(2) = 7 + 2 = 9.$$

4 Concluding remarks

The number of k's in all the partitions of n has been considered in order to provide a new proof of the classical identity

$$n\, p(n) = \sum_{k=1}^{n} \sigma(k)\, p(n-k).$$

Taking into accunt the generating function of $S_{n,k}$, we can write

$$\sum_{n=0}^{\infty} (S_{n+1,1} - S_{n,1})\, q^n = \sum_{n=0}^{\infty} S_{n+1,1}\, q^n - \sum_{n=0}^{\infty} S_{n,1}\, q^n$$
$$= \frac{1}{q} \sum_{n=0}^{\infty} S_{n,1}\, q^n - \sum_{n=0}^{\infty} S_{n,1}\, q^n = \frac{1-q}{q} \sum_{n=0}^{\infty} S_{n,1}\, q^n = \frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)\, q^n.$$

Thus we deduce that

$$p(n) = S_{n+1,1} - S_{n,1}.$$

In this context, Theorem 1 can be rewritten without Euler's partition function $p(n)$.

Theorem 2. *Let n be a positive integer. If $g(n) = \sum_{d|n} f(d)$, then*

$$\sum_{k=1}^{n} f(k)\, S_{n,k} = \sum_{k=1}^{n} g(k)\, (S_{n+1-k,1} - S_{n-k,1}). \tag{8}$$

The following result in partition theory has been widely attributed to Richard Stanley, although it is a particular case of a more general result that had been established by Nathan Fine 15 years earlier [4].

Theorem 3. *The number of 1's in the partitions of n is equal to the number of parts that appear at least once in a given partition of n, summed over all the partitions of n.*

Other results related to the number of 1's in all the partitions of n can be seen in [2, 7].

Replacing $p(n)$ by $S_{n+1,1} - S_{n,1}$ in Corollary 4, we obtain a new identity for the number of 1's in all the partitions of n. This new identity involves the parts greater than 1 in all the partitions of $n+1$.

Corollary 11. *The number of 1's in the partitions of n is equal to*

$$-\sum_{k=2}^{n+1} \mu(k)\, S_{n+1,k}.$$

Example 11. *We have $S_{5,1} = 12$, because the partitions of 5 that contain 1 as a part are:*

$$(4,1),\ (3,1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1).$$

According to (3) and Corollary 11, we also have

$$S_{5,1} = S_{6,2} + S_{6,3} + S_{6,5} - S_{6,6} = 8 + 4 + 1 - 1 = 12.$$

Finally combinatorial proofs of our corollaries would be very interesting.

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