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EVOLUTION OF CONVEX HYPERSURFACES BY A FULLY NONLINEAR MIXED VOLUME PRESERVING CURVATURE FLOW*

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Abstract

In this paper we study the evolution of closed convex hypersurfaces under the mixed volume preserving curvature flow in Euclidean space with the speed given by reversed function that is symmetric and homogeneous of degree one. We prove that the hypersurfaces preserve convexity under the flow, the maximum existence time is infinite and the hypersurfaces asymptotically approach to sphere.

MSC: 53C44

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1 Introduction

Let M_0 be a smooth, strictly convex hypersurface without boundary. Suppose M_0 is given by a smooth embedding $X_0 : \mathbb{S}^n \to M_0 \subset \mathbb{R}^{n+1}$. Let $X_t = X(.,t)$ evolving according to

$$\begin{aligned}\frac{\partial}{\partial t}X(x,t) &= k(x,t)\nu(x,t) \\ X(x,0) &= X_0(x)\end{aligned} \tag{1}$$

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where $k(x,t) = \{\frac{1}{\mathrm{F}(\mathcal{W}(x,t))} - h(t)\}$, $\mathcal{W}(x,t)$ is the matrix of Weingarten map of $M_t = X_t(\mathbb{S}^n)$ at the point $X_t(x)$, $\nu(x,t)$ is the outer unit normal vector to M_t at $X_t(x)$, $h(t)$ is a global term to be specified and the function F has the following properties:
i) $\mathrm{F}(\mathcal{W}) = f(\kappa(\mathcal{W}))$where $\kappa(\mathcal{W})$ gives the eigenvalues of $\mathcal{W}$ and f is a smooth symmetric function defined on the positive cone

$$\Gamma = \{\kappa = (\kappa_1, ..., \kappa_n) \in \mathbb{R}^n : \kappa_i > 0 \text{ for all } i = 1, \cdots, n\};$$

ii) f is strictly increasing in each argument that is $\frac{\partial f}{\partial \kappa_i} > 0$ for each $i = 1, \cdots, n$ at every point of Γ;
iii) f is homogenous of degree one: $f(\alpha\kappa) = \alpha f(\kappa)$ for any $\kappa > 0$;
iv) f is strictly positive on Γ and normalized: $f(1, 1, \ldots, 1) = 1$;
v) f is concave;
vi) f is inverse concave that is the function $f_*(x_1, \ldots, x_n) = f(x_1^{-1}, \ldots, x_n^{-1})^{-1}$ is concave;
vii) f_* vanishes on the boundary of Γ.

Problems of this kind have been studied widely from different points of view. Brakke in [6] studied the motion of surfaces by their mean curvature using the methods of geometric measure theory. In 1984 Huisken in [9] considered $k(x,t) = -H(x,t)$, $h = 0$ where H is the mean curvature of hypersurfaces and the initial hypersurface is convex closed. He showed that in this case the flow has unique smooth solution in $[0,T)$ where T is a maximal time that the flow exists, and as $t \to T$, the hypersurfaces M_t converge to a point. Moreover if $\tilde{M}_t$ is obtained from M_t by a homothety about the point p kipping the area of $\tilde{M}_t$ constant, then the hypersurfaces $\tilde{M}_t$ converge to a sphere as $t \to T$.

When in 1 we let $f(x,t) = H(x,t)$ and $h = 0$, where $H(x,t)$ the mean curvature of evolving hypersurfaces, the flow is called inverse mean curvature flow, which is an important tool in general relativity. Huisken and Ilmanen in [11] studied the inverse mean curvature flow in 3-dimensional Riemannian manifold with nonnegative scalar curvature instead of $\mathbb{R}^{n+1}$. They develop the theory of weak solutions and used it to prove the Riemannian Penrose Inequality, that interpreted as an optimal lower bound for total energy of the system measured at spacelike infinity in terms of the size of the largest blackhole contained inside.

Huisken and Ilmanen in [12] also proved a sharp lower bound for the mean curvature under the inverse mean curvature flow for star-shaped surfaces, independently of the initial mean curvature. They also proved that the solution of the inverse mean curvature flow is smooth if the mean curvature bounded from below.

In 1991 Urbas studied the evolution of smooth closed uniformly convex hypersurfaces in the case $k(x,t) = f(x,t)$, $h = 0$, where f satisfies the conditions (i) - (vii)([17]). He proved that the hypersurfaces remain smooth and uniformly convex for all time and assymptotically become rounded. He also studied the evolution of starshaped hypersurfaces when $k(x,t) = f(x,t)^{-1}$, where f has the properties (i)-(v) and obtained the same results [16].

Various flows with nonzero $h(t)$ have been considered by different authors. Huisken in 1987 also studied the volume preserving mean curvature flow taking

$$h(t) = \frac{\int_{M_t} H d\mu}{\int_{M_t} d\mu}.$$

Mixed volume preserving curvature flow for strictly convex closed hypersurfaces has been studied by McCoy in 2005 in [14] in the case that the speed of the flow is $h(t) - F(\mathcal{W}(x,t))$ where

$$h(t) = \frac{\int_{M_t} F(\mathcal{W}) E_{k+1} d\mu}{\int_{M_t} E_{k+1} d\mu}.$$

The preservation of convexity, short time and long time existence and asymptotically convergence to a sphere have been proved during the paper. He also included some generalizations for the flow in [15] and extended some of previous results when the the function $F(\mathcal{W})$ is a homogeneous of degree $\alpha > 0$.
The mixed volume of a convex hypersurface M is written as

$$V_{n-k}(\Phi) = \begin{cases} Vol(\Phi) & k = -1 \\ \left\{(n+1)\binom{n}{k}\right\}^{-1} \int_M E_k \, d\mu & k = 0, 1, \ldots, n-1, \end{cases}$$

where Φ is the $(n+1)$-dimensional region contained inside M, $\partial \Phi = M$, and E_l is the lth elementary symmetric function of $\kappa_1, \ldots, \kappa_n$, the priciple curvature of M

$$E_l = \begin{cases} 1 & l = 0 \\ \sum_{1 \le i_1 < \cdots < i_l \le n} \kappa_{i_1} \kappa_{i_2} \ldots \kappa_{i_l} & l = 0, \ldots, n, \end{cases}$$

for $k = -1$ and $k = 0$, V_{n-1} and V_n correspond to the volume and area of M respectively.

We consider the general mixed volume preserving curvature flow and prove the following theorem during the paper.

Theorem 1.1. *Let M_0 be a smooth, compact and strictly convex hypersurface and suppose that F satisfies conditions (i)-(vii), then the evolution equation 1 with*

$$h(t) = \frac{\int_{M_t} F(\mathcal{W})^{-1} E_{k+1} d\mu}{\int_{M_t} E_{k+1} d\mu.} \tag{2}$$

has a unique smooth solution M_t in $[0, \infty)$ and if $\tilde{M}_t$ are rescaled hypersurfaces of M_t parametrized by $\tilde{X}_t(.,t) = e^{-t}X(.,t)$ then $\tilde{M}_t$'s converge as $t \to \infty$ in the $C^\infty-$ topology, to a sphere and the volume of V_{n-k} of the sphere is the same as M_0.

In section 2 we state some definitions and some general time independent results that will be required in other sections. In section 3 we prove the short time existence of the flow using the theory of fully nonlinear parabolic equations, we show that if the initial hypersurface M_0 is strictly convex, then then the flow 1 has a unique smooth solution in some interval $[0, T)$ where $0 < T \leq \infty$. In section 4 using Maximum Principle we prove that the strictly convexity is preserved by the flow. Some corollaries also are obtained. The aim of section 5 is proving the existence of the solution of the flow in $[0, \infty)$. We use the method of support function for hypersurfaces which Urbas used it in [17]. Some results from theory of fully nonlinear partial differential equations are required in this section. We try to show that the support function has uniform $C^{2,\alpha}$ bound which leads to long time existence of the solution. Finally in section 6 using a parametrization we prove that the hypersurfaces converge in a C^∞ to a sphere.

2 Preliminaries

Similar notation as in [9, 16] will be used here. If M is a hypersurface as in Section 1, the metric g_{ij} on M_t is given by

$$g_{ij} = \left\langle \frac{\partial X}{\partial x_i}(x,t), \frac{\partial X}{\partial x_j}(x,t) \right\rangle,$$

the second fundamental form $A = \{h_{ij}\}$ is

$$h_{ij} = -\left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}(x,t), \nu \right\rangle = \left\langle \frac{\partial X}{\partial x_i}(x,t), \frac{\partial \nu}{\partial x_j}(x,t) \right\rangle,$$

the Weingarten map is $\mathcal{W} = \{h^i_j\}$ with $h^i_j = g^{ik} h_{kj}$, the mean curvature of M_t is

$$H = g^{ij} h_{ij} = h^i_i,$$

and the norm of second fundamental form is

$$|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl},$$

where g^{ij} is the inverse of the metric g_{ij}. The Riemann curvature of hypersurface is given by

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}. \tag{3}$$

We also recall the Codazzi equations which say that

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij}. \tag{4}$$

Throughout this paper indices are summed from 1 to n and raised indices indicate contraction with the metric unless otherwise indicated.
Let $(\dot{F}^{kl})$ be the matrix of first partial derivative of F with respect to the components of its arguments that given by

$$\frac{\partial}{\partial} F(A + sB)\Big|_{s=0} = \dot{F}^{kl}(A) B_{kl}.$$

Similarly for the second partial derivative of F

$$\frac{\partial^2}{\partial t^2} F\left(A + sB\right)\Big|_{s=0} = \ddot{F}^{kl,rs}(A) B_{kl} B_{rs}.$$

We will use the notations

$$\dot{f}_i(\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa) \qquad \text{and} \qquad \ddot{f}_{ij}(\kappa) = \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j}(\kappa).$$

The following two Lemmas will be required in next sections see [17] for proof.

Lemma 2.1. *for any concave function f satisfying (i)-(iv) and for any $\kappa \in \Gamma$,*

$$f(\kappa) \leq \frac{1}{n} H, \tag{5}$$

$$\mathcal{T} = trace(\dot{F}^{kl}) = \sum_k f_k \geq 1. \tag{6}$$

Lemma 2.2. *Let f satisfies (i)-(iv) and $\kappa = (\kappa_1, \ldots, \kappa_n) \in \Gamma$ and suppose that $\kappa_1 \leq \cdots \leq \kappa_n$, then*

$$\kappa_n - f(\kappa_1 \ldots \kappa_n) \geq \frac{1}{n} \sum_{1}^{n-1} (\kappa_n - \kappa_i). \tag{7}$$

Lemma 2.3. *([5]) Fore any flow of the form 1,*

$$\frac{d}{dt} \int_{M_t} E_l d\mu = \begin{cases} (l+1) \int_{M_t} (\frac{1}{F} - h) E_{l+1} d\mu & l = 0, 1, \ldots n-1 \\ 0 & l = n, , \end{cases}$$

From this lemma we have

Corollary 2.4. *The flow 1 preserves $\int_{M_t} E_k d\mu$.*

Urbas in [16] and [17] used some norms and function space on $\mathbb{S}^n$ and $\mathbb{S}^n \times [0, T)$ that will be needed here.
Let $C^k(\mathbb{S}^n)$ is the Banch space of real valued functions on $\mathbb{S}^n$ which are k-times continuously differentiable with respect to x, and equipped with the norm

$$\|u\|_{C^k(\mathbb{S}^n)} = \sum_{|\beta| \leq k} \sup_{\mathbb{S}^n} |\overline{\nabla}^{\beta} u|,$$

where β is a standard multi-index for partial derivatives and $\overline{\nabla}$ is the derivative on $\mathbb{S}^n$.
For $\alpha \in (0, 1]$, let $C^{k,\alpha}$ be the space of functions $u \in C^k(\mathbb{S}^n)$ such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n)} = \|u\|_{C^k(\mathbb{S}^n)} + \sup_{|\beta| = k} \sup_{x,y \in \mathbb{S}^n} \frac{\left|\overline{\nabla}^{\beta} u(x) - \overline{\nabla}^{\beta} u(y)\right|}{|x - y|^{\alpha}}$$

is finite, and $|x - y|$ here is the distance between x and y in $\mathbb{S}^n$. We denote by $C^k(\mathbb{S}^n \times I)$ the space of real valued functions u which are k-times continuously differentiable with respect to x and $[\frac{k}{2}]$-times continuously differentiable with respect to t such that the norm

$$\|u\|_{C^k(\mathbb{S}^n \times I)} = \sum_{|\beta| + 2r \leq k} \sup_{\mathbb{S}^n \times I} \left|\overline{\nabla}^{\beta} D_t^r u\right|$$

is finite. Here $[\frac{k}{2}]$ is the largest integer not grater than $\frac{k}{2}$.
We also denote by $C^{k,\alpha}(\mathbb{S}^n \times I)$ the space of functions in $C^k(\mathbb{S}^n \times I)$ such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n\times I)} = \|u\|_{C^k(\mathbb{S}^n\times I)} + \sup_{|\beta|+2r=k} \sup_{\substack{(x,s),(y,t)\in\mathbb{S}^n\times I \\ (x,s)\neq(y,t)}} \frac{\left|\overline{\nabla}^{\beta} D_t^r u(x,s) - \overline{\nabla}^{\beta} D_t^r u(y,t)\right|}{\left(|x-y|^2 + |s-t|\right)^{\frac{\alpha}{2}}}.$$

3 Short time existence

In this section we prove the short-time existence of the solution of the flow. For any compact strictly convex hypersurface M the gauss map $\nu : M \to \mathbb{S}^n$ is diffeomorphism. So we can reparametrize the convex hypersurface by the inverse gauss map

$$X(x) = X(\nu^{-1}(x)), \quad x \in \mathbb{S}^n.$$

The support function of hypersurface M_t is defined as flows

$$s(x,t) = \langle X(x,t), x \rangle ,$$

where x is outer unit normal to M_t at $X(x,t)$ for all t in the interval of existence. All information about the hypersurfaces can be obtained from support function. The entries of $\mathcal{W}^{-1}$, the matrix of inverse Weingarten map, are given by

$$r_{ij} = \overline{\nabla}_i \overline{\nabla}_j s + \bar{g}_{ij} s, \tag{8}$$

where $\overline{\nabla}$ is the covariant derivative and $\bar{g}_{ij}$ the standard metric on $\mathbb{S}^n$. See [2] and [17] for more details.

We consider $F_* \left(\mathcal{W}^{-1}(r_{ij})\right) = f_*(r_1, \ldots, r_n)$, where $r_i = \frac{1}{k_i}$ for $i = 1, \ldots, n$, are the principle radii of hypersurface M_t. We have the following evolution equations in terms of r_{ij},

Lemma 3.1. *The following evolution equations are hold,*

$$\frac{\partial}{\partial t} s = F_*(r_{ij}) - h, \tag{9}$$

$$\frac{\partial}{\partial t} F_* = \mathcal{L} F_* + (F_* - h)\, trace(\dot{F}_*), \tag{10}$$

$$\frac{\partial}{\partial t} r_{ij} = \mathcal{L} r_{ij} + \ddot{F}_*^{kl,pq} \overline{\nabla}_i r_{pq} \overline{\nabla}_j r_{kl} + (trace \dot{F}_*) r_{ij} - \bar{g}_{ij} h, \tag{11}$$

where $\mathcal{L} = \dot{F}_*^{kl} \overline{\nabla}_k \overline{\nabla}_l$.

Proof. The first equation is a result of the definition of support function and and equation 1.
For the second equation we have

$$\begin{aligned}\frac{\partial}{\partial t}F_* &= \dot{F}_*^{ij}\frac{\partial}{\partial t}r_{ij} = \dot{F}_*^{ij}\frac{\partial}{\partial t}\left(\overline{\nabla}_i\overline{\nabla}_j s + s\bar{g}_{ij}\right)\\ &= \dot{F}_*^{ij}\overline{\nabla}_i\overline{\nabla}_j F_* + \dot{F}_*^{ij}\left(\frac{1}{F}-h\right)\bar{g}_{ij}\\ &= \mathcal{L}F_* + \left(F_*-h\right) trace\dot{F}_*.\end{aligned}$$

For the last equation we have

$$\begin{aligned}\frac{\partial}{\partial t}r_{ij} &= \overline{\nabla}_i\overline{\nabla}_j\frac{\partial s}{\partial t} + \bar{g}_{ij}\frac{\partial s}{\partial t} = \overline{\nabla}_i\overline{\nabla}_j F_* + \bar{g}_{ij}\left(F_*-h\right)\\ &= \dot{F}_*^{kl}\overline{\nabla}_i\overline{\nabla}_j r_{kl} + \ddot{F}_*^{kl,pq}\overline{\nabla}_i r_{pq}\overline{\nabla}_j r_{kl} + \bar{g}_{ij}\left(F_*-h\right),\end{aligned}$$

using formulas for interchanging the order of covariant derivative and the relation $\overline{\nabla}_i r_{jk} = \overline{\nabla}_j r_{ik}$ ([17],[4])we have

$$\begin{aligned}\frac{\partial}{\partial t}r_{ij} &= \dot{F}_*^{kl}(\overline{\nabla}_k\overline{\nabla}_i r_{jl} - R_{ikl}^m r_{mj} - R_{ikj}^m r_{ml})\ddot{F}_*^{kl,pq}\overline{\nabla}_i r_{pq}\overline{\nabla}_j r_{kl} + \bar{g}_{ij}(F_*-h),\\ &= \mathcal{L}r_{ij} + \dot{F}_*^{kl}\left(-\bar{g}_{il}\delta_k^m r_{mj} + \bar{g}_{kl}\delta_i^m r_{mj} - \bar{g}_{ij}\delta_k^m r_{ml} + \bar{g}_{kj}\delta_i^m r_{ml}\right)\\ &+ \ddot{F}_*^{kl,pq}\overline{\nabla}_i r_{pq}\overline{\nabla}_j r_{kl} + \bar{g}_{ij}\left(F_*-h\right)\\ &= \mathcal{L}r_{ij} + \ddot{F}_*^{kl,pq}\overline{\nabla}_i r_{pq}\overline{\nabla}_j r_{kl} + (trace\dot{F}_*)r_{ij} - \bar{g}_{ij}F_* + \bar{g}_{ij}\left(F_*-h\right),\end{aligned}$$

thus the last equation holds and the Lemma is proved. □

We end this section by proving the short time existence for the equation

$$\frac{\partial s}{\partial t} = F_*(r_{ij}) - h. \tag{12}$$

Theorem 3.2. *Suppose M_0 is a compact and strictly convex hypersurface that evolves under the equation 1 then the flow 1 has a unique unique solution in a short time that is there is a $0 < T \leq \infty$ such that the flow 1 has a smooth solution on $[0,T)$.*

Proof. From 8 we have

$$\frac{\partial s}{\partial t} = F_*(r_{ij}) - h(t) := G(r_{ij}).$$

Since the eigenvalues of r_{ij} are r_i, $i = 1, \ldots, n$, then the eigenvalues of $\dot{F}_*^{ij}$ are $\frac{\partial f_*}{\partial r_i}$ for $i = 1, \ldots, n$ (see [17]) moreover we can write by direct computation

$$\dot{f}_*^i = \frac{\partial}{\partial r_i} f_*(r_1, \ldots, r_n) = f^{-2} \dot{f}_i > 0,$$

therefore $\dot{G}$ is positive definite then we can conclude that the equation 12 is strictly parabolic and the short time existence and the uniqueness of 1 follows from the theory of nonlinear parabolic equation([13]). □

4 Preservation of convexity

Since the initial hypersurface is strictly convex and compact we have $h_{ij} \geq \varepsilon F$. In this section we prove that this inequality holds during the flow(probably for another constant $\varepsilon > 0$). If r_i are the principle radii of initial hypersurface then the inequality $r_i \leq cf$ holds for some $c > 0$. We require the following generalization of maximum principle accommodate to the elliptic operator $\mathcal{L}$.

Theorem 4.1. *([7] ,[3]) Let S_{ij} be a smooth time-varying symmetric tensor field on a compact manifold M satisfying*

$$\frac{\partial}{\partial t} S_{ij} = a^{kl} \nabla_k \nabla_l S_{ij} + u^k \nabla_k S_{ij} + N_{ij},$$

where a^{kl} and u are smooth, ∇ is a smooth symmetric connection and a^{kl} is also positive definite every where. Suppose $N_{ij} v^i v^j \leq 0$ whenever $S_{ij} \leq 0$ and $S_{ij} v^j = 0$. If $S_{ij} \leq 0$ on M at time $t = 0$ then $S_{ij} \leq 0$ on $[0, T)$.

Theorem 4.2. *If $r_i \leq cf_*$ at time $t = 0$ for some $c > 0$, then this is true for any time that the flow 1 exists.*

Proof. Let $T_{ij} = r_{ij} - cF_* \bar{g}_{ij}$ and $T_{ij} \leq 0$ at $t = 0$. Using 3.1 the evolution equation for T_{ij}

$$\frac{\partial}{\partial t} T_{ij} = \mathcal{L} T_{ij} + N_{ij},$$

where $N_{ij} = \ddot{F}_*^{kl,pq} \overline{\nabla}_i r_{pq} \overline{\nabla}_j r_{kl} + \left(trace \dot{F}_* \right) T_{ij} - h(1 + trace \dot{F}) \bar{g}_{ij}$.

Suppose t_0 be the first time that T_{ij} has a null eigenvector in the existence interval at point (x_0, t_0), then at point (x_0, t_0) we have

$$N_{ij} v^i v^j = \ddot{F}_*^{kl,pq} \overline{\nabla}_i r_{pq} \overline{\nabla}_j r_{kl} v^i v^j + \left(trace \dot{F}_* \right) T_{ij} v^i v^j - h(1 + trace \dot{F}_*) \bar{g}_{ij} v^i v^j.$$

Using the concavity of f_* the first term on the right hand is nonpositive, the second term vanishes and the last term is negative using 6, so we have $N_{ij}v^iv^j \leq 0$, then we have $T_{ij} \leq 0$ for any time that the flow exists. □

Remark 4.3. *If we choose coordinates such that $r_{ij} = diag(r_1, \ldots, r_n)$ then it can be concluded that $r_i < Cf_*$ for any $i = 1, \ldots, n$ and then $r_{max} < Cf_*$ so we have $\kappa_{min} > C'f$ where $C' = \frac{1}{C}$.*

Now we can state the following Lemma (see [4] for the proof).

Lemma 4.4. *If f_* approaches zero on $\partial\Gamma$ then for any $C > 0$ there exists $C' > 0$ such that if $r \in \Gamma$ and $r_{max} \leq Cf_*$, then $r_{max} \leq C'r_{min}$*

Remark 4.5. *In view of Lemma 4.4 there is a $C' > 0$ such that $\kappa_{max} < C'\kappa_{min}$ and this in turn implies that $h_{ij} \geq \varepsilon F g_{ij}$ is maintained under the flow for some $\varepsilon > 0$ depending on n and M_0.*

We end this section with two corollaries.

Corollary 4.6. *([2]) There are constants $0 < \underline{C} \leq \overline{C}$ depending on n, F and M_0 such that for any $t \in [0, T)$,*

$$\underline{C}Id \leq \dot{F}(\mathcal{W}(x,t)) \leq \overline{C}Id. \tag{13}$$

Corollary 4.7. *([14]), For $t \in [0, T)$*
i) there is a $c = c(M_0, F)$ such that $|\overline{\nabla} s| \leq c$,
ii) there is a $d = d(M_0, F)$ such that $M_0 \subset B_d(0)$.

5 Priori estimates and long time existence

Like [17] it is convenient to work with the scaling hypersurface $\tilde{M}_t = e^{-t}M_t$ assuming $\tilde{s} = e^{-t}s$ rather than s itself to find priori estimates. Using homogeneity of F we have

$$\frac{\partial}{\partial t}\tilde{s} = F_*(\overline{\nabla}_{ij}\tilde{s} + \bar{g}_{ij}\tilde{s}) - \tilde{s} - e^{-t}h(t) \tag{14}$$

in the remainder of this section s will denote the solution of the normalized problem 14 rather than 12. In the following two Lemmas a lower bound and also an upper bound for r_{ij} that is required to derive long time existence, will be proved.

Lemma 5.1. *The upper bound for the eigenvalues of $r_{ij} = \overline{\nabla}_{ij}s + \bar{g}_{ij}s$ is preserved by the flow. In other words if at $t = 0$ we have*

$$\overline{\nabla}_{ij}s + \bar{g}_{ij}s \leq K\bar{g}_{ij},$$

for some positive K, then this remains true for all $t \in [0, T)$.

Proof. Let $T_{ij} = r_{ij} - K\bar{g}_{ij}$ like [17] we find the evolution of r_{ij} in terms of s and its derivatives we have

$$\frac{\partial}{\partial t}s = F_*(r_{ij}) - s - e^{-t}h(t), \tag{15}$$

$$\frac{\partial}{\partial t}\overline{\nabla}_l s = \dot{F}_*^{ij}\overline{\nabla}_l r_{ij} - \overline{\nabla}_l s, \tag{16}$$

$$\frac{\partial}{\partial t}\overline{\nabla}_{kl}s = \dot{F}_*^{klij}\overline{\nabla}_{ij}s + \ddot{F}_*^{ij,pq}\overline{\nabla}_k r_{pq}\overline{\nabla}_l r_{ij} - \overline{\nabla}_{kl}s. \tag{17}$$

Using standard formulas for interchanging the order of covariant differentiation and the gauss formula (see [17] for more details) we have

$$\dot{F}_*^{ij}\overline{\nabla}_{klij}s = \dot{F}_*^{ij}(\overline{\nabla}_{ijkl}s + 2\bar{g}_{kl}\overline{\nabla}_{ij}s - \bar{g}_{ij}\overline{\nabla}_{kl}s + \bar{g}_{jk}\overline{\nabla}_{il}s - \bar{g}_{il}\overline{\nabla}_{jk}s). \tag{18}$$

Use of 18 in 17 we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\nabla}_{kl}s = {} & \dot{F}_*^{ij}\overline{\nabla}_{ijkl}s + 2\bar{g}_{ij}\dot{F}_*^{ij}\overline{\nabla}_{ij}s - (trace\dot{F}_*)\overline{\nabla}_{kl}s \\ & + \bar{g}_{jk}\dot{F}_*^{ij}\overline{\nabla}_{il}s - \bar{g}_{il}\dot{F}_*^{ij}\overline{\nabla}_{jk}s + \dot{F}_*^{ij}\overline{\nabla}_l r_{ij} - \overline{\nabla}_{kl}s.\end{aligned} \tag{19}$$

Using degree one homogeneity of F_* in 15 we have

$$\bar{g}_{kl}\frac{\partial}{\partial t}s = \bar{g}_{kl}\dot{F}_*^{ij}\overline{\nabla}_{ij}s + \bar{g}_{kl}(trace\dot{F}_* - 1) - \bar{g}_{kl}e^{-t}h(t). \tag{20}$$

Adding 19 to 20 we have

$$\begin{aligned}\frac{\partial}{\partial t}r_{kl} = {} & \mathcal{L}r_{kl} + 2\bar{g}_{kl}F_* - (trace\dot{F}_* + 1)r_{kl} + F_{*k}^{i}\overline{\nabla}_{il}s \\ & - F_{*l}^{i}\overline{\nabla}_{ik}s + \ddot{F}_*^{ij,pq}\overline{\nabla}_k r_{pq}\overline{\nabla}_l r_{ij} - \bar{g}_{kl}e^{-t}h(t),\end{aligned} \tag{21}$$

then the evolution of T_{ij} is given by $\frac{\partial}{\partial t}T_{ij} = \frac{\partial}{\partial t}r_{kl} = \mathcal{L}T_{kl} + N_{kl}$, where

$$\begin{aligned}N_{kl} = {} & 2\bar{g}_{kl}F_* - (trace\dot{F}_* + 1)r_{kl} + F_{*k}^{i}\overline{\nabla}_{il}s \\ & - F_{*l}^{i}\overline{\nabla}_{ik}s + \ddot{F}_*^{ij,pq}\overline{\nabla}_k r_{pq}\overline{\nabla}_l r_{ij} - \bar{g}_{kl}e^{-t}h(t).\end{aligned}$$

Now suppose that v is a null eigenvector of T_{ij} at (x_0, t_0) for some $t_0 > 0$. Choosing coordinate at this point such that $(r_{kl}) = diag(r_1, \dots, r_n)$ and $r_i > r_j$ with $i < j$, the null eigenvector condition implies that the largest eigenvalue of r_{kl} is K with the corresponding eigenvector v and $r_{kl}v^k = K\bar{g}_{kl}v^k$ then we have

$$\begin{aligned} N_{kl}v^k v^l \leq 2F_*|v|^2 - (trace\dot{F}_* + 1)r_{kl}v^k v^l &= 2F_*|v|^2 - (trace\dot{F}_* + 1)K\bar{g}_{kl}v^k v^l \\ &= (2F_* - (trace\dot{F}_* + 1)K)|v|^2 \end{aligned}$$

using 6, we have at (x, t)

$$N_{kl} \leq 2(F_* - K) \leq 0.$$

then the result follows from theorem 4.1. □

Corollary 5.2. *$F_* \leq K$ where K is the maximum of r_i at $t = 0$, for $i = 1, \dots, n$.*

Proof. The result is concluded from the previous Lemma and the fact that f_* is homogenous and increasing function. □

Corollary 5.3. *$h(t) \leq K$, where K is the maximum of r_i at $t = 0$, for $i = 1, \dots, n$.*

Proof. Using definition of h and the existence of upper bound for f_* the result will be obtained. □

Lemma 5.4. *Let s be the solution of 15 on $\mathbb{S}^n \times [0, T)$ then for any $t \in [0, T)$ we have*

$$0 < s(t) \leq \max_{\mathbb{S}^n} s(0). \tag{22}$$

Proof. At a point that s attain a maximum we have $\overline{\nabla}_{ij}s \leq 0$ using orthonormal frame at such point and homogeneity of F_* we have

$$\begin{aligned} \frac{\partial}{\partial t}s &= F_*(\overline{\nabla}_{ij}s + \delta_{ij}s) - s - e^{-t}h(t) \\ &\leq F_*(\delta_{ij}s) - s - e^{-t}h(t) \\ &= -e^{-t}h(t) < 0. \end{aligned}$$

Using Hamiltonian Maximum principle the right side of 22 is proved.
The left side of the inequality is a result of strong maximum principle. □

Now we prove a lower bound for the eigenvalues of r_{ij} using the method that Urbas used in [17].

Lemma 5.5. *Let s be the solution of 15 on $\mathbb{S}^n \times [0,T)$ then for $t \in [0,T)$ if at $t = 0$ we have $r_{ij} \geq \varepsilon\delta_{ij}$ then this remains true for all $t \in [0,T)$.*

Proof. Instead of proving lower bound for the eigenvalues of (r_{ij}) we estimate its inverse matrix. If we denote the partial derivatives $\partial b^{pq}/\partial r_{kl}$ and $\partial^2 b^{pq}/\partial r_{mn}\partial r_{kl}$ by b^{pq}_{kl} and $b^{pq}_{kl,mn}$ then we have

$$b^{pq}_{kl} = -b^{pk}b^{ql}, \tag{23}$$

$$b^{pq}_{kl,mn} = b^{m}b^{kn}b^{ql} + b^{pk}b^{qm}b^{ln}, \tag{24}$$

$$\overline{\nabla}_j b^{pq} = b^{pq}_{kl}\overline{\nabla}_j r_{kl}, \tag{25}$$

$$\overline{\nabla}_{ij} b^{pq} = b^{pq}_{kl}\overline{\nabla}_{ij} r_{kl} + b^{pq}_{kl,mn}\overline{\nabla}_i r_{mn}\overline{\nabla}_j r_{kl}. \tag{26}$$

Using 21, 23, 24 and 26 we have

$$\begin{aligned}\frac{\partial}{\partial t}b^{pq} = b^{pq}_{kl}\frac{\partial}{\partial t}r_{kl} = &- b^{pk}b^{ql}(\dot{F}^{ij}_{*}\overline{\nabla}_{ij}r_{kl} + 2\bar{g}_{kl}F_{*} - (trace\dot{F}_{*} + 1)r_{kl} + F^{i}_{*k}\overline{\nabla}_{il}s \\ &- F^{i}_{*l}\overline{\nabla}_{ik}s + \ddot{F}^{ij,mn}_{*}\overline{\nabla}_k r_{mn}\overline{\nabla}_l r_{ij} - \bar{g}_{kl}e^{-t}h(t)),\end{aligned}$$

using 26 we have

$$\begin{aligned}\frac{\partial}{\partial t}b^{pq} &= \dot{F}^{ij}_{*}(\overline{\nabla}_{ij}b^{pq} - b^{pq}_{kl,mn}\overline{\nabla}_i r_{mn}\overline{\nabla}_j r_{nl}) - 2b^{pk}b^{ql}\bar{g}_{kl}F_{*} + (trace\dot{F}_{*} + 1)b^{pq} \\ &- b^{pk}b^{ql}(F^{i}_{*k}\overline{\nabla}_{il}s - F^{i}_{*l}\overline{\nabla}_{ik}s) - b^{pk}b^{ql}\ddot{F}^{ij,mn}_{*}\overline{\nabla}_k r_{mn}\overline{\nabla}_l r_{ij} + e^{-t}h(t)b^{pk}b^{ql}\bar{g}_{kl} \\ &= \mathcal{L}b^{pq} - 2F_{*}b^{pk}b^{ql}\bar{g}_{kl} + (trace\dot{F}_{*} + 1)b^{pq} - b^{pk}b^{ql}(F^{i}_{*k}\overline{\nabla}_{il}s - F^{i}_{*l}\overline{\nabla}_{ik}s) \\ &- b^{pk}b^{ql}\ddot{F}^{ij,mn}_{*}\overline{\nabla}_k r_{mn}\overline{\nabla}_l r_{ij} - \dot{F}^{ij}_{*}(b^{pm}b^{kn}b^{ql} - b^{pk}b^{qm}b^{ln})\overline{\nabla}_i r_{mn}\overline{\nabla}_j r_{kl} \\ &+ e^{-t}h(t)b^{pk}b^{ql}\bar{g}_{kl}.\end{aligned} \tag{27}$$

By a directional computation it is proved that $\overline{\nabla}_i r_{jk} = \overline{\nabla}_j r_{ik}$ which implies

$$\begin{aligned}&b^{pk}b^{ql}\ddot{F}^{ij,mn}_{*}\overline{\nabla}_k r_{mn}\overline{\nabla}_l r_{ij} - \dot{F}^{ij}_{*}(b^{pm}b^{kn}b^{ql} + b^{pk}b^{qm}b^{ln})\overline{\nabla}_i r_{mn}\overline{\nabla}_j r_{kl} \\ &\quad = b^{pk}b^{ql}(\ddot{F}^{ij,mn}_{*} + 2\dot{F}^{im}_{*}b^{jn})\overline{\nabla}_k r_{mn}\overline{\nabla}_l r_{ij},\end{aligned}$$

replacing to the 27 gives

$$\begin{aligned}\frac{\partial}{\partial t}b^{pq} = &\mathcal{L}b^{pq} - 2F_{*}b^{pk}b^{ql}\bar{g}_{kl} + (trace\dot{F}_{*} + 1)b^{pq} \\ &- b^{pk}b^{ql}(\ddot{F}^{ij,mn}_{*} + 2\dot{F}^{im}_{*}b^{jn})\overline{\nabla}_k r_{mn}\overline{\nabla}_l r_{ij} \\ &- b^{pk}b^{ql}(F^{i}_{*k}\overline{\nabla}_{il}s - F^{i}_{*l}\overline{\nabla}_{ik}s) + b^{pk}b^{ql}\bar{g}_{kl}e^{-t}h(t).\end{aligned}$$

If we set $p = q$ then

$$\begin{aligned}\frac{\partial}{\partial t} b^{pp} &= \mathcal{L} b^{pp} - 2F_* b^{pk} b^{pl} \bar{g}_{kl} + (trace \dot{F}_* + 1) b^{pp} \\ &- b^{pk} b^{pl} (\ddot{F}_*^{ij,mn} + 2\dot{F}_*^{im} b^{jn}) \overline{\nabla}_k r_{mn} \overline{\nabla}_l r_{ij} \\ &+ b^{pk} b^{pl} \bar{g}_{kl} e^{-t} h(t)\end{aligned}$$

But it has proved in [17] that

$$(\ddot{F}_*^{ij,mn} + 2\dot{F}_*^{im} b^{jn}) \eta_{ij} \eta_{mn} \geq 0.$$

If we suppose that (b^{ij}) attains its maximum eigenvalue at a point like x_t with unit eigenvector $\xi_t \in \mathbb{S}^n$, using a rotation on the orthonormal frame $e_1, \ldots, e_n$ at x_t we may assume that $\xi_t = e_1$, then we have

$$\begin{aligned}\frac{\partial}{\partial t} b^{11} &\leq 2F_* (b^{11})^2 + (trace \dot{F}_* + 1) b^{11} + e^{-t} h(t) (b^{11})^2 \\ &\leq -2(trace \dot{F}_*) b^{11} + (trace \dot{F}_* + 1) b^{11} + e^{-t} h(t) (b^{11})^2 \\ &\leq -(trace \dot{F}_* - 1) b^{11} + e^{-t} h(t) (b^{11})^2 \leq e^{-t} h(t) (b^{11})^2 \\ &\leq r_{max}(0) (b^{11})^2 = K (b^{11})^2,\end{aligned}$$

using maximum principle we have

$$b^{11}_{max}(0) \leq \frac{1}{K e^{-t} - K + (b^{11}_{max}(0))^{-1}}.$$

Since at $t = 0$, the following equality holds

$$b^{11}_{max} = \kappa_{max}(0) = \frac{1}{r_{max}(0)},$$

then we have

$$b^{11}_{max}(t) \leq \frac{1}{K e^{-t}} = \frac{1}{K} e^t,$$

and this complete the proof a of Lemma. □

From Lemma 5.4 and Lemma 5.5 we conclude the following Lemma to get uniform $C^{k,\alpha}$ bound for s.

Lemma 5.6. *([17]) Let s be a solution of 14 on $\mathbb{S}^n \times [0, T]$. Then for any $t \in (0, T)$, any positive integer k and any $\alpha \in (0, 1)$ we have*

$$\|s\|_{C^{k,\alpha}(\mathbb{S}^n \times [t,T])} \leq C,$$

where C depends only on n, k, α, $r_{min}(0)$, $r_{max}(0)$, t^{-1}, F and $\|s\|_{C^2(\mathbb{S}^n \times [0,T])}$.

Now following the Urbas method in [17] we can extend the time of solution to infinity and the long time existence of 14 is derived.

6 Assymptotic behavior

In this section we investigate the assymptotic behavior of evolving hypersurfaces under the flow 1.

Theorem 6.1. *Let $\tilde{M}_t = e^{-t}M_t$ are the rescaling hypersurfaces then as $t \to \infty$, the hypersurfaces $\tilde{M}_t$ converge in C^∞ topology to a sphere*

Proof. To prove the theorem let $\tau_{ij} = \overline{\nabla}_{ij}s + s\bar{g}_{ij} - F\bar{g}_{ij}$, then using 21 we have

$$\begin{aligned}\frac{\partial}{\partial t}\tau_{ij} &= \mathcal{L}r_{ij} + 2\bar{g}_{kl}F_* - (trace\dot{F}_* + 1)r_{kl} + F^i_{*k}\overline{\nabla}_{il}s \\ &- F^i_{*l}\overline{\nabla}_{ik}s + \ddot{F}^{ij,pq}_*\overline{\nabla}_k r_{pq}\overline{\nabla}_l r_{ij} - \bar{g}_{kl}e^{-t}h(t) \\ &- \overline{\nabla}_{ij}\dot{F}^{ij}_*\overline{\nabla}_{ij}F_* - (trace\dot{F}_* - 1)F_*\bar{g}_{kl} \\ &= \mathcal{L}\tau_{ij} - (trace\dot{F}_* + 1)\tau_{kl} + F^i_{*k}\overline{\nabla}_{il}s - F^i_{*l}\overline{\nabla}_{ik}s \\ &+ \ddot{F}^{ij,pq}_*\overline{\nabla}_k r_{pq}\overline{\nabla}_l r_{ij} - \bar{g}_{kl}e^{-t}h(t).\end{aligned}$$

Consider the maximum eigenvalue of τ over $\mathbb{S}^n$ at time t is attained at point $x_t \in \mathbb{S}^n$ with unit eigenvector $\xi_t \in T_{x_t}\mathbb{S}^n$. By a rotation on the orthonormal frame $e_1, \ldots, e_n$ we may assume $\xi_t = e_1$ at x_t. Let $k = l = 1$, then

$$\begin{aligned}\frac{\partial}{\partial t}\tau_{11} &\leq \dot{F}^{ij}_*\overline{\nabla}_{ij}\tau_{11} - (trace\dot{F}_* + 1)\tau_{11} - (trace\dot{F}_* + 1)e^{-t}h(t) \\ &\leq -2\tau_{11}\end{aligned}$$

The definition of τ_{ij} implies

$$\max[r_{max}(x,t) - f_*(r(x,t))] \leq C_1 e^{-2t}, \tag{28}$$

where $r(x,t) = (r_1(x,t), \ldots r_n(x,t))$ is the eigenvalues of $\overline{\nabla}_{ij}s + \bar{g}_{ij}s$ at (x,t), $r_{max}(x,t)$ is maximum eigenvalue of $\overline{\nabla}_{ij}s + \bar{g}_{ij}s$ at point (x,t), and C_1 is a positive constant depending on s_0 and f_*. Using 7 we have

$$(r_{max}(x,t) - r_{min}(x,t)) \leq nC_1 e^{-2t}, \tag{29}$$

where $r_{min}(x,t)$ is the minimum eigenvalue of $\overline{\nabla}_{ij}s + \bar{g}_{ij}s$ at point (x,t), therefore for any $x \in \mathbb{S}^n$ we have

$$dist(r(x,t), \mathcal{D}) \leq Ce^{-2t}, \tag{30}$$

where $\mathcal{D} = \{(r_1, \ldots, r_n) \in \Gamma | r_1 = \cdots = r_n\}$. Since$f_*$ is smooth and the eigenvalues of $\overline{\nabla}_{ij}s + \bar{g}_{ij}s$ remain in a fixed compact convex subset K of Γ and also $\frac{\partial f_*}{\partial t} = \frac{1}{n}$ on $\mathcal{D}$, then we have

$$|\frac{\partial f_*}{\partial r_i}(r(x,t) - \frac{1}{n}| = \leq \sup_K |D^2 f| dist(r(x,t), \mathcal{D}) \leq Ce^{-2t}. \tag{31}$$

Using 5 we also have

$$f_*(r_1, \ldots, r_n) \leq \frac{1}{n}\sum_{i=1}^{n} r_i = \frac{1}{n} trace(\overline{\nabla}_{ij}s + \bar{g}_{ij}s) = \frac{1}{n}\Delta s + s, \tag{32}$$

and from 29 we obtain

$$\frac{1}{n}\Delta s + s - Ce^{-2t}h(t) \leq f_*, \tag{33}$$

and hence we have

$$\frac{1}{n}\Delta s - Ce^{-2t} - e^{-t}h(t) \leq \frac{\partial s}{\partial t} \leq \frac{1}{n}\Delta s - e^{-t}h(t), \tag{34}$$

since $h(t) \leq K$ we have

$$\frac{1}{n}\Delta s - C'e^{-t} \leq \frac{\partial s}{\partial t} \leq \frac{1}{n}\Delta s. \tag{35}$$

Now suppose $\bar{s}(t) = (1/|\mathbb{S}^n|)\int_{\mathbb{S}^n} s(x,t)dx$ where $|\mathbb{S}^n| = \int_{\mathbb{S}^n} d\mu$. Integrating 35 over $\mathbb{S}^n$ gives

$$-C'e^{-t} \leq \frac{\partial \bar{s}}{\partial t} \leq 0. \tag{36}$$

Integrating again over any interval $[t_1, t_2] \subset [0, \infty)$ implies

$$0 \leq \bar{s}(t_1) - \bar{s}(t_2) \leq C'e^{-t_1}.$$

Since $\bar{s}$ is nonincreasing by 36 and bounded from below then $s^* = \lim_{t\to} \bar{s}(t)$ exists and

$$|\bar{s}(t) - s^*| \leq C'e^{-t}.$$

Multiplying 35 by s and integrating over $\mathbb{S}^n$ and using Poicare inequality wehave

$$\frac{d}{dt}\int_{\mathbb{S}^n} s^2 \leq -\frac{2}{n}\int_{\mathbb{S}^n} |\overline{\nabla s}^2| \leq \int_{\mathbb{S}^n} |\bar{s} - s|^2,$$

hence

$$\frac{d}{dt}\int_{\mathbb{S}^n}(s - \bar{s})^2 = \frac{d}{dt}\int_{\mathbb{S}^n} |s^2 - \bar{s}^2| \leq -\frac{2}{n}\int_{\mathbb{S}^n} +2Ce^{-t},$$

which implies

$$\int_{\mathbb{S}^n} (s - s^*)^2 \leq C(\gamma) e^{-\gamma t}, \tag{37}$$

for any $\gamma < 2$. Using interpolation inequality ([17]) we have

$$\int_{\mathbb{S}^n} |\overline{\nabla}^k s|^2 \leq C_k(\gamma, \tilde{\gamma}) e^{-\tilde{\gamma} t}, \tag{38}$$

for any $\tilde{\gamma} \leq \gamma \leq 2$. By Sobolev embedding theorem on $\mathbb{S}^n$ (see [1], §2.7) we have

$$\|s - s^*\|_{C^l(\mathbb{S}^n)} \leq (\int_{\mathbb{S}^n} |\overline{\nabla}^k s|^2 + |s - s^*|^2)^{1/2}, \tag{39}$$

for any $k > l + n/2$. The convergence of s in C^k norms to s^* follows from 37, 38 and 39. So the hypersurface $\tilde{M}_t$ converge in the C^∞ topology to a sphere of radius s^* centered at the origin. □

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