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BEREZIN TRANSFORM OF INVERTIBLE POSITIVE OPERATORS*

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Abstract

In this paper we introduce a class $\mathcal{A} \subset L^\infty(\mathbb{D})$ such that if $\phi \in \mathcal{A}$ and satisfies certain positive-definite condition, then there exists a $\psi \in \mathcal{A}$ such that $\phi(z) \leq \alpha e^{\psi(z)}$, for some constant $\alpha > 0$. Further, if $\phi(z) = \langle Ak_z, k_z \rangle$, for some bounded positive, invertible operator A from the Bergman space $L^2_a(\mathbb{D})$ into itself then $\psi(z) = \langle (\log A)k_z, k_z \rangle$. Here $k_z,\ z \in \mathbb{D}$ are the normalized reproducing kernel of $L^2_a(\mathbb{D})$. Applications of these results are also discussed.

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1 Introduction

Let $dA(z)$ be the area measure on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ normalized so that the area of the disk is 1. That is, $dA(z) = \frac{1}{\pi}dxdy$. Let

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$L^2(\mathbb{D}, dA)$ be the Hilbert space of Lebesgue measure functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z), \quad f, g \in L^2(\mathbb{D}, dA).$$

The Bergman space $L_a^2(\mathbb{D})$ is the set of of those functions in $L^2(\mathbb{D}, dA)$ that are analytic on $\mathbb{D}$. The space $L_a^2(\mathbb{D})$ is a closed subspace [5] of $L^2(\mathbb{D}, dA)$ and so there is an orthogonal projection P from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. If the analytic function f on $\mathbb{D}$ has power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

Let $K(z, \bar{w})$ be the function on $\mathbb{D}\times\mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L_a^2(\mathbb{D})$ because the formula

$$f(z) = \int_{\mathbb{D}} f(w)K(z, \bar{w})dA(w)$$

reproduces each f in $L_a^2(\mathbb{D})$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$, then $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$ and

$$K(z, \bar{w}) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)} = \frac{1}{(1 - z\bar{w})^2}.$$

Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. For any $a \in \mathbb{D}$, let ϕ_a be the analytic mapping on $\mathbb{D}$ defined by $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is

$$J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4}.$$

Let $L^\infty(\mathbb{D}, dA)$ be the Banach space of all essentially bounded measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\}$$

and $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert space H into itself and $\mathcal{LC}(H)$ be the subspace of $\mathcal{L}(H)$ consisting of all compact operators from the Hilbert space H into itself. Let $I_{\mathcal{L}(H)}$ denotes the identity operator in $\mathcal{L}(H)$. We define $\rho : \mathcal{L}(L^2_a(\mathbb{D})) \longrightarrow L^\infty(\mathbb{D})$ by

$$\rho(T)(z) = \widetilde{T}(z) = \langle Tk_z, k_z\rangle, \quad z \in \mathbb{D}.$$

Let $V(\mathbb{D}) = \{\phi \in L^\infty(\mathbb{D}) : \text{ess} \lim_{|z|\longrightarrow 1^-} \phi(z) = 0\}$. If $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ then $\rho(T) \in L^\infty(\mathbb{D})$ and $\|\rho(T)\|_\infty \leq \|T\|$ as $|\rho(T)(z)| = |\langle Tk_z, k_z\rangle| \leq \|T\|$ for all $z \in \mathbb{D}$. Further, if $T \in \mathcal{LC}(L^2_a(\mathbb{D}))$, then as $k_z \longrightarrow 0$ weakly, hence $\rho(T) \in V(\mathbb{D})$. One may also notice that if $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is diagonal with respect to the basis $\{e_n\}_{n=0}^\infty$, then $\rho(T)$ is radial. If $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ then [22], $T \cong 0$ if and only if $\widetilde{T}(z) = 0$ for all $z \in \mathbb{D}$. Let $T \in \mathcal{L}(L^2_a(\mathbb{D}))$. If $0 < mI_{\mathcal{L}(L^2_a)} \leq T \leq MI_{\mathcal{L}(L^2_a)}$ then it follows from Kantorovich inequality [14], [16] that $\widetilde{T}(z)\widetilde{T^{-1}}(z) \leq \frac{(m+M)^2}{4mM} = C(\text{say})$ for all $z \in \mathbb{D}$. The constant C is called the Kantorovich constant. It is also well known [23] that $\widetilde{T^2}(z) \leq C\left(\widetilde{T}(z)\right)^2$. If $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ are positive and invertible operators whose spectrums are contained in $[m, M]$ with $0 < m < M$, then the geometric mean $S\sharp T$ of S and T is defined [19] and [12] as $S\sharp T = S^{\frac{1}{2}}\left(S^{\frac{-1}{2}}TS^{\frac{-1}{2}}\right)^{\frac{1}{2}}S^{\frac{1}{2}}$ and $\widetilde{S}(z)\widetilde{T}(z) \leq \frac{(m+M)^2}{4mM}\widetilde{S\sharp T}(z)$ for all $z \in \mathbb{D}$. The Toeplitz operator T_ψ with symbol ψ in $L^\infty(\mathbb{D})$ is defined on $L^2_a(\mathbb{D})$ by $T_\psi f = P(\psi f)$. It is well known [22] that each bounded linear operator on $L^2_a(\mathbb{D})$ is uniquely determined by its Berezin transform and the behavior of the operators can be analyzed by exploring the corresponding Berezin transform.

The natural question that arises at this point is: Given a function $\phi \in L^\infty(\mathbb{D})$ does there exist an operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\widetilde{T}(z) = \phi(z)$ and given two operators $S, T \in \mathcal{L}(L^2_a(\mathbb{D}))$ when $\widetilde{S}(z) \geq \widetilde{T}(z)$ for all $z \in \mathbb{D}$?

The organization of this paper is as follows: In Section 2, we discuss some of the algebraic properties of the Berezin transform $\rho(T)$ and the map $\sigma_z(T) = e^{\widetilde{\log T}(z)}$ where $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is positive and invertible. Section 3, is devoted to minimax approximation and in this section we obtain an estimate for $\rho(T) - \sigma_z(T)$. In Section 4, we introduce a class $\mathcal{A} \subset L^\infty(\mathbb{D})$ and establish that if $\phi \in \mathcal{A}$ and satisfies certain positive-definite condition, then there exists a $\psi \in \mathcal{A}$ such that $\phi(z) \leq \alpha e^{\psi(z)}$, for some constant $\alpha > 0$. These results also gives us an idea about the domination of Berezin transform of bounded positive invertible operators defined on $L^2_a(\mathbb{D})$.

2 Invertible positive operators on $L^2_a(\mathbb{D})$

An operator $T \in \mathcal{L}(H)$ is said to be positive if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. In short, we write $T \geq 0$. If further $T \in \mathcal{L}(H)$ is positive and invertible then we write $T > 0$. If $T > 0$, then $\log T = \lim\limits_{\alpha\to+0} \dfrac{T^\alpha - I}{\alpha}$ and $T = \lim\limits_{n\to\infty}\left(1 + \dfrac{\log T}{n}\right)^n$. If $S, T \in \mathcal{L}(H)$ and $S \geq T \geq 0$ then by Löwner-Heinz inequality $S^\alpha \geq T^\alpha$ for $\alpha \in [0, 1]$ and if $S \geq T > 0$ then $\log S \geq \log T$. The last relation is called $\log S$ majorizes $\log T$.

For $A > 0$, the exponential map on $\mathcal{L}(H)$, denoted exp, is defined as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

The absolute convergence of this series is established just as in the scalar case from whence follows the continuity of exp. If $A, B \in \mathcal{L}(H)$ and $AB = BA$, then by multiplying the series defining $\exp(A)$ and $\exp(B)$ and rearranging one can verify that $\exp(A + B) = \exp(A)\exp(B)$. Further, if $A \in \mathcal{L}(H)$ and $\|I - A\| < 1$, then there exists $B \in \mathcal{L}(H)$ such that $A = \exp(B)$. Let G be the set of all positive, invertible operators in $\mathcal{L}(L^2_a(\mathbb{D}))$. Define for $z \in \mathbb{D}$, $\sigma_z : G \longrightarrow \mathbb{C}$ as $\sigma_z(A) = e^{\widetilde{\log A}(z)}$ and $\rho : \mathcal{L}(L^2_a(\mathbb{D})) \longrightarrow L^\infty(\mathbb{D})$ as $\rho(A)(z) = \widetilde{A}(z)$. Thus $\sigma_z(A) = e^{\rho(\log A)(z)}$. In this section, we shall discuss some of the algebraic properties of $\sigma_z(A)$ and the Berezin transform $\rho(A)$ for $A \in G$.

Proposition 2.1. *Let $z \in \mathbb{D}$ and $A, B \in G$. The following hold:*

(i) $\sigma_z(sA) = s\sigma_z(A)$ *for all* $s > 0$.

(ii) $\sigma_z(A^{-1}) = (\sigma_z(A))^{-1}$.

(iii) *If* $\alpha > 0, \beta > 0$ *and* $\alpha + \beta = 1$, *then* $\sigma_z(\alpha A + \beta B) \geq (\sigma_z(A))^\alpha(\sigma_z(B))^\beta$.

(iv) *If* $\{A\}' = \{B \in \mathcal{L}(L^2_a(\mathbb{D})) : BA = AB\}$ *the commutant of* A *then*

$$\begin{aligned}\sigma_z(A) &= \inf\{\widetilde{AB}(z) | \sigma_z(B) \geq 1, B \in \{A\}'\}\\ &= \inf\{\rho(AB)(z) | \sigma_z(B) \geq 1, B \in \{A\}'\}.\end{aligned}$$

(v) *If* $AB = BA$, *then* $\sigma_z(A + B) \geq \sigma_z(A) + \sigma_z(B)$.

Proof. Since $\sigma_z(s) = e^{\widetilde{\log sI}(z)} = e^{\langle (\log s)k_z, k_z\rangle} = e^{\log s} = s$ for all $s > 0$, hence

$$\begin{aligned}\sigma_z(sA) &= e^{\langle \log(sA)k_z, k_z\rangle}\\ &= e^{\langle (\log s + \log A)k_z, k_z\rangle}\\ &= e^{\log s}\sigma_z(A) = s\sigma_z(A).\end{aligned}$$

This proves **(i)**. Now we shall prove **(ii)**. Notice that

$$\begin{aligned}\sigma_z(A^{-1}) &= e^{(\widetilde{\log A^{-1}})(z)}\\ &= e^{\langle (\log A^{-1})k_z, k_z\rangle}\\ &= e^{\langle -(\log A)k_z, k_z\rangle} = e^{-\widetilde{\log A}(z)}\\ &= \frac{1}{e^{(\widetilde{\log A})(z)}} = \frac{1}{\sigma_z(A)} = (\sigma_z(A))^{-1}.\end{aligned}$$

To prove **(iii)**, let $\alpha > 0, \beta > 0$ and $\alpha + \beta = 1$. Then it follows from the operator concavity of the logarithm [10] that

$$\begin{aligned}\sigma_z(\alpha A + \beta B) &= e^{\langle (\log(\alpha A + \beta B))k_z, k_z\rangle}\\ &\geq e^{\langle (\alpha \log A + \beta \log B)k_z, k_z\rangle}\\ &= e^{\langle \alpha \log A k_z, k_z\rangle} e^{\langle \beta \log B k_z, k_z\rangle}\\ &= e^{\alpha\langle (\log A)k_z, k_z\rangle} e^{\beta\langle (\log B)k_z, k_z\rangle}\\ &= (\sigma_z(A))^{\alpha} (\sigma_z(B))^{\beta}.\end{aligned}$$

To prove **(iv)**, we shall first show that $\sigma_z(AB) = \sigma_z(A)\sigma_z(B)$ if $AB = BA$. Notice that $\log(AB) = \log(A) + \log(B)$ if $AB = BA$. Hence $e^{\langle (\log(AB))k_z, k_z\rangle} = e^{\langle (\log A + \log B)k_z, k_z\rangle} = \sigma_z(A)\sigma_z(B)$ if $AB = BA$. Now suppose a positive operator B commutes with $A \in G$ and assume $\sigma_z(A) \geq 1$. Then $\langle (AB)k_z, k_z\rangle \geq \sigma_z(AB) = \sigma_z(A)\sigma_z(B) \geq \sigma_z(A)$. Consider, in particular $B = \sigma_z(A)A^{-1}$. Then $\sigma_z(B) = \sigma_z(A)\sigma_z(A^{-1}) = 1$. Further

$$\langle (AB)k_z, k_z\rangle = \langle A(\sigma_z(A)A^{-1})k_z, k_z\rangle = \sigma_z(A)\langle AA^{-1}k_z, k_z\rangle = \sigma_z(A).$$

The assertion **(iv)** follows. To prove **(v)**, assume that $AB = BA$. Then

$$\begin{aligned}&\sigma_z(A+B) = \inf\{\langle ((A+B)C)k_z, k_z\rangle |\ \sigma_z(C) \geq 1, (A+B)C = C(A+B)\}\\ &= \inf\{\langle ACk_z, k_z\rangle + \langle BCk_z, k_z\rangle |\ \sigma_z(C) \geq C, AC + BC = CA + AB\}\\ &\geq \inf\{\langle ACk_z, k_z\rangle |\ \sigma_z(C) \geq 1, AC = CA\}\\ &+ \inf\{\langle BCk_z, k_z\rangle |\ \sigma_z(C) \geq 1, BC = CB\}\\ &= \sigma_z(A) + \sigma_z(B).\end{aligned}$$

$\square$

Proposition 2.2. *Let $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$ be positive and invertible. Then the following hold:*

(i) If $A \leq B$ then $\sigma_z(A) \leq \sigma_z(B)$ and $\dfrac{1}{\widetilde{A^{-1}}(z)} \leq \sigma_z(A) \leq \widetilde{A}(z)$, $z \in \mathbb{D}$.

(ii) $\|A^{-1}\|^{-1} \leq \sigma_z(A) \leq r(A) = \|A\|$ where $r(A)$ is the spectral radius of A, and $z \in \mathbb{D}$.

Proof. To prove **(i)**, assume $A \leq B$. Then it follows from the operator monotonicity of the logarithm that $\log A \leq \log B$ and

$$\sigma_z(A) = e^{\widetilde{\log A}(z)} = e^{\langle (\log A) k_z, k_z \rangle} \leq e^{\langle (\log B) k_z, k_z \rangle} = \sigma_z(B).$$

Now, let $A \in G$ and $A = \sum_{i=1}^{n} s_i E_i$ be the spectral decomposition of A. Then

$$e^{\widetilde{(\log A)}(z)} = \sigma_z(A) = \sigma_z\left(\sum_{i=1}^{n} s_i E_i\right) = \prod_{i=1}^{n} s_i^{\widetilde{E_i}(z)}$$

for the projections E_i with $\sum_{i=1}^{n} E_i = 1$. By considering the simple functions $A_n = \sum_{i=1}^{n} s_i^{(n)} E_i^{(n)}$ of A converging uniformly to $A = \int_m^M s dE_s$, (where $0 < m \leq A \leq M$ for positive numbers $m < M$) we define

$$\prod \int_m^M s d\langle E_s k_z, k_z \rangle = \lim_{n \longrightarrow \infty} \prod_{i=1}^{n} {s_i^{(n)}}^{\langle E_i^{(n)} k_z, k_z \rangle}.$$

This definition makes sense and it also shows that $\sigma_z(A) = \prod \int_m^M s d\langle E_s k_z, k_z \rangle$. Thus

$$\sigma_z(A) \leq \widetilde{A}(z) \tag{2.1}$$

since $\widetilde{A}(z)$ is the continuous weighted arithmetic mean and $\sigma_z(A)$ is the continuous weighted geometric mean with the weight k_z and (2.1) follows from the arithmetic- geometric mean inequality.(Kubo, F, Ando, T, Means of positive operators). Equality holds in (2.1) if and only if k_z is an eigenvector of A. From the harmonic-geometric-arithmetic mean inequality it follows that

$$\frac{1}{\widetilde{A^{-1}}(z)} \leq \sigma_z(A) \leq \widetilde{A}(z). \tag{2.2}$$

This proves **(i)**. Now it follows from [8] that $\sigma_z(A) \leq \langle Ak_z, k_z\rangle \leq \|A\| = r(A)$ and $\sigma_z(A) \geq \frac{1}{\langle A^{-1}k_z, k_z\rangle} \geq \frac{1}{\|A^{-1}\|}$. Thus the result **(ii)** follows. □

A real-valued continuous function f on $(0,\infty)$ is said to be operator monotone if, for any positive operators S, T the relation $S \leq T$ always implies $f(S) \leq f(T)$. It is well-known [9] that such a function f has the unique integral representation

$$f(s) = \alpha + \beta s - \int_0^\infty \left(\frac{1}{t+s} - \frac{t}{t^2+1}\right) d\gamma(t),$$

where α is real, $\beta \geq 0$ and γ is a positive measure on $(0,\infty)$ satisfying $\displaystyle\int_0^\infty \frac{d\gamma(t)}{t^2+1} < \infty$. The most important examples of operator monotone functions [20] are $\log s$ and $s^r (0 \leq r \leq 1)$ with integral representations

$$\log s = -\frac{\sin(s\pi)}{\pi} \int_0^\infty \left(\frac{1}{t+s} - \frac{t}{t^2+1}\right) dt,$$

and for $0 < r < 1$,

$$s^r = \cos\left(\frac{r\pi}{2}\right) - \frac{\sin(r\pi)}{\pi} \int_0^\infty \left(\frac{1}{t+s} - \frac{t}{t^2+1}\right) t^r dt.$$

Let us further assume $\lim_{s \longrightarrow 0^+} f(s) = 0$ (so let us set $f(0) = 0$.) It is easy to see that

$$0 = f(0) = \alpha - \int_0^\infty \left(\frac{1}{t} - \frac{t}{t^2+1}\right) d\gamma(t)$$

and

$$\begin{aligned} f(s) &= \beta s + \int_0^\infty \left(\frac{1}{t} - \frac{1}{t+s}\right) d\gamma(t) \\ &= \beta s + \int_0^\infty \frac{s}{t+s} \frac{d\gamma(t)}{t}. \end{aligned}$$

It is clear from this expression that f is concave (operator concave). The function $f : (a, b) \longrightarrow \mathbb{R}$ is said to be convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, $0 < \lambda < 1$. The function f is concave if $-f$ is convex. If S, T are positive operators in $\mathcal{L}(L^2_a(\mathbb{D}))$, then it follows from [18] that for any operator monotone function f with $f(0) = 0$ we have

$$\|f(S) - f(T)\| \leq f\left(\|S - T\|\right).$$

The following is also true:

Theorem 2.3. *If $S, T \in \mathcal{L}(L^2_a(\mathbb{D})$ are positive and $\|S - T\| > a > 0$ for some constant a then*

$$e^{\|\log(S+a)-\log(T+a)\|} \leq \left(\frac{e}{a}\|S - T\|\right).$$

Proof. Since $\log s = -\int_0^\infty \left(\frac{1}{t+s} - \frac{t}{t^2+1}\right) dt$, it follows that

$$\log(S + a) - \log(T + a) = \int_0^\infty \left[(T + a + t)^{-1} - (S + a + t)^{-1}\right] dt.$$

Now if $b > 0$, we have

$$\begin{aligned}\log(S + a) - \log(T + a) &= \int_0^b \left[(T + a + t)^{-1} - (S + a + t)^{-1}\right] dt \\ &\quad + \int_b^\infty \left[(T + a + t)^{-1} - (S + a + t)^{-1}\right] dt.\end{aligned}$$

Thus

$$\begin{aligned}\|\log(S + a) - \log(T + a)\| &\leq \int_0^b \left\|(T + a + t)^{-1} - (S + a + t)^{-1}\right\| dt \\ &\quad + \int_b^\infty \left\|(T + a + t)^{-1} - (S + a + t)^{-1}\right\| dt.\end{aligned}$$

To estimate the first integral on the right, we notice that

$$\begin{aligned}\left\|(T + a + t)^{-1} - (S + a + t)^{-1}\right\| &\leq \max\left\{\|(T + a + t)^{-1}\|, \|(S + a + t)^{-1}\|\right\} \\ &\leq \frac{1}{t + a}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^b \left\|(T + a + t)^{-1} - (S + a + t)^{-1}\right\| dt &\leq \int_0^b \frac{dt}{t + a} \\ &= \log\left(\frac{a + b}{a}\right).\end{aligned}$$

To estimate the second integral, we notice that

$$\begin{aligned}\left\|(T + a + t)^{-1} - (S + a + t)^{-1}\right\| &= \left\|(S + a + t)^{-1}(S - T)(T + a + t)^{-1}\right\| \\ &\leq \frac{1}{(t + a)^2}\|S - T\|.\end{aligned}$$

Hence

$$\int_b^\infty \left\|(T+a+t)^{-1}-(S+a+t)^{-1}\right\| dt \le \int_b^\infty \frac{1}{(t+a)^2}\|S-T\|dt = \frac{\|S-T\|}{a+b}.$$

Therefore,

$$\|\log(S+a)-\log(T+a)\| \le \log\left(\frac{a+b}{a}\right)+\frac{\|S-T\|}{a+b}.$$

But as a function of b, the expression $\log\left(\frac{a+b}{a}\right)+\frac{\|S-T\|}{a+b}$ attains its minimum at $b=\|S-T\|-a$. Hence

$$\|\log(S+a)-\log(T+a)\| \le \log\left(\frac{\|S-T\|}{a}\right)+1 = \log\left(\frac{e}{a}\|S-T\|\right)$$

Thus

$$e^{\|\log(S+a)-\log(T+a)\|} \le \frac{e}{a}\|S-T\|.$$

□

Remark 2.4. Hence in Theorem 2.3, taking $a=1$, we obtain

$$e^{\|\log(S+I)-\log(T+I)\|} \le e\|S-T\|$$

if $\|S-T\|>1$.

Corollary 2.5. *Let $A,B\in G$ and suppose $A\le B$ and $0<a=\|A-B\|$. Assume that $0<m_1I\le A\le M_1I$ and $0<m_2I\le B\le M_2I$. Then*

$$|\sigma_z(A)-\sigma_z(B)| \le \frac{2e}{r}\|B\|\|A-B\|$$

where $r=M_2-m_1$.

Proof. Notice that $0<m_1I\le A\le B\le M_2I$. If $r=M_2-m_1$, then $r>0$. Let $S=A-\frac{r}{2}I$ and $T=B-\frac{r}{2}I$. Hence

$$\|S-T\| = \left\|\left(A-\frac{r}{2}I\right)-\left(B-\frac{r}{2}I\right)\right\| = \|A-B\| = a>0.$$

Let $k = a - \frac{r}{2}$. Thus by Theorem 2.3,

$$\begin{aligned}
e^{\|\log(A+k)-\log(B+k)\|} &= e^{\|\log(A+(a-\frac{r}{2})I)-\log(B+(a-\frac{r}{2})I)\|} \\
&= e^{\|\log(A-\frac{r}{2}I+a-\log(B-\frac{r}{2}I+a)\|} \\
&= e^{\|\log(S+a)-\log(T+a)\|} \\
&\leq \frac{e}{a}\|S-T\| \\
&= \frac{e}{a}\|A-B\| \\
&= \frac{e}{k+\frac{r}{2}}\|A-B\| \\
&\leq \frac{e}{\frac{r}{2}}\|A-B\| \\
&= \frac{2e}{r}\|A-B\|.
\end{aligned}$$

Letting $k \longrightarrow 0$ we obtain $e^{\|\log A-\log B\|} \leq \frac{2e}{r}\|A-B\|$. Now from Proposition 2.2, it follows that

$$\begin{aligned}
|\sigma_z(A)-\sigma_z(B)| &= |e^{\widetilde{(\log A)}(z)} - e^{\widetilde{(\log B)}(z)}| \\
&= |e^{\langle(\log B)k_z,k_z\rangle}||e^{\langle(\log A)k_z,k_z\rangle-\langle(\log B)k_z,k_z\rangle} - 1| \\
&\leq \|B\|e^{|\widetilde{(\log A)}(z)-\widetilde{(\log B)}(z)|} \\
&\leq \|B\|e^{|\rho(\log A-\log B)(z)|} \\
&\leq \|B\|e^{\|\log A-\log B\|} \\
&\leq \frac{2e}{r}\|B\|\|A-B\|.
\end{aligned}$$

The result follows. □

Given $1 \leq p < \infty$, we define the Schatten p-class of the Hilbert space H, denoted by S_p is the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p (the pth summable sequence space). It is known that S_p is a Banach space with the norm

$$\|T\|_p = \left[\sum_n |\lambda_n|^p\right]^{1/p}.$$

The space S_1 is also called the trace class of H. If T is in S_1, then the series $\sum_{n=1}^{\infty}\langle Te_n, e_n\rangle$ converges absolutely for any orthonormal basis $\{e_n\}$ of H and the sum is independent of the choice of the orthonormal basis. We call this value the trace of T and denote it by $\mathrm{tr}(T)$.

Theorem 2.6. *Let $A \in G$. Then the map $h : \mathbb{D} \longrightarrow \mathbb{C}$ defined by $h(z) = \sigma_z(A)$ satisfies the following:*

$$|h(z) - h(w)| \leq \|A\| e^{2\sqrt{2}\|\log A\|\beta(z,w)}$$

where $\beta(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|$, the pseudohyperbolic metric on $\mathbb{D}$.

Proof. For $A \in G$, we shall first show that

$$\|\widetilde{A}(z) - \widetilde{A}(w)\| \leq 2\sqrt{2}\|A\|\beta(z,w).$$

From [7], we have

$$\widetilde{T}(z) - \widetilde{T}(w) = \mathrm{trace}[T(P_z - P_w)]$$

where $P_z(f) = \langle f, k_z\rangle k_z, \quad f \in L^2_a(\mathbb{D})$. It is known [3], for $X \in S_1$, $T \in \mathcal{L}(L^2_a(\mathbb{D}))$, $TX \in S_1$ and $|\mathrm{trace}(TX)| \leq \|T\|\|X\|_{\mathrm{trace}}$. Thus

$$|\widetilde{T}(z) - \widetilde{T}(w)| \leq 2\|T\| \left\{1 - |\langle k_z, k_w\rangle|^2\right\}^{1/2}.$$

By direct calculation, using $K(z,a) = \frac{1}{(1-\bar{a}z)^2}$, we see that

$$\begin{aligned}
1 - |\langle k_z, k_w\rangle|^2 &= 1 - \left|\left\langle k_z, \frac{K_w}{\|K_w\|}\right\rangle\right|^2 = 1 - \frac{1}{\|K_w\|^2}|k_z(w)|^2 \\
&= 1 - \frac{(1-|w|^2)^2(1-|z|^2)^2}{|1-\bar{z}w|^4} \\
&= 1 - \left(1 - \frac{|z-w|^2}{|1-\bar{w}z|^2}\right)^2 \\
&= 1 - \left(1 + \frac{|z-w|^4}{|1-\bar{w}z|^4} - 2\frac{|z-w|^2}{|1-\bar{w}z|^2}\right) \\
&= -\frac{|z-w|^4}{|1-\bar{w}z|^4} + 2\frac{|z-w|^2}{|1-\bar{w}z|^2} \\
&= \frac{|z-w|^2}{|1-\bar{w}z|^2}\left(2 - \frac{|z-w|^2}{|1-\bar{w}z|^2}\right) \\
&\leq 2\frac{|z-w|^2}{|1-\bar{w}z|^2} = 2(\beta(z,w))^2.
\end{aligned}$$

Thus

$$\left\{1 - |\langle k_z, k_w\rangle|^2\right\}^{1/2} \le \sqrt{2}\beta(z,w).$$

Hence

$$|\widetilde{T}(z) - \widetilde{T}(w)| \le 2\sqrt{2}\|T\|\beta(z,w).$$

Thus

$$\begin{aligned}
|h(z) - h(w)| &= |\sigma_z(A) - \sigma_w(A)| \\
&= \left| e^{\widetilde{\log A}(z)} - e^{\widetilde{\log A}(w)} \right| \\
&= \left| e^{\langle (\log A)k_z, k_z\rangle} - e^{\langle (\log A)k_w, k_w\rangle} \right| \\
&= \left| e^{\langle (\log A)k_w, k_w\rangle} \right| \left| e^{\langle (\log A)k_z, k_z\rangle - \langle (\log A)k_w, k_w\rangle} - 1 \right| \\
&\le \|A\| \left| e^{\widetilde{\log A}(z) - \widetilde{\log A}(w)} \right| \\
&\quad (\text{since} |e^{\langle (\log A)k_z, k_z\rangle}| \le |\langle Ak_z, k_z\rangle| \le \|A\|) \\
&\le \|A\|\, e^{\left|\widetilde{(\log A)}(z) - \widetilde{(\log A)}(w)\right|} \\
&\le \|A\|\, e^{2\sqrt{2}\|\log A\|\beta(z,w)}.
\end{aligned}$$

The result follows. □

Proposition 2.7. *The sequence* $(\widetilde{A^s}(z))^{\frac{1}{s}}$ *converges monotone decreasingly (respectively, increasingly) to* $\sigma_z(A)$ *as* $s \downarrow 0$ *(respectively* $s \uparrow 0$*). That is,* $(\rho(A^s)(z))^{\frac{1}{s}}$ *converges monotone decreasingly(respectivly, increasingly) to* $\sigma_z(A)$ *as* $s \downarrow 0$ *(respectivly,* $s \uparrow 0$*).*

Proof. To prove the proposition, let $0 \le t \le s$. Then

$$\left(\widetilde{A^s}(z)\right)^{\frac{t}{s}} = (\langle A^s k_z, k_z\rangle)^{\frac{t}{s}} \le \langle A^t k_z, k_z\rangle.$$

Using L'Hospital's rule, we obtain

$$\begin{aligned}
\lim_{s\downarrow 0} \log\langle A^s k_z, k_z\rangle^{\frac{1}{s}} &= \lim_{s\downarrow 0} \frac{\log\langle A^s k_z, k_z\rangle}{s} \\
&= \lim_{s\downarrow 0} \frac{\frac{d\langle A^s k_z, k_z\rangle}{ds}}{\langle A^s k_z, k_z\rangle} \\
&= \lim_{s\downarrow 0} \frac{\langle A^s(\log A)k_z, k_z\rangle}{\langle A^s k_z, k_z\rangle} \\
&= \langle (\log A)k_z, k_z\rangle.
\end{aligned}$$

Hence $e^{\widetilde{\log A}(z)} = \sigma_z(A) = \lim_{s\downarrow 0}\left(\widetilde{A^s}(z)\right)^{\frac{1}{s}}$. This completes the prove. □

3 Minimax approximation and the map $\sigma_z(A)$

In this section we shall discuss minimax approximation and we obtain an estimate for $\rho(T)-\sigma_z(T)$. Let f be a real-valued continuous function on $[a,b]$. Let $\rho_n(f)=\inf\limits_{\deg q\leq n}\|f-q\|_\infty$. Let $q_n^*(x)$ be the unique polynomial of degree less than equal to n such that $\|f-q_n^*\|=\rho_n(f)$. The approximation q_n^* is called the minimax approximation [4] to $f(x)$ on $[a,b]$. Let $E_n(f,x)=\max\limits_{a\leq x\leq b}|f(x)-q_n^*(x)|$ and $\epsilon(x)=f(x)-q_n^*(x)$. Then by Chebyshev equioscillation theorem [15] there are at least $n+2$ points $a=x_0<x_1<x_2<\ldots<x_n<x_{n+1}=b$ where $\epsilon(x_i)=\pm E_n,\ i=0,1,2,\ldots,n+1,\ \epsilon(x_i)=-\epsilon(x_{i+1}),\ i=0,1,2,\ldots n$ and $\epsilon^{'}(x_i)=0, i=1,\ldots n$.

Lemma 3.1. *Let $f\in C^2[a,b]$ with $f^{''}(x)>0$ for $a\leq x\leq b$. If $q_1^*(x)=a_0+a_1x$ is the linear minimax approximation to $f(x)$ on $[a,b]$, then*

$$a_1=\frac{f(b)-f(a)}{b-a}, a_0=\frac{f(a)+f(c)}{2}-\left(\frac{a+c}{2}\right)\left[\frac{f(b)-f(a)}{b-a}\right] \tag{3.1}$$

where c is the unique solution of $f^{'}(c)=\frac{f(b)-f(a)}{b-a}$.

Proof. Since $f^{''}(x)>0$ on $[a,b]$, hence f is convex on $[a,b]$. Let

$$\rho_1(f)=\inf_{\deg q\leq 1}\|f-q\|_\infty.$$

Let $\epsilon(x)=f(x)-(a_0+a_1x)$. The function f is convex on $[a,b]$ as $f^{''}(x)>0$. Then by Chebyshev equioscillation theorem, there exists a point $x_1\in[a,b]$ such that $\epsilon(a)=\rho_1, \epsilon(b)=\rho_1$, $\epsilon(x_1)=-\rho_1$ and $\epsilon^{'}(x_1)=0$. That is,

$$f(a)-(a_0+a_1a)=\rho_1, \tag{3.2}$$

$$f(b)-(a_0+a_1b)=\rho_1, \tag{3.3}$$

$$f(x_1)-(a_0+a_1x_1)=-\rho_1, \tag{3.4}$$

$$f^{'}(x_1)-a_1=0. \tag{3.5}$$

Hence $a_1=f^{'}(x_1)$. Now subtracting (3.2) from (3.3) gives

$$f(b)-f(a)-a_1(b-a)=0.$$

Hence $a_1=\frac{f(b)-f(a)}{b-a}=f^{'}(x_1)$. Thus $x_1=c$. From (3.4), it follows that

$$f(c)-(a_0+a_1c)=-\rho_1. \tag{3.6}$$

Adding (3.2) and (3.6), we obtain $f(c)+f(a)-2a_0-a_1(c+a)=0$. Hence $a_0=\frac{f(a)+f(c)}{2}-\left[\frac{f(b)-f(a)}{b-a}\right]\left(\frac{a+c}{2}\right)$. The result follows. □

Lemma 3.2. *Let f be monotone increasing and differentiable on $[a,b]$. Assume f is concave on $[a,b]$. Let $q_1^*(x) = -a_0 - a_1 x$ is the unique minimax approximation to f on $[a,b]$ and c is the unique solution of the equation*

$$f^{'}(x) = \frac{f(b)-f(a)}{b-a} = a_1 \tag{3.7}$$

Let $g(t) = t - f^{-1}(a_1 t + d)$ where $d = 2a_0 - f(c) - cf^{'}(c)$ and a_0 is as defined in (3.1). Then

$$g(t) \leq \frac{f(c)(b-a) + f(a)b - f(b)a}{f(b)-f(a)} - c$$

Proof. From Lemma 3.1, it follows that $a_1 = \frac{f(b)-f(a)}{b-a} = f^{'}(c)$ and

$$\begin{aligned} a_0 &= \frac{f(a)+f(c)}{2} - \left[\frac{f(b)-f(a)}{b-a}\right]\left(\frac{a+c}{2}\right) \\ &= \frac{1}{2}\left[\frac{bf(a)-af(b)}{b-a} + f(c) + cf^{'}(c)\right]. \end{aligned}$$

Thus $d = 2a_0 - f(c) - cf^{'}(c) = \frac{bf(a)-af(b)}{b-a}$. Let $t_0 = \frac{f(c)-d}{a_1}$. Then $t_0 \in [a,b]$ and $f(c) = a_1 t_0 + d$. Since f^{-1} is convex, hence g is concave and

$$g^{'}(t_0) = 1 - \frac{a_1}{f^{'}\left(f^{-1}(a_1 t_0 + d)\right)} = 1 - \frac{a_1}{f^{'}(c)} = 0.$$

Hence t_0 is a point of maximum of g and

$$g(t_0) = \frac{f(c)(b-a) + f(a)b - f(b)a}{f(b)-f(a)} - c.$$

□

Theorem 3.3. *Suppose A is a positive operator and $0 < m \leq A \leq M$. Then*

$$\widetilde{A}(z) - \sigma_z(A) \leq L(M,m)\left(\log L(M,m) + \frac{M\log m - m\log M}{M-m} - 1\right)$$

where $L(M,m) = \frac{M-m}{\log M - \log m}$ is the logarithmic mean.

Proof. Let $f(t) = \log t$ on $[m,M]$ in Lemma 3.2. Putting $t = \widetilde{A}(z)$, we have

$$\widetilde{A}(z) - \sigma_z(A) \leq t - f^{-1}(a_1 t + d) = t - e^{a_1 t + d}.$$

Then $f^{'}(c) = \frac{1}{c}$ and therefore

$$c = \frac{1}{a_1} = \frac{M-m}{\log M - \log m} = L(M,m)$$

and

$$\begin{aligned}
\widetilde{A}(z) - \sigma_z(A) &\leq \frac{\log L(M,m)(M-m) + (\log m)M - m\log M}{\log M - \log m} - c \\
&= \frac{\log L(M,m)(M-m) + M\log m - m\log M}{\log M - \log m} - L(M,m) \\
&= \left[\frac{\log L(M,m) + \frac{(M\log m - m\log M)}{M-m}}{\frac{\log M - \log m}{M-m}}\right] - L(M,m) \\
&= L(M,m)\left[\log L(M,m) + \frac{M\log m - m\log M}{M-m} - 1\right].
\end{aligned}$$

□

4 On the range of Berezin transform

Ahern, Flores and Rudin [1] and Englis [11] established that a function $\phi \in h^{\infty}(\mathbb{D})$ if and only if $\phi(z) = \widetilde{T_\phi}(z)$ for every $z \in \mathbb{D}$. Ahren [2] showed that if f and g are non-constant holomorphic functions on $\mathbb{D}$ then there exists a function $u \in L^1(\mathbb{D}, dA)$ such that $f\overline{g} = \rho(T_u)$. Ahren also established that there are very few such triples (f, g, u). Cuckovic and Li [6] Considered functions of the form $f_1\overline{g_1} + h$ where f_1 and g_1 are holomorphic on the unit disk $\mathbb{D}$ and h is either harmonic or of the form $f_2\overline{g_2}$ for some holomorphic functions f_2 and $g_2(z) = z^n$ with $n \geq 1$. They characterized all such functions f_1, g_1, h for which it is possible to find $u \in L^1(\mathbb{D}, dA)$ such that $\rho(T_u) = f_1\overline{g_1} + h$ and give precise relations between f_1, f_2, g_1 and $g_2(z) = z^n$ with $n \geq 1$. N.V. Rao [21] described all functions in the range of ρ which are of the form $\sum_{i=1}^{N} f_i\overline{g_i}$ where f_i, g_i are all holomorphic in $\mathbb{D}$. In fact, Rao gave a complete description of all such $u \in L^1(\mathbb{D}, dA)$ and the corresponding f_i, g_i $1 \leq i \leq N$ such that $\rho(T_u) = \sum_{i=1}^{N} f_i\overline{g_i}$. In this section we shall introduce a class $\mathcal{A} \subset L^{\infty}(\mathbb{D})$ and establish that if $\phi \in \mathcal{A}$ and satisfies certain positive-definite condition, then there exists a $\psi \in \mathcal{A}$ such that $\phi(z) \leq \alpha e^{\psi(z)}$, for some constant $\alpha > 0$.

Definition 4.1. A function $g(x, \bar{y})$ on $\mathbb{D} \times \mathbb{D}$ is called of positive type (or positive definite), written $g \gg 0$, if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} g(x_j, \overline{x_k}) \geq 0 \tag{4.1}$$

for any n-tuple of complex numbers $c_1, \ldots, c_n$ and points $x_1, \ldots, x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$. We shall say $\Upsilon \in \mathcal{A}$ if $\Upsilon \in L^\infty(\mathbb{D})$ and is such that

$$\Upsilon(z) = \Theta(z, \bar{z}) \tag{4.2}$$

where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y and if there exists a constant $c > 0$ such that

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y}) K(x, \bar{y}) \gg 0 \text{ for all } x, y \in \mathbb{D}.$$

It is a fact that (see [13], [17]) Θ as in (4.2), if it exists, is uniquely determined by Υ.

Theorem 4.2. *If $\phi \in \mathcal{A}$ and $0 \leq \phi$ then there exist a positive operator $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\phi(z) = \widetilde{S}(z)$ for all $z \in \mathbb{D}$. Further, if $0 < m \leq A \leq M$, $r = L(M, m)$ and $s = \frac{m \log M - M \log m}{\log M - \log m}$ then $\widetilde{A}(z) \leq r e^{\frac{s-r}{r}} \sigma_z(A)$, and equality holds if and only if M and m are eigenvalues of A, $\widetilde{\log A}(z) = \frac{r-s}{r}$ and k_z is a linear combination of eigenvectors corresponding to eigenvalues m and M.*

Proof. For the first part of the proof it suffices to show that $0 \leq \phi \in \mathcal{A}$ if and only if there exists a positive operator $S \in \mathcal{L}(L_a^2)$ such that $\phi(z) = \langle Sk_z, k_z \rangle$ for all $z \in \mathbb{D}$. So let $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ be a positive operator. Let $\Theta(x, \bar{y}) = \frac{\langle SK_y, K_x \rangle}{\langle K_y, K_x \rangle}$ where $K_x = K(., \bar{x})$ is the unnormalized reproducing kernel at x. Then $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. Let $\phi(z) = \Theta(z, \bar{z})$.

Then $\phi(z) = \langle Sk_z, k_z \rangle$ for all $z \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{D})$ as S is bounded. Now let $f = \sum_{j=1}^{n} c_j K_{x_j}$ where c_j's are constants, $x_j \in \mathbb{D}$ for $j = 1, 2, \ldots, n$. Since S is bounded and positive there exists a constant $c > 0$ such that

$0 \le \langle Sf, f\rangle \le c\|f\|^2$. But

$$\begin{aligned}\langle Sf, f\rangle &= \left\langle S\left(\sum_{j=1}^{n} c_j K_{x_j}\right), \sum_{j=1}^{n} c_j K_{x_j}\right\rangle \\ &= \sum_{j,k=1}^{n} c_j \bar{c_k} \langle SK_{x_j}, K_{x_k}\rangle \\ &= \sum_{j,k=1}^{n} c_j \bar{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j})\end{aligned}$$

and $c\|f\|^2 = c\langle f, f\rangle = c\sum_{j,k=1}^{n} c_j \bar{c_k} K(x_k, \bar{x_j})$. Hence we obtain that

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y}) K(x, \bar{y}) \gg 0.$$

Thus $\phi \in \mathcal{A}$.

Now let $\phi \in \mathcal{A}$ and $\phi(z) = \Theta(z, \bar{z})$ where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. We shall prove the existence of a positive, bounded operator $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\phi(z) = \langle Sk_z, k_z\rangle$. Let

$$Sf(x) = \int_{\mathbb{D}} f(z)\Theta(x, \bar{z}) K(x, \bar{z}) dA(z). \tag{4.3}$$

Indeed,

$$\begin{aligned}Sf(x) &= \langle Sf, K_x\rangle \\ &= \langle f, S^* K_x\rangle \\ &= \int_{\mathbb{D}} f(z)\overline{\langle S^* K_x, K_z\rangle} dA(z) \\ &= \int_{\mathbb{D}} f(z)\langle SK_z, K_x\rangle dA(z) \\ &= \int_{\mathbb{D}} f(z)\Theta(x, \bar{z}) K(x, \bar{z}) dA(z).\end{aligned}$$

Then

$$\begin{aligned}\langle SK_y, K_x\rangle &= \int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z}) K(x, \bar{z}) dA(z) \\ &= \int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z}) \overline{K_x(z)} dA(z) \\ &= \overline{\langle \overline{\Theta(x, \bar{z})} K_x, K_y\rangle} \\ &= \overline{\overline{\Theta(x, \bar{y})}\langle K_x, K_y\rangle} \\ &= \Theta(x, \bar{y})\langle K_y, K_x\rangle.\end{aligned}$$

Hence $\Theta(x,\bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle}$ and $\phi(z) = \Theta(z,\bar{z}) = \langle Sk_z, k_z\rangle$. We shall now prove that S is positive, bounded. That is, there exists a constant $c > 0$ such that $0 \le \langle Sf, f\rangle \le c\|f\|^2$ for all $f \in L_a^2(\mathbb{D})$. Since $\phi \in \mathcal{A}$, there exists a constant $c > 0$ such that for all $x, y \in \mathbb{D}$,

$$cK(x,\bar{y}) \gg \Theta(x,\bar{y})K(x,\bar{y}) \gg 0. \tag{4.4}$$

Let $f = \sum_{j=1}^{n} c_j K_{x_j}$ where c_j are constants, $x_j \in \mathbb{D}$ for $j = 1, 2, \ldots, n$. Then from (4.4) it follows that $\langle Sf, f\rangle = \sum_{j,k=1}^{n} c_j \bar{c_k}\Theta(x_k, \bar{x_j})K(x_k, \bar{x_j}) \ge 0$ and

$$\begin{aligned}\langle Sf, f\rangle &= \sum_{j,k=1}^{n} c_j \bar{c_k}\Theta(x_k, \bar{x_j})K(x_k, \bar{x_j})\\ &\le c\sum_{j,k=1}^{n} c_j \bar{c_k}K(x_k, \bar{x_j})\\ &= c\|f\|^2.\end{aligned}$$

Since the set of vectors $\left\{\sum_{j=1}^{n} c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2, \ldots, n\right\}$ is dense in $L_a^2(\mathbb{D})$, hence $0 \le \langle Sf, f\rangle \le c\|f\|^2$ for all $f \in L_a^2(\mathbb{D})$ and thus S is bounded and positive. The point to note is that e^t is a convex function and the line $rt + s$ crosses e^t at $t = \log m$ and $t = \log M$. Thus

$$e^t \le rt + s \le re^{\frac{s-r}{r}}e^t \tag{4.5}$$

on $[\log m, \log M]$ since the function $G(t) = re^{\frac{s-r}{r}}e^t - rt - s$ is a convex function, $G^{'}(t) = 0$ at $t = \frac{r-s}{r} \in [\log m, \log M]$ as $m \le L(m,M) \le M$ and the point $t = \frac{r-s}{r}$ is a local minimum of G.

Now let $R = \log A$. Then from (4.5) it follows that

$$\langle e^R k_z, k_z\rangle \le \langle (rR + s)k_z, k_z\rangle = r\langle Rk_z, k_z\rangle + s \le re^{\frac{s-r}{r}}e^{\langle Rk_z, k_z\rangle}.$$

This implies,

$$\begin{aligned}\langle Ak_z, k_z\rangle &\le re^{\frac{s-r}{r}}e^{\langle(\log A)k_z, k_z\rangle}\\ &= re^{\frac{s-r}{r}}\sigma_z(A).\end{aligned}$$

This establishes the inequality.
Now since $e^t < rt + s$ for $t \in (\log m, \log M)$, the equality $\langle e^R k_z, k_z\rangle = \langle (rR+s)k_z, k_z\rangle$ holds if and only if k_z is a linear combination of eigenvectors corresponding to m and M. Further, $\frac{r-s}{s}$ is the only zero of G and the equality

$$r\langle Rk_z, k_z\rangle + s = re^{\frac{s-r}{r}} e^{\langle Rk_z, k_z\rangle}$$

holds if and only if $\widetilde{R}(z) = \frac{r-s}{r}$. □

Remark 4.3. Let $K = \frac{M}{m}$. Then it is not difficult to verify that $r = \frac{(K-1)m}{\log K}$ and $\frac{b}{a} = \frac{\log(Km^{1-K})}{K-1}$. Thus $re^{\frac{s-r}{r}} = \frac{(K-1)K^{\frac{1}{K-1}}}{e\log K}$.

Hence it follows from the Theorem 4.2 that $\widetilde{A}(z) \leq \frac{(K-1)K^{\frac{1}{K-1}}}{e\log K}\sigma_z(A)$. To verify the equaity let v and w be the unit eigenvectors corresponding to eigenvalues m and M of A respectively. Now if $k_z = \sqrt{1-t^2}v + tw$ for some t lying in $(0,1)$, then

$$\log m^{1-t^2}M^{t^2} = \langle Rk_z, k_z\rangle = \frac{r-s}{r} = 1 - \frac{\log(Km^{1-K})}{K-1} = 1 + \log\left(K^{\frac{1}{1-K}}m\right).$$

That is, $t^2\log K = 1 + \log K^{\frac{1}{1-K}}$. Hence $t^2 = \frac{1}{\log K} - \frac{1}{K-1}$. Thus

$$k_z = \sqrt{\frac{k}{k-1} - \frac{1}{\log k}}v + \sqrt{\frac{1}{\log k} - \frac{1}{k-1}}w.$$

Theorem 4.4. *The function $\phi \in \mathcal{A}$ and satisfies*

$$CK(x,\bar{y}) \gg \Theta(x,\bar{y})K(x,\bar{y}) \gg mK(x,\bar{y}) \gg 0 \tag{4.6}$$

for all $x, y \in \mathbb{D}$ and some constants $C, m > 0$ if and only if there exists a positive, invertible operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\phi(z) = \langle Ak_z, k_z\rangle$ for all $z \in \mathbb{D}$.

Proof. Suppose $\phi \in \mathcal{A}$ and (4.6) holds. Then from Theorem 4.2, it follows that there exists a positive linear operator $A \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $\phi(z) = \langle Ak_z, k_z\rangle$. Now let $f = \sum_{j=1}^{n} c_j K_{x_j}$ where c_j's are constants, $x_j \in \mathbb{D}$ for

$j = 1, 2, \ldots, n$. Since

$$\begin{aligned}\langle Af, f\rangle &= \left\langle A\left(\sum_{j=1}^{n} c_j K_{x_j}\right), \sum_{j=1}^{n} c_j K_{x_j}\right\rangle \\ &= \sum_{j,k=1}^{n} c_j \bar{c_k} \Theta(x_k, \bar{x_j}) K(x_k, , \bar{x_j})\end{aligned}$$

and

$$m\|f\|^2 = m\langle f, f\rangle = m \sum_{j,k=1}^{n} c_j \bar{c_k} K(x_k, \bar{x_j}),$$

it follows from (4.6) that $\langle Af, f\rangle \geq m\|f\|^2$. As the set of vectors

$$\left\{\sum_{j=1}^{n} c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2, \ldots, n\right\}$$

is dense in $L_a^2(\mathbb{D})$, hence $0 \leq \langle Af, f\rangle \geq m\|f\|^2$ for all $f \in L_a^2(\mathbb{D})$. That is, $A \geq mI$ where I is the identity operator in $\mathcal{L}(L_a^2(\mathbb{D}))$. Hence A is invertible. Conversely, suppose A is a bounded, positive operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ which is also invertible. Then from Theorem 4.2, it follows that $\phi(z) = \langle Ak_z, k_z\rangle \in \mathcal{A}$ and there exists a constant $m > 0$ such that $A \geq mI$. Hence if $f = \sum_{j=1}^{n} c_j K_{x_j}$ where c_j's are constants, $x_j \in \mathbb{D},\ j = 1, 2, \ldots, n$, then $\langle Af, f\rangle = \sum_{j,k=1}^{n} c_j \bar{c_k} \Theta(x_k, \bar{x_j}) K(x_k, \bar{x_j})$ and $m\|f\|^2 = m\langle f, f\rangle = m \sum_{j,k=1}^{n} c_j \bar{c_k} K(x_k, \bar{x_j})$. As $\langle Af, f\rangle \geq m\|f\|^2$, hence $\Theta(x, \bar{y}) K(x, \bar{y}) \gg mK(x, \bar{y})$ for all $x, y \in \mathbb{D}$. The theorem follows. $\square$

Corollary 4.5. *Let $\phi \in \mathcal{A}$ and satisfies*

$$CK(x, \bar{y}) \gg \Theta(x, \bar{y}) K(x, \bar{y}) \gg mK(x, \bar{y}) \gg 0$$

for all $x, y \in \mathbb{D}$ and some constant $C, m > 0$. Then there exists $\psi \in \mathcal{A}$ such that $\phi(z) \leq \alpha e^{\psi(z)}$, for some constant α. If $\phi(z) = \widetilde{A}(z),\ A \in G$ then $\psi(z) = \widetilde{\log A}(z)$ and $\rho(A)(z) \leq \alpha e^{\rho(\log A)(z)}$ for all $z \in \mathbb{D}$.

Proof. Let $\phi \in \mathcal{A}$. Then from Theorem 4.4, it follows that there exists a positive, invertible operator $A \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\phi(z) = \langle Ak_z, k_z\rangle$

for all $z \in \mathbb{D}$. Hence $A \in G$. Let $\psi(z) = \langle (\log A)k_z, k_z\rangle, \quad z \in \mathbb{D}$. Since $\log A \in \mathcal{L}(L^2_a(\mathbb{D}))$ is positive and $\psi(z) = \rho(\log A)(z) \in \mathcal{A}$. From the Theorem 4.2, it follows that $\rho(A)(z) = \widetilde{A}(z) \leq \alpha e^{\rho(\log A)(z)}$ and $\phi(z) \leq \alpha e^{\psi(z)}$ for all $z \in \mathbb{D}$. □

Corollary 4.6. *Let $\phi \in \mathcal{A}$ and satisfies*

$$CK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg mK(x, \bar{y}) \gg 0$$

for all $x, y \in \mathbb{D}$ and some constant $C, m > 0$. The following hold:

(i) Suppose $\phi(z) = \langle Ak_z, k_z\rangle, \quad z \in \mathbb{D}$, $0 < m \leq A \leq M$ and $t \in \mathbb{R}$. Then

$$\rho(A^t)(z) \leq \alpha e^{\rho(\log A^t)(z)}$$

for all $z \in \mathbb{D}$ and where α is a constant depending on m, M and t. In fact, $\alpha = \frac{(K^t-1)K^{\frac{t}{K^t-1}}}{e \log K^t}$.

(ii) For each t real, there exists a function $\psi_t \in \mathcal{A}$ such that $\psi_t \leq \alpha e^{t\phi}$ where α is a constant depending on t and ϕ. In fact, if $\phi(z) = \langle Bk_z, k_z\rangle = \widetilde{B}(z)$ $z \in \mathbb{D}$, then $\psi_t(z) = \rho(e^{tB})(z)$ and $\psi_t(z) \leq \alpha e^{t\phi(z)}$ where

$$\alpha = \frac{e^{tL} - e^{tl}}{te(L-l)} \exp\left(\frac{t(Le^{tl} - le^{tL})}{e^{tL} - e^{tl}}\right)$$

and $0 < l \leq B \leq L$ and $\rho(e^{tB})(z) \leq \alpha e^{t\widetilde{B}(z)}$.

Proof. Let $K = \frac{M}{m}$. Then $m^t \leq A^t \leq M^t$ for $t \geq 0$. The inequality **(i)** follows from Theorem 4.2 for $t \geq 0$ since $K^t = \frac{M^t}{m^t}$. For $t < 0$, $M^t \leq A^t \leq m^t$ and $K^{-t} = \left(\frac{M}{m}\right)^{-t} = \frac{m^t}{M^t}$. Thus by Theorem 4.2,

$$\begin{aligned}\langle A^t k_z, k_z\rangle &\leq \frac{(K^{-t}-1)K^{\frac{-t}{K^{-t}-1}}}{e \log K^{-t}}\sigma_z(A^t)\\ &= \frac{(K^t-1)K^{\frac{t}{K^t-1}}}{e \log K^t}\sigma_z(A^t).\end{aligned}$$

To establish **(ii)**, let $B = \log A, l = \log m$ and $L = \log M$. From **(i)**, it follows that $\langle e^{tB}k_z, k_z\rangle = \langle A^t k_z, k_z\rangle$ and $e^{t\widetilde{B}(z)} = \sigma_z(A^t)$. The inequality **(ii)** follows. □

References

[1] P. Ahren, M. Flores and W. Rudin. An invariant volume-mean-value property. *J. Funct. Anal.* **111**(1993), 380-397.

[2] P. Ahren. On the range of the Berezin transform. *J. Funct. Anal.* **215**(1)(2004), 206-216.

[3] N. I. Akhiezer and I. M. Glazman. *Theory of linear operators in Hilbert space.* Monographs and studies in Mathematics, Pitman, **9**, 1981.

[4] K. E. Atkinson. *An introduction to Numerical analysis.* 2nd edition, John wiely and sons, **1989**.

[5] J. B. Conway. *A course in Functional Analysis.* 2nd edition, Springer-Verlag, New York, **1990**.

[6] *Ž. Č*učković and B. Li. Berezin transform, Mellin transform and Toeplitz operators. *Complex Anal. Oper. Theory.* **6**(1)(2010), 189-218.

[7] N. Das, C. K. Mohapatra and R. P. Lal. On the range of Berezin transform. *J. Int. Acad. Phys. Sci.* **10**(2006), 9-15.

[8] R. G. Douglas. *Banach algebra techniques in operator theory.* Academic Press, New York, **1972**.

[9] W. F. Donoghue. *Monotone matrix functions and analysis continuation*, Springer Verlag, New York, **1970**.

[10] S. S. Dragomir. *Inequalities for Functions of Selfadjoint Operators on Hilbert Spaces.* arXiv:1203.1667v1 [math.FA] **2012**.

[11] M. Englis. Functions invariant under Berezin transform. *J. Funct. Anal.* **121**(1994), 223-254.

[12] M. Fujii, S. Izumino, R. Nakamato and Y. Seo. Operator inequalities related to Cauchy-Schwarz and Holder-Mecarthy inequalities. *Nihonkai Math. J.* **8**(1997), 117-122.

[13] G. M. Golusin. *Geometric theory of functions of complex variable.* Nauka, **1966**.

[14] W. Greub and W. Rheinboldt. On a generalization of an inequality of L.V. Kantorovich. *Proc. Amer. Math. Soc.* **10**(1959), 407-415.

[15] M. K. Jain, S. R. K. Iyenger and R. K. Jain. *Numerical methods for scientific and engineering computation.* 3rd edition, New Age International (P) Ltd. **1993**.

[16] L. V. Kantorovich. Functional analysis and applied mathematics(in Russian). *Uspekhi Mat. Nauk(N. S.).* **3**(1948), 89-185.

[17] S. G. Krantz. *Function theory of several complex variables.* John Wiley, New York, **1982**.

[18] F. Kittaneh and H. Kosaki. Inequalities for the Schatten p-norm V. *Publ. RIMS, Kyoto Univ.* **23**(1986), 433-443.

[19] F. Kubo and T. Ando. Means of positive linear operators. *Math. Ann.* **246**(1980), 205-224.

[20] C. S. Lin and S. S. Dragomir. On high-power operator inequaalities and spectral radii of operators. *Publ. RIMS, Kyoto Univ.* **42**(2006), 391-397.

[21] N. V. Rao. *Range of Berezin transform.* Arxiv preprint arXiv:1003.3939, **2010**.

[22] K. Stroethoff. The Berezin transform and operators on spaces of analytic functions. *Linear operators, Banach Center Publ.* **38**(1997), 361-380.

[23] T. Yamazaki. An extension of Kantorovich inequality to n-operators via the geometric mean by Ando-Li-Mathias. *Linear Algebra Appl.* **416**(2006), 688-695.