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ON AN OBLIQUE PROJECTION METHOD FOR SOLVING THE EIGENVALUE PROBLEM OF THE COMPANION MATRIX*

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Abstract

In the present research, we take another look at the relatively new method for computing eigenvalues and eigenvectors of the Frobenius companion matrix. The purpose of the paper is to interpret the method considered in terms of oblique projection methods, i.e. as a Galerkin type method. Based on this dependence, we derive some new theoretical results. We establish certain error estimates, which will contribute to further studies of the convergence analysis of the method under consideration.

MSC: 65H04, 65F15

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1 Introduction

We consider the particular eigenvalue problem: find a scalar $\lambda \in \mathbf{C}$ and a nonzero vector $\mathbf{x} \in \mathbf{C}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x}, \tag{1}$$

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where A is the *Frobenius companion matrix* of order n with complex entries and with only simple eigenvalues

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}. \tag{2}$$

Let us denote by $\lambda_1, \lambda_2, \dots, \lambda_n$ the two by two distinct eigenvalues of A. Then it is immediate to verify that the eigenvector associated with λ_s has the form

$$\mathbf{x}_s = (1, \lambda_s, \lambda_s^2, \dots, \lambda_s^{n-1})^T,$$

for $s = 1, \dots, n$. Therefore the eigenvector matrix V of A is the well-known *Vandermonde matrix*

$$V = V(\lambda) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \dots & \dots & \ddots & \dots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix}. \tag{3}$$

Which implies that Fobenius companion matrix (2) has complete *biorthogonal systems* of right eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ (the columns of V) and left eigenvectors $\mathbf{y}_1^*, \mathbf{y}_2^*, \dots, \mathbf{y}_n^*$ (the rows of V^{-1}). Thus we have the following eigenvalue decomposition of (2)

$$A = V\Lambda V^{-1}, \tag{4}$$

where $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$.

In our previous work [1] we developed a new iterative scheme for finding all the eigenvalues and corresponding left, and right eigenvectors of the companion matrix (2), see *Algorithm-1.*

Where $z_i^{(k)}$ is the k-th approximation of the eigenvalue λ_i $(i = 1, \dots, n)$. The right and left approximate eigenvectors associated with eigenvalue λ_i are denoted by $\mathbf{v}_i^{(k)}$ and $\mathbf{w}_i^{(k)}$, respectively. On step 6 of *Algorithm-1*, the two-sided *Rayleigh quotient*

$$\rho(\mathbf{x}, \mathbf{y}) = \rho(A, \mathbf{x}, \mathbf{y}) = \frac{\mathbf{y}^* A\mathbf{x}}{\mathbf{y}^*\mathbf{x}} \tag{5}$$

is used for computing the approximate eigenvalue. Formula (5) is introduced for the first time by Ostrowski in [9] (see also [10, 11]).

Algorithm-1: Diagonalization of the Frobenius companion matrix

Input: A, initial vector $\mathbf{z}^{(0)}$(where $z_i^{(0)} \neq z_j^{(0)}$ for $i \neq j$), tolerance $\epsilon \ll 1$.
Output: Approximate eigenvalues and eigenvector matrices:
$\lambda = (\lambda_1, \ldots, \lambda_n)$, $V(\lambda)$, $W(\lambda)$.

Set k=0.
While not converged **do**
Compute the Vandermonde matrix $V_k = V(\mathbf{z}^{(k)})$.
Compute the inverse Vandermonde matrix $W_k^* = V(\mathbf{z}^{(k)})^{-1}$.
For i=1:n **do**
Compute next eigenvalue estimate
$z_i^{(k+1)} = \rho_k = (\mathbf{w}_i^{(k)})^* A \mathbf{v}_i^{(k)}$,
where $\mathbf{v}_i^{(k)}$ is the i-*th* column vector of V_k,
and $(\mathbf{w}_i^{(k)})^*$ is the i-*th* row vector of W_k^*
End for
Set k=k+1.
If $\|\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)}\|_2 < \epsilon$ **then**
Set $\lambda = \mathbf{z}^{(k+1)}$, $V = V_k$, $W = W_k^*$.
break
End If
End While

In [1] local convergence analysis of the introduced algorithm is provided. It is proved that the asymptotic rate of convergence of the new scheme is quadratic. The main result in [1] is the following theorem.

Theorem 1 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be right and left eigenvectors associated with the eigenvalue* λ *of the Frobenius companion matrix* F_p*, which has* n *distinct eigenvalues. Then* $\lim_{k\to\infty} \mathbf{v}_k = \mathbf{x}$ *and* $\lim_{k\to\infty} \mathbf{w}_k = \mathbf{y}$ *if and only if* $\mathbf{z}_{k+1} = \rho_k = \rho(\mathbf{v}_k, \mathbf{w}_k)$ *approaches* λ *and the asymptotic convergence rate is quadratic.*

Our main objective in this work is to interpret the aforementioned algorithm in terms of projection methods. By using this dependence we will establish new error bounds for this algorithm. The rest of the paper is organized as follows. In Section 2 we give a short introduction to projection methods for solving the eigenvalue problem. Interpretation of the *Algorithm-1* as a projection method is shown in Section 3. In Section 4 we derive some error bounds and the conclusion is in Section 5.

2 Projection methods for eigenvalue problems

Given two $m-$dimensional subspaces $\mathcal{K}$ and $\mathcal{L}$ of $\mathbf{C}^n$, the basic idea of projection methods is to approximate the exact eigenpair $(\lambda, \mathbf{x})$ to the eigenvalue problem (1), by an approximate eigenpair $(\hat{\lambda}, \hat{\mathbf{x}})$, with $\hat{\lambda} \in \mathbf{C}$ and $\hat{\mathbf{x}} \in \mathcal{K}$, such that the following *Petrov-Galerkin condition* is satisfied

$$A\hat{\mathbf{x}} - \hat{\lambda}\hat{\mathbf{x}} \perp \mathcal{L} \tag{6}$$

for some given inner product. The subspace $\mathcal{K}$ reffered to as the subspace of approximants or the *right subspace* and $\mathcal{L}$ as the *left subspace.*

There are two main classes of projection methods: *orthogonal projection* (Ritz-Galerkin) and *oblique projection* methods (see for more details [2, 3, 5, 6]). In the case of orthogonal projection methods the subspace $\mathcal{K}$ is the same as $\mathcal{L}$, while in oblique projection techniques subspace $\mathcal{K}$ is different from $\mathcal{L}$.

We can translate in matrix form the approximate eigenvector $\hat{x}$ in some basis and express the Petrov-Galerkin condition. Let assume that we have a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of $\mathcal{K}$ and denote by V the matrix with column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. By analogy, let $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be a basis of $\mathcal{L}$ and denote by W the matrix with column vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$. Usually, the following additional assumption is required

$$W^*V = I, \tag{7}$$

which means that the two bases are biorthogonal, i.e. $(\mathbf{v}_i, \mathbf{w}_j) = \delta_{ij}$. Now we can translate the approximate problem into this basis and solve it numerically. Letting

$$\hat{\mathbf{x}} = V\mathbf{u}$$

the condition (6) becomes

$$\langle (A - \hat{\lambda} I)V\mathbf{u}, \mathbf{w}_j \rangle = 0, \ j = 1, \ldots, m\,.$$

Therefore, $\mathbf{u}$ and $\hat{\lambda}$ must satisfy

$$B_m\mathbf{u} = \hat{\lambda}\mathbf{u}, \tag{8}$$

where

$$B_m = W^*AV. \tag{9}$$

In this way, we reduced the initial eigenvalue problem to the smaller $m \times m$ eigenvalue problem. Each eigenvalue $\hat{\lambda}$ of B_m is called a *Ritz value*, and $V\mathbf{u}$ is called *Ritz vector*, where $\mathbf{u}$ is the eigenvector of B_m associated with $\hat{\lambda}$.

In the case of orthogonal projection methods the matrix B_m has the form $B_m = V^*AV$. In this case, the method for solving the reduced eigenvalue problem (8) is also known as the *Rayleigh-Ritz procedure.*

We can also reformulate projection methods in terms of projection operators as follows.

Case 1. Orthogonal projection methods. Let $P_{\mathcal{K}}$ be the orthogonal projector onto $\mathcal{K}$, then condition (6) has the form

$$A\hat{\mathbf{x}} - \hat{\lambda}\hat{\mathbf{x}} \perp \mathcal{K} \tag{10}$$

or equivalently,

$$P_{\mathcal{K}}(A - \hat{\lambda}I)\hat{\mathbf{x}} = 0, \ (\hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K})$$

or

$$P_{\mathcal{K}}A\hat{\mathbf{x}} = \hat{\lambda}\hat{\mathbf{x}}. \tag{11}$$

Hence, we have replaced the initial problem (1) by an eigenvalue problem for the linear transformation $P_{\mathcal{K}}A_{|\mathcal{K}}$ which is from $\mathcal{K}$ to $\mathcal{K}$. The last equation is equivalent to

$$P_{\mathcal{K}}AP_{\mathcal{K}}\hat{\mathbf{x}} = \hat{\lambda}\hat{\mathbf{x}}, \ (\hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K}), \tag{12}$$

which involves the operator

$$A_m = P_{\mathcal{K}}AP_{\mathcal{K}}. \tag{13}$$

This is an extension of $P_{\mathcal{K}}A_{|\mathcal{K}}$ to the whole space. Equation (11) will be referred to as the Galerkin approximation problem.

Case 2. Oblique projection methods. Let $Q_{\mathcal{K}}^{\mathcal{L}}$ be the oblique projector onto $\mathcal{K}$ and orthogonal to $\mathcal{L}$. This projector has the following properties:

$$Q_{\mathcal{K}}^{\mathcal{L}}\mathbf{z} \in \mathcal{K} \ \text{ and } \ (I - Q_{\mathcal{K}}^{\mathcal{L}})\mathbf{z} \perp \mathcal{L}. \tag{14}$$

for any vector $\mathbf{z} \in \mathbf{C}^n$. Then the orthogonality condition (6) is equivalent to

$$Q_{\mathcal{K}}^{\mathcal{L}}(A - \hat{\lambda}I)\hat{\mathbf{x}} = 0, \ (\hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K})$$

or

$$Q_{\mathcal{K}}^{\mathcal{L}}A\tilde{\mathbf{x}} = \tilde{\lambda}\tilde{\mathbf{x}}. \tag{15}$$

Thus, the eigenvalue problem for matrix A is replaced by the eigenvalue problem of the linear operator $Q_{\mathcal{K}}^{\mathcal{L}}A_{|\mathcal{K}}$. In this case, we can define an extension A_m by analogy to the definition of (13) in two different ways. One possible presentation could be

$$A_m = Q_{\mathcal{K}}^{\mathcal{L}}AQ_{\mathcal{K}}^{\mathcal{L}} \tag{16}$$

or, the more useful extension

$$A_m = Q_{\mathcal{K}}^{\mathcal{L}} A P_{\mathcal{K}}. \tag{17}$$

Then by analogy to (12) we get the approximation problem

$$A_m \hat{\mathbf{x}} = \hat{\lambda} \hat{\mathbf{x}}, \ (\ \hat{\lambda} \in \mathbf{C}, \hat{\mathbf{x}} \in \mathcal{K}) \, . \tag{18}$$

Therefore in both cases (12) and (18), the Ritz values are actually the eigenvalues of the operator A_m, and the Ritz vectors are the corresponding eigenvectors belonging to $\mathcal{K}$.

There are different ways to construct the subspaces $\mathcal{K}$ and $\mathcal{L}$. We can work with subspaces of both fixed and variable dimensions. Some of the well-known examples when the subspace dimension is fixed to one are the *Power method* and *Rayleigh quotient iteration* (see [3, 10]). In other projection methods the dimension of the subspace $\mathcal{K}$ is increasing, usually one starts with a subspace of dimension one, and the dimension increases by one at each iteration step. Some of the most popular methods of this type use the so-called *Krylov subspace*. Such examples are Arnoldi method, Lanczos method and incomplete orthogonalization (see [2, 4]). Methods like *Davidson method* and *Jacobi-Davidson method* are examples with increasing subspace dimension and without using Krylov subspaces (see [7, 8]).

3 *Algorithm-1* as a projection method

In this section, we will show that the presented *Algorithm-1* can be formulated in terms of projection methods. Without loss of generality we fix the value of index i on step 6 of *Algorithm-1*. Let denote by

$$\mathcal{K}_i = \mathrm{span}\{\mathbf{v}_i^{(k)}\} \tag{19}$$

the right subspace and by

$$\mathcal{L}_i = \mathrm{span}\{(\mathbf{w}_i^{(k)})^*\} \tag{20}$$

the left subspace associated with λ_i at iteration step k. It is obvious that the biorthogonality condition (7) is valid. Then the corresponding reduced eigenvalue problem (8)-(9) of (1) has the following trivial form

$$(\mathbf{w}_i^{(k)})^* A \mathbf{v}_i^{(k)} \mathbf{y} = \hat{\lambda}_i \mathbf{y} \, ,$$

where the matrix B_m is a scalar (m=1) and it is identical to the Rayleigh quotient (5). In other words, we unambiguously obtain the Ritz value

$$\hat{\lambda}_i = z_i^{(k+1)} = \rho(\mathbf{v}_i^{(k)}, \mathbf{w}_i^{(k)}) .$$

which is the next approximation of λ_i. Furthermore, it is immediate to verify that the residual vector of approximate eigenpair $(\hat{\lambda}_i, \mathbf{v}_i^{(k)})$

$$\mathbf{r}_i^{(k)} = A\mathbf{v}_i^{(k)} - \hat{\lambda}_i \mathbf{v}_i^{(k)}$$

is orthogonal to $\mathcal{L}_i$, i.e. the Petrov-Galerkin condition (6) is satisfied. With this, we confirmed that the approach of *Algorithm-1* is an oblique projection method with subspaces $\mathcal{K}_i$ and $\mathcal{L}_i$ having dimensions fixed to one.

4 Convergence analysis

Our main purpose in this section is to apply some of the known convergence results on Projection methods to the considered *Algorithm-1.* We aim to show the relation between the solutions of the main eigenvalue problem (1) and the approximate problem (18). A natural way to derive error bounds is by using the approach of residual vectors. In our case we can investigate either of the residuals $(A_m - \lambda I)\mathbf{x}$ or $(A - \hat{\lambda} I)\hat{\mathbf{x}}$, where $(\lambda, \mathbf{x})$ is the exact eigenpair and $(\hat{\lambda}, \hat{\mathbf{x}})$ is the corresponding approximate eigenpair, and A_m is defined by (16) or (17).

The quantity which plays an important role in convergence analysis of projection methods is the distance between the exact eigenvector x and the right subspace $\mathcal{K}$ defined by $\|(I - P_{\mathcal{K}})\mathbf{x}\|$, where $P_{\mathcal{K}}$ is the orthogonal projector onto $\mathcal{K}$. This quantity can also be interpreted as the $\sin(\alpha)$ (in the case of $\|\mathbf{x}\| = 1$) of the acute angle between the eigenvector $\mathbf{x}$ and $\mathcal{K}$.

Let now consider *Algorithm-1* for the fixed value of the index i. The orthogonal projector $P_{\mathcal{K}_i}$ onto $\mathcal{K}_i$ defined by (19) has the form

$$P_{\mathcal{K}_i} = \frac{\mathbf{v}_i^{(k)}(\mathbf{v}_i^{(k)})^*}{\|\mathbf{v}_i^{(k)}\|^2} \tag{21}$$

and the oblique projector $Q_{\mathcal{K}_i}^{\mathcal{L}_i}$ onto $\mathcal{K}_i$ along the orthogonal complement of $\mathcal{L}_i$ defined by (20) has the form

$$Q_{\mathcal{K}_i}^{\mathcal{L}_i} = \mathbf{v}_i^{(k)}(\mathbf{w}_i^{(k)})^* . \tag{22}$$

If we consider the exact eigenpair $(\lambda_i, \mathbf{x}_i)$ as an approximate eigenpair of the operator A_m, then the following estimates hold for the corresponding residual vector. For the sake of simplicity, we omit the superscript (k).

Theorem 2 *Let denote by* $\gamma_i = \|Q_{\mathcal{K}_i}^{\mathcal{L}_i}\|_2\|A\|_2$*, where* $Q_{\mathcal{K}_i}^{\mathcal{L}_i}$ *and* $P_{\mathcal{K}_i}$ *are defined by (21) and (22) respectively. Then the following inequalities are hold*

$$(i)\quad \|(A_m - \lambda_i I)P_{\mathcal{K}_i}\boldsymbol{x}_i\| \leq \gamma_i\|(I - P_{\mathcal{K}_i})\boldsymbol{x}_i\|\,, \tag{23}$$

$$(ii)\quad \|(A_m - \lambda_i I)\boldsymbol{x}_i\| \leq \sqrt{|\lambda_i|^2 + \gamma_i^2}\,\|(I - P_{\mathcal{K}_i})\boldsymbol{x}_i\|\,, \tag{24}$$

for $i = 1, \ldots, n$*, where* $A_m = Q_{\mathcal{K}_i}^{\mathcal{L}_i} A P_{\mathcal{K}_i}$*.*

Proof: (i). It is immediate to verify that

$$(A - \lambda_i I)P_{\mathcal{K}_i}\mathbf{x}_i = (A - \lambda_i I)(P_{\mathcal{K}_i} - I)\mathbf{x}_i$$

and since

$$Q_{\mathcal{K}_i}^{\mathcal{L}_i} P_{\mathcal{K}_i} = P_{\mathcal{K}_i}\,, \tag{25}$$

we get

$$(A_m - \lambda_i I)P_{\mathcal{K}_i}\mathbf{x}_i = Q_{\mathcal{K}_i}^{\mathcal{L}_i}(A - \lambda_i I)P_{\mathcal{K}_i}\mathbf{x}_i = -Q_{\mathcal{K}_i}^{\mathcal{L}_i}(A - \lambda_i I)(I - P_{\mathcal{K}_i})\mathbf{x}_i\,.$$

Because of $(I - P_{\mathcal{K}_i})$ is an orthogonal projector, it follows that $\|I - P_{\mathcal{K}_i}\| = 1$. Using that and the Cauchy-Schwarz inequality we get the first result.

(ii). The following equalities are hold

$$\begin{aligned}(A_m - \lambda_i I)\mathbf{x}_i &= (A_m - \lambda_i I)(P_{\mathcal{K}_i}\mathbf{x}_i + (I - P_{\mathcal{K}_i})\mathbf{x}_i)\\ &= (A_m - \lambda_i I)P_{\mathcal{K}_i}\mathbf{x}_i + (A_m - \lambda_i I)(I - P_{\mathcal{K}_i})\mathbf{x}_i)\,.\end{aligned}$$

Using the equality $A_m(I - P_{\mathcal{K}_i}) = 0$ we get

$$(A_m - \lambda_i I)\mathbf{x}_i = (A_m - \lambda_i I)P_{\mathcal{K}_i}\mathbf{x}_i - \lambda_i(I - P_{\mathcal{K}_i})\mathbf{x}_i)\,.$$

It easy to verify that the two terms in the right-hand side are orthogonal. Then by using the Pythaga's formula, we get the second result, which proves the theorem.

This theorem is analogous of a result obtained by Saad in [2]. We remark that the Theorem 2 states the accuracy of the exact eigenpair with respect to the approximate problem. The following corollary is straightforward.

Corollary 1 *The statements* (i) *and* (ii) *in Theorem 3 remain valid if we replace the matrix* A_m *with* $\tilde{A}_m = Q_{\mathcal{K}_i}^{\mathcal{L}_i} A Q_{\mathcal{K}_i}^{\mathcal{L}_i}$*.*

Proof: It repeats the proof of Theorem 3 by using (25) and the equality $A_m P_{\mathcal{K}_i} = \tilde{A}_m P_{\mathcal{K}_i}$.

To investigate the distance between the Ritz value and the exact eigenvalue we can use the following theorem.

Theorem 3 *Let A be diagonalizable with eigenvalue decomposition (4)*

$$A = V \Lambda V^{-1},$$

where $\Lambda = diag\{\lambda_1, \dots, \lambda_n\}$. Let $(\hat{\lambda}_i, \hat{\boldsymbol{x}}_i)$ be an approximate eigenpair of A computed by Algorithm-1 *with the corresponding residual vector*

$$\boldsymbol{r}_i = (A - \hat{\lambda}_i I)\hat{\boldsymbol{x}}_i\,,\ (i = 1, \dots, n)\,.$$

Then there exists an eigenvalue λ_i of A such that

$$|\lambda_i - \hat{\lambda}_i| \leq k(V)\|\boldsymbol{r}_i\|\,, \tag{26}$$

where $k(V) = \|V\|\|V^{-1}\|$ denotes the condition number of V.

Proof: The result is obvious if $\lambda_i = \hat{\lambda}_i$. Let assume that $\lambda_i \neq \hat{\lambda}_i$, then the matrix $A - \lambda_i I$ is nonsingular and we have

$$(A - \hat{\lambda}_i I)^{-1} = V(\Lambda - \hat{\lambda}_i I)^{-1} V^{-1}\,,$$

which implies

$$1 \leq \|\hat{\mathbf{x}}_i\| = \|(A - \hat{\lambda}_i I)^{-1}\mathbf{r}_i\| = \|V(\Lambda - \hat{\lambda}_i I)^{-1}V^{-1}\mathbf{r}_i\| \leq k(V)\frac{\|\mathbf{r}_i\|}{\min_j |\lambda_j - \hat{\lambda}_i|}\,.$$

Rearranging the terms we get the result of the theorem.

A similar approach to Theorem 3 for studying the convergence of Ritz values is the next theorem, where both left and right residuals are used. This is the two-sided analogous of the known Bauer-Fike theorem (see [2, 11]).

Theorem 4 *Let we have the eigenvalue decomposition (4) of A*

$$A = V \Lambda V^{-1},$$

where $\Lambda = diag\{\lambda_1, \dots, \lambda_n\}$. Let $(\hat{\lambda}_i, \hat{\boldsymbol{x}}_i, \hat{\boldsymbol{y}}_i)$ be an approximate eigentriplet of A computed by Algorithm-1 *with the corresponding residual vectors*

$$\boldsymbol{r}_i = (A - \hat{\lambda}_i I)\hat{\boldsymbol{x}}_i \quad and \quad \boldsymbol{s}_i^* = \hat{\boldsymbol{y}}_i^*(A - \hat{\lambda}_i I)\,,\ (i = 1, \dots, n)\,.$$

Then there exists an eigenvalue λ_i of A such that

$$|\lambda_i - \hat{\lambda}_i| \leq \sqrt{k(V)\|\boldsymbol{r}_i\|\|\boldsymbol{s}_i\|}\,, \tag{27}$$

where $k(V)$ denotes the condition number of V.

Proof: If $\hat{\lambda}_i$ is an eigenvalue of A the result is obvious. Let suppose the opposite. Then the matrix $A - \lambda_i I$ is nonsingular and we have

$$1 = |\hat{\mathbf{y}}_i^* \hat{\mathbf{x}}_i| = |\mathbf{s}_i^*(A - \hat{\lambda}_i I)^{-2}\mathbf{r}_i| = |\mathbf{s}_i^* V(\Lambda - \hat{\lambda}_i I)^{-2} V^{-1}\mathbf{r}_i| \leq$$

by using that the matrix $(\Lambda - \hat{\lambda}_i I)$ is diagonal

$$\leq k(V)\|\mathbf{s}_i\|\|\mathbf{r}_i\|\|(\Lambda - \hat{\lambda}_i I)^{-2}\|\,,$$

which implies by rearranging the terms

$$\min_{\lambda_i} |\lambda_i - \hat{\lambda}_i| \leq k(V)\|\mathbf{s}_i\|\|\mathbf{r}_i\|.$$

From the last expression, we get the result of the theorem.

It is important to remark that the estimates obtained in Theorem 3 and Theorem 4 depend on the quantity $k(V)$ which is not known, and so these are *a priori* results. Such estimates are of great value in assessing the relative performance of algorithms.

5 Conclusion

Our goal in this study was a further exploration of a relatively new iterative method that we introduced in a previous work. We have shown that this method can be considered as an oblique projection method. Using this relation we proved some analogs of known results from the theory of projection methods. We established an a posterior error bound and error bounds by using the residuals. Although the resulting bounds are not easy to calculate, they give considerable insight. We believe that the obtained results will contribute to further investigation of the considered method.

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