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FREDHOLM TOEPLITZ OPERATORS ON THE WEIGHTED BERGMAN SPACES*

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Abstract

In this paper we have shown that if $\phi \in (L^2_h(dA_\alpha))^\perp \cap L^\infty(\mathbb{D})$ and Range$T^{(\alpha)}_\phi$ is closed, then the Toeplitz operator $T^{(\alpha)}_\phi \in \mathcal{L}\left(L^2_a(dA_\alpha)\right)$ is a Fredholm operator of index zero and $T^{(\alpha)}_\phi$ is not of finite rank. Several applications of the result were also obtained. We further show that if $\phi \in L^\infty_{M_n}(\mathbb{D})$ is such that T_ϕ is Fredholm and of index zero in $\mathcal{L}\left(L^{2,\mathbb{C}^n}_a(dA_\alpha)\right)$ then there exists $\psi \in E_{n\times n} = E \otimes M_n$ such that $T_{\phi+\delta\psi}$ is invertible for all sufficiently small nonzero δ. Here E is a total subspace of $L^\infty(\mathbb{D})$ and M_n is the set of all $n \times n$ matrices with complex entries.

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1 Introduction

Let $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ be the normalized area measure on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. For $\alpha > -1$, let

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$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z),\ z \in \mathbb{D}$. Let $L^2(\mathbb{D}, dA_\alpha)$ be the space of all absolutely square-integrable Lebesgue measurable functions on $\mathbb{D}$. Let the weighted Bergman space of the disk $\mathbb{D}$, $L^2_a(dA_\alpha)$ be the subspace consisting of all analytic functions of $L^2(\mathbb{D}, dA_\alpha)$. The space $L^2(\mathbb{D}, dA_\alpha)$ is a Hilbert space with respect to the inner product defined by

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z), \quad f, g \in L^2(\mathbb{D}, dA_\alpha).$$

The weighted Bergman space $L^2_a(dA_\alpha)$ is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$ [4] and hence a reproducing kernel Hilbert space with the reproducing kernel given by $K^{(\alpha)}(z, \overline{w}) = \frac{1}{(1-z\overline{w})^{\alpha+2}},\ z, w \in \mathbb{D}$. The orthogonal projection P_α from the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ onto the closed subspace $L^2_a(dA_\alpha)$ is given by $P_\alpha f(z) = \int_{\mathbb{D}} K^{(\alpha)}(z, \overline{w})f(w)dA_\alpha(w)$. Let $k_z(w) = \frac{1-|z|^2}{(1-w\overline{z})^2}$. The functions $k_z^{1+\frac{\alpha}{2}} = \frac{K^{(\alpha)}(z,w)}{\sqrt{K^{(\alpha)}(w,\overline{w})}} = \frac{(1-|w|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{w}z)^{2+\alpha}}$ are the normalized reproducing kernels of $L^2_a(dA_\alpha)$. The sequence of functions $\{\frac{z^n}{\gamma_{n,\alpha}}\}$ form an orthonormal basis [13] for $L^2_a(dA_\alpha)$ where $\gamma^2_{n,\alpha} = \|z^n\|^2 = (\alpha+1)\int_{\mathbb{D}} |z|^{2n}(1-|z|^2)^\alpha dA(z) = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \sim (n+1)^{-\alpha-1}$. Let $L^\infty(\mathbb{D}, dA)$ be the space of all essentially bounded Lebesgue measurable functions on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, define $\|\varphi\|_\infty = ess\ sup\{|\varphi(z)| : z \in \mathbb{D}\} < \infty$. The space $L^\infty(\mathbb{D})$ is a Banach space with respect to the $\|\cdot\|_\infty$. Let $H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$ and $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. $L^2_h(dA_\alpha) = L^2_a(dA_\alpha) \oplus \left(\overline{L^2_a(dA_\alpha)}\right)_0$ where $\left(\overline{L^2_a(dA_\alpha)}\right)_0 = \{f \in L^2_a(dA_\alpha) : f(0) = 0\}$.

Given a function $\varphi \in L^\infty(\mathbb{D})$, we define an operator $T^{(\alpha)}_\varphi$ on $L^2_a(dA_\alpha)$ by $T^{(\alpha)}_\varphi f = P_\alpha(\varphi f), \quad f \in L^2_a(dA_\alpha)$. The operator $T^{(\alpha)}_\varphi$ is called the Toeplitz operator on the weighted Bergman space with symbol φ. Since the projection P_α has norm 1, we have $\|T^{(\alpha)}_\varphi\| \leq \|\varphi\|_\infty$. We can write $T^{(\alpha)}_\varphi$ as an integral operator as, $T^{(\alpha)}_\varphi f(z) = \int_{\mathbb{D}} \varphi(w)K^{(\alpha)}(z, w)f(w)dA_\alpha(w) = \int_{\mathbb{D}} \frac{\varphi(w)f(w)}{(1-z\overline{w})^{\alpha+2}}dA_\alpha(w)$. Let $\overline{L^2_a(dA_\alpha)} = \{\overline{f} : f \in L^2_a(dA_\alpha)\}$. The space $\overline{L^2_a(dA_\alpha)}$ is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$. The little Hankel operator $h^{(\alpha)}_\phi$ with symbol ϕ is defined by $h^{(\alpha)}_\phi f = \overline{P_\alpha}(\phi f), f \in L^2_a(dA_\alpha)$ where $\overline{P_\alpha}$ is the orthogonal projection from the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$ onto $\overline{L^2_a(dA_\alpha)}$. Clearly, $\|h^{(\alpha)}_\phi\| \leq \|\phi\|_\infty$. Define J_α from $L^2(\mathbb{D}, dA_\alpha)$ into itself by $(J_\alpha f)(z) = f(\overline{z}),\ z \in \mathbb{D}$. The operator J_α is an unitary operator. For $\phi \in L^\infty(\mathbb{D})$,

define $S_\phi^{(\alpha)}$ from $L_a^2(dA_\alpha)$ into itself by $S_\phi^{(\alpha)} f = P_\alpha\left(J_\alpha(\phi f)\right)$. The operator $S_\phi^{(\alpha)}$ is a linear operator and $\|S_\phi^{(\alpha)}\| \leq \|\phi\|_\infty$. It is not difficult to verify that $h_\phi^{(\alpha)} = J_\alpha S_\phi^{(\alpha)}$. Thus we shall refer in the sequel, both the operators $h_\phi^{(\alpha)}$ and $S_\phi^{(\alpha)}$ as little Hankel operators on $L_a^2(dA_\alpha)$.
Let $L_a^{2,\mathbb{C}^n}(dA_\alpha) = L_a^2(dA_\alpha) \otimes \mathbb{C}^n$ and $L_{M_n}^\infty(\mathbb{D}) = L^\infty(\mathbb{D}) \otimes M_n$ where $M_n(\mathbb{C}) = M_n, n \geq 1$ is the set of all $n \times n$ matrices with entries in $\mathbb{C}$. The space $L_a^{2,\mathbb{C}^n}(dA_\alpha), n \geq 1$ is called the vector-valued weighted Bergman space. The inner product on $L_a^{2,\mathbb{C}^n}(dA_\alpha)$ is defined as

$$\langle f, g \rangle_{L_a^{2,\mathbb{C}^n}(dA_\alpha)} = \int_{\mathbb{D}} \langle f(z), g(z) \rangle_{\mathbb{C}^n} dA_\alpha(z).$$

With this inner product $L_a^{2,\mathbb{C}^n}(dA_\alpha)$ is a Hilbert space. The norm defined on $L_a^{2,\mathbb{C}^n}(dA_\alpha)$ is given by

$$\|f\|^2_{L_a^{2,\mathbb{C}^n}(\mathbb{D}, dA_\alpha)} = \int_{\mathbb{D}} \|f(z)\|^2_{\mathbb{C}^n} dA_\alpha(z).$$

It is a closed subspace of $L^{2,\mathbb{C}^n}(\mathbb{D}, dA_\alpha) = L^2(\mathbb{D}, dA_\alpha) \otimes \mathbb{C}^n$. Let $P_\alpha^{\mathbb{C}^n}$ denote the orthogonal projection from $L^{2,\mathbb{C}^n}(\mathbb{D}, dA_\alpha)$ onto $L_a^{2,\mathbb{C}^n}(dA_\alpha)$. For $\phi \in L_{M_n}^\infty(\mathbb{D})$, we define the Toeplitz operator T_ϕ from $L_a^{2,\mathbb{C}^n}(dA_\alpha)$ into itself as $T_\phi f = P_\alpha^{\mathbb{C}^n}(\phi f)$ and the Hankel operator H_ϕ from $L_a^{2,\mathbb{C}^n}(dA_\alpha)$ into $\left(L_a^{2,\mathbb{C}^n}(dA_\alpha)\right)^\perp = L^{2,\mathbb{C}^n}(\mathbb{D}, dA_\alpha) \ominus L_a^{2,\mathbb{C}^n}(dA_\alpha)$ as $H_\phi f = (I - P_\alpha^{\mathbb{C}^n})(\phi f)$.
For $\phi \in L_{M_n}^\infty(\mathbb{D})$, define $\|\phi\|_\infty = ess\ sup_{z \in \mathbb{D}} \|\phi(z)\|$. If $\phi \in L_{M_n}^\infty(\mathbb{D})$, then it is not difficult to see that $\|T_\phi\| \leq \|\phi\|_\infty$ and $\|H_\phi\| \leq \|\phi\|_\infty$. This is so as $\|P_\alpha^{\mathbb{C}^n}\| \leq 1$ and $\|I - P_\alpha^{\mathbb{C}^n}\| \leq 1$.
For $\phi \in L_{M_n}^\infty(\mathbb{D})$, we define the little Hankel operator S_ϕ from $L_a^{2,\mathbb{C}^n}(dA_\alpha)$ into itself as $S_\phi f = P_\alpha^{\mathbb{C}^n} J_\alpha^n(\phi f)$ where $J_\alpha^n : L^{2,\mathbb{C}^n}(\mathbb{D}, dA_\alpha) \to L^{2,\mathbb{C}^n}(\mathbb{D}, dA_\alpha)$ is defined as $J_\alpha^n f(z) = f(\overline{z})$. The map J_α^n is unitary. There are also many equivalent ways of defining little Hankel operators. Let $\overline{L_a^{2,\mathbb{C}^n}(dA_\alpha)} = \overline{L_a^2(dA_\alpha)} \otimes \mathbb{C}^n$. For $\phi \in L_{M_n}^\infty(\mathbb{D})$, define h_ϕ from $L_a^{2,\mathbb{C}^n}(dA_\alpha)$ into $\overline{L_a^{2,\mathbb{C}^n}(dA_\alpha)}$ as $h_\phi f = \overline{P_\alpha^{\mathbb{C}^n}}(\phi f)$ where $\overline{P_\alpha^{\mathbb{C}^n}}$ is the orthogonal projection from $L^{2,\mathbb{C}^n}(\mathbb{D}, dA_\alpha)$ onto $\overline{L_a^{2,\mathbb{C}^n}(dA_\alpha)}$. It is not difficult to verify that $h_\phi = J_\alpha^n S_\phi$.

Let $\mathcal{L}(H)$ be the space of all bounded linear operators from the Hilbert space H into itself. Let $\mathcal{LC}(H)$ be the set of all compact operators in $\mathcal{L}(H)$. The quotient algebra $\mathcal{L}(H)/\mathcal{LC}(H)$ is a Banach algebra called the Calkin algebra. The natural homomorphism from $\mathcal{L}(H)$ onto $\mathcal{L}(H)/\mathcal{LC}(H)$ is denoted by π. An operator T in $\mathcal{L}(H)$ is a Fredholm operator if $\pi(T)$ is an invertible element of $\mathcal{L}(H)/\mathcal{LC}(H)$. The collection of Fredholm operators on H is denoted by $\mathcal{F}(H)$. It follows from a theorem of Atkinson [1] that, an operator T in $\mathcal{L}(H)$ is a Fredholm operator if and only if the range of T is closed, $dim\ kerT$ is finite and $dim\ kerT^*$ is finite. Let $\mathcal{F}(L_a^2(dA_\alpha))$ be the set of all Fredholm operators in $\mathcal{L}(L_a^2(dA_\alpha))$. The classical index j is the function from $\mathcal{F}(L_a^2(dA_\alpha))$ to $\mathbb{Z}$ defined as $j(T) = dim\ kerT - dim\ kerT^*$. In this paper we have shown that if $\phi \in (L_h^2(dA_\alpha))^\perp \cap L^\infty(\mathbb{D})$ and Range$T_\phi^{(\alpha)}$ is closed, then $T_\phi^{(\alpha)}$ is a Fredholm operator of index zero and $T_\phi^{(\alpha)}$ is not of finite rank. The organization of the paper is as follows: In section 2, we establish certain preliminary results. We have shown that $kerh_\phi^{(\alpha)} = \{0\}$ if and only if $\overline{Range\ h_\phi^{(\alpha)}} = \overline{L_a^2(dA_\alpha)}$ and $kerH_{\overline{f}H}^{E_p^\alpha} = \{0\}$ if and only if $\overline{\text{Range}H_{\overline{f}H}^{E_p^\alpha}} = E_p^\alpha$ where $\phi \in L^\infty(\mathbb{D}),\ H, f \in L_a^2(dA_\alpha)$. We relate the concept of common zero set and the rank of a Toeplitz operator. In section 3, we present the main results of the paper and some applications of these results. We establish that if $\phi \in (L_h^2(dA_\alpha))^\perp \cap L^\infty(\mathbb{D})$ and Range$T_\phi^{(\alpha)}$ is closed, then $T_\phi^{(\alpha)}$ is a Fredholm operator of index zero and $T_\phi^{(\alpha)}$ is not of finite rank. Further if $\phi \in L_{M_n}^\infty(\mathbb{D})$ is such that T_ϕ is Fredholm and of index zero in $\mathcal{L}\left(L_a^{2,\mathbb{C}^n}(dA_\alpha)\right)$ then there exists $\psi \in E_{n\times n} = E \otimes M_n$ such that $T_{\phi+\delta\psi}$ is invertible for all sufficiently small nonzero δ.

2 Preliminaries

In this section we present some elementary results that will be used in establishing the main results of the paper. We relate the kernel of little Hankel operators and intermediate Hankel operators with the kernel of their adjoints. We relate the concept of common zero set and the rank of a Toeplitz operator. For $p \geq 0$, let $E_p^\alpha = \overline{\text{span}}\{|z|^{2k}\overline{z}^n, k = 0, ..., p\ ;\ n = 0, 1, 2, ...\}$. For $p \geq 0$, the spaces E_p^α are closed subspaces of $L_a^2(dA_\alpha)$. For $\phi \in L^\infty(\mathbb{D})$, we define the intermediate Hankel operator $H_\phi^{E_p^\alpha} : L_a^2(dA_\alpha) \to E_p^\alpha$ by $H_\phi^{E_p^\alpha}(f) = P_p^{(\alpha)}(\phi f), f \in L_a^2(dA_\alpha)$ where $P_p^{(\alpha)}$ is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto E_p^α. Notice that $\overline{L_a^2(dA_\alpha)} \subseteq E_p^\alpha \subseteq ((L_a^2(dA_\alpha))_0)^\perp$ where $(L_a^2(dA_\alpha))_0 = \{g \in L_a^2(dA_\alpha) : g(0) = 0\}$.

Lemma 2.1. *Let $\phi \in L^\infty(\mathbb{D})$. Then $kerh_\phi^{(\alpha)} = \{0\}$ if and only if $\overline{Range\ h_\phi^{(\alpha)}} = \overline{L_a^2(dA_\alpha)}$.*

Proof. Observe that $\left(S_\phi^{(\alpha)}\right)^* = S_{\phi^+}^{(\alpha)}$ where $\phi^+(z) = \overline{\phi(\overline{z})}$. It is easy to see that $f \in kerS_\phi^{(\alpha)}$ if and only if $f^+ \in kerS_{\phi^+}^{(\alpha)}$. This implies if $kerS_\phi^{(\alpha)} = \{0\}$ then $ker\left(S_\phi^{(\alpha)}\right)^* = \{0\}$. Thus $\overline{Range\ S_\phi^{(\alpha)}} = L_a^2(dA_\alpha)$. Conversely, if $\overline{Range\ S_\phi^{(\alpha)}} = L_a^2(dA_\alpha)$ then $ker\left(S_\phi^{(\alpha)}\right)^* = \{0\}$ which implies $kerS_\phi^{(\alpha)} = \{0\}$.
Clearly $h_\phi^{(\alpha)} f = 0$ for $f \in kerh_\phi^{(\alpha)}$. This implies $J_\alpha S_\phi^{(\alpha)} f = 0$ which further implies $S_\phi^{(\alpha)} f = 0$. Thus $f^+ \in kerS_{\phi^+}^{(\alpha)}$. Hence $J_\alpha f^+ \in kerS_{\phi^+}^{(\alpha)} J_\alpha = ker\left(h_\phi^{(\alpha)}\right)^*$ as $S_{\phi^+}^{(\alpha)} J_\alpha = \left(J_\alpha S_\phi^{(\alpha)}\right)^* = \left(h_\phi^{(\alpha)}\right)^*$. Let $\overline{g} \in \overline{L_a^2(dA_\alpha)}$ and $\overline{g} \in ker\left(h_\phi^{(\alpha)}\right)^*$. This implies $J_\alpha \overline{g} \in ker\left(S_\phi^{(\alpha)}\right)^*$. Hence $(J_\alpha \overline{g})^+ \in h_\phi^{(\alpha)}$. Therefore, $(J_\alpha \overline{g})^+ \in h_\phi^{(\alpha)}$. That is, $g \in kerh_\phi^{(\alpha)}$. Thus if $f \in L_a^2(dA_\alpha)$ then $f \in kerh_\phi^{(\alpha)}$ if and only if $\overline{f} \in ker\left(h_\phi^{(\alpha)}\right)^*$. Hence $kerh_\phi^{(\alpha)} = \{0\}$ if and only if $ker\left(h_\phi^{(\alpha)}\right)^* = \{0\}$ and this is true if and only if $\overline{Range\ h_\phi^{(\alpha)}} = \overline{L_a^2(dA_\alpha)}$.
The result follows. $\square$

Lemma 2.2. *Let $H \in L_a^2(dA_\alpha)$ and assume that $f \in L_a^2(dA_\alpha)$ is not a polynomial. Then $kerH_{\overline{f}H}^{E_p^\alpha} = \{0\}$ if and only if $\overline{RangeH_{\overline{f}H}^{E_p^\alpha}} = E_p^\alpha$.*

Proof. Notice that

$$\begin{aligned} kerH_{\overline{f}H}^{*E_p^\alpha} &= \{g \in E_p^\alpha : P_\alpha(f\overline{H}g) = 0\} \\ &\supset ker\left(h_{\overline{f}H}^{(\alpha)}\right)^* \\ &= \{g \in \overline{L_a^2(dA_\alpha)} : P_\alpha(f\overline{H}g) = 0\}. \end{aligned}$$

If $kerH_{\overline{f}H}^{*E_p^\alpha} = \{0\}$ then $ker\left(h_{\overline{f}H}^{(\alpha)}\right)^* = \{0\}$. Hence $kerh_{\overline{f}H}^{(\alpha)} = \{0\}$. Since for $h \in L_a^2(dA_\alpha)$, $H_{\overline{f}H}^{E_p^\alpha} h = h_{\overline{f}H}^{(\alpha)} h + P_{E_p^\alpha \ominus (\overline{L_a^2(dA_\alpha)})}(\overline{f}Hh)$, we obtain $kerH_{\overline{f}H}^{E_p^\alpha} = \{0\}$.
Now suppose $kerH_{\overline{f}H}^{E_p^\alpha} = \{0\}$. This implies $kerh_{\overline{f}H}^{(\alpha)} = \{0\}$. Because if $kerh_{\overline{f}H}^{(\alpha)} \neq \{0\}$ then [5], [6], [7] there exists an inner function $G_\alpha \in L_a^2(dA_\alpha)$ such that $G_\alpha \in kerh_{\overline{f}H}^{(\alpha)}$. That is, $h_{\overline{f}HG_\alpha}^{(\alpha)} \equiv 0$. This is so as $kerh_{\overline{f}H}^{(\alpha)} = kerS_{\overline{f}H}^{(\alpha)}$ is an invariant subspace of z. Observe that for $\psi \in L^\infty(\mathbb{D})$, $h_\psi^{(\alpha)} T_z^{(\alpha)} = h_{\psi z}^{(\alpha)}$ and $\left(T_{\overline{z}}^{(\alpha)} h_\psi^{(\alpha)}\right)^* = \left(h_\psi^{(\alpha)}\right)^* T_z^{(\alpha)} = S_\psi^{(\alpha)} + J_\alpha T_z^{(\alpha)}$. Further for $g \in$

$L^2_a(dA_\alpha)$, $S^{(\alpha)}_{\psi^+}J_\alpha T^{(\alpha)}_z g = S^{(\alpha)}_{\psi^+}(J_\alpha(zg)) = P_\alpha(J_\alpha(\psi^+\overline{z}J_\alpha g)) = P_\alpha(\overline{\psi}zg) = \left(h^{(\alpha)}_{\psi\overline{z}}\right)^* g$. Similarly one can show that $\left(T^{(\alpha)}_{\overline{z}^k}h^{(\alpha)}_{\psi}\right)^* = \left(h^{(\alpha)}_{\psi\overline{z}^k}\right)^*$ for all $k = 0, 1, ..., p$. Thus $h^{(\alpha)}_{\overline{f}HG_\alpha} \equiv 0$ implies $\left(T^{(\alpha)}_{\overline{z}^k}h^{(\alpha)}_{\overline{f}HG_\alpha}\right)^* \equiv 0$. Hence $\left(h^{(\alpha)}_{\overline{f}HG_\alpha\overline{z}^k}\right)^* \equiv 0$. This implies $h^{(\alpha)}_{\overline{f}HG_\alpha\overline{z}^k} \equiv 0$ for all $k = 0, 1, ..., p$. Hence $\overline{f}HG_\alpha\overline{z}^k \in \left(\overline{L^2_a(dA_\alpha)}\right)^\perp$. That is, $\langle\overline{f}HG_\alpha\overline{z}^k, \overline{z}^k\overline{g}\rangle = 0$ for all $g \in L^2_a(dA_\alpha)$ and $k = 0, 1, ..., p$. Thus $\overline{f}HG_\alpha \in (E^\alpha_p)^\perp$. Hence $ker H^{E^\alpha_p}_{\overline{f}H} \neq \{0\}$. Thus $ker H^{E^\alpha_p}_{\overline{f}H} = \{0\}$ implies $ker h^{(\alpha)}_{\overline{f}H} = \{0\}$. Hence $ker\left(h^{(\alpha)}_{\overline{f}H}\right)^* = \{0\}$. This implies $ker H^{*E^\alpha_p}_{\overline{f}H} = \{0\}$. Because if $ker H^{*E^\alpha_p}_{\overline{f}H} \neq \{0\}$ then there exist $0 \neq g \in E^\alpha_p \cap L^\infty$ such that $\overline{f}Hg \in (L^2_a(dA_\alpha))^\perp$. That is, $\overline{f}H\overline{g} \in (\overline{L^2_a(dA_\alpha)})^\perp$. This implies $h^{(\alpha)}_{\overline{f}H\overline{g}} \equiv 0$ and therefore $ker h^{(\alpha)}_{\overline{f}H} \supseteq \overline{g}L^2_a(dA_\alpha) \cap L^2_a(dA_\alpha) \neq \{0\}, \overline{g} \in \overline{E^\alpha_p} \cap L^\infty = H^\infty$. Thus $ker H^{E^\alpha_p}_{\overline{f}H} = \{0\}$ if and only if $ker H^{*E^\alpha_p}_{\overline{f}H} = \{0\}$. □

If N is a subspace of $L^2_a(dA_\alpha)$, let $\mathcal{Z}(N) = \{z \in \mathbb{D} : f(z) = 0$ for all $f \in N\}$ which is called the zero set of functions in N. Here if z_1 is a zero of multiplicity at most n of all functions in N then z_1 appears n times in the set $\mathcal{Z}(N)$ and they are treated as distinct elements of $\mathcal{Z}(N)$. In the following proposition, we have shown that if $\phi \in L^\infty(\mathbb{D})$ is such that $T^{(\alpha)}_\phi$ is a finite rank Toeplitz operator and Card $\mathcal{Z}\left(ker\left(T^{(\alpha)}_\phi\right)^*\right)$ = Rank of $T^{(\alpha)}_\phi$ then $\phi \equiv 0$. With the following result we begin to link the ideas of subspaces and zero-sets.

Proposition 2.1. *If N is a subspace of $L^2_a(dA_\alpha)$ of finite codimension in $L^2_a(dA_\alpha)$ then*

$$\mathcal{Z}(N) = \{z \in \mathbb{D} : f(z) = 0 \textit{ for all } f \in N\}$$

is a finite set.

Proof. Suppose $\mathcal{Z}(N)$ is an infinite set. Let $\{z_j\}^\infty_{j=1}$ be distinct points of $\mathcal{Z}(N)$ and let $f_1, f_2, f_3, ..., f_n$ be functions in $L^2_a(dA_\alpha)$ such that

$$f_i(z_1) = ... = f_i(z_{i-1}) = 0, f_i(z_i) = 1 \ \text{ for all } \ i \geq 2.$$

For example, we could take the functions (f_i) to be the polynomials. Then

$f_1, f_2, ...$ are linearly independent modulo N i.e. if

$$\alpha_1 f_1 + \alpha_2 f_2 + ... + \alpha_n f_n \in N$$

where $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ then $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. This contradicts the assumption that N has finite codimension in $L^2_a(dA_\alpha)$. Since each zero of an analytic function has finite multiplicity, the result is proved. □

Let $k_z^{1+\frac{\alpha}{2}}$ be the normalised reproducing kernel for the Bergman space $L^2_a(dA_\alpha)$. When $|z| \to 1$, $k_z^{1+\frac{\alpha}{2}} \to 0$ weakly and the normalised reproducing kernels $k_z^{1+\frac{\alpha}{2}}, z \in \mathbb{D}$ span $L^2_a(dA_\alpha)$ [13].

Theorem 2.3. *If $\phi \in L^\infty(\mathbb{D})$ is such that $T_\phi^{(\alpha)}$ is a finite rank Toeplitz operator and Card $\mathcal{Z}\left(ker\left(T_\phi^{(\alpha)}\right)^*\right) =$ Rank of $T_\phi^{(\alpha)}$ then $\phi \equiv 0$.*

Proof. Suppose $T_\phi^{(\alpha)}$ is a finite rank. Then Range$T_\phi^{(\alpha)}$ is finite dimensional and is a closed subspace of $L^2_a(dA_\alpha)$. Let $n = \dim \text{Range}T_\phi^{(\alpha)}$. Thus $ker\left(T_\phi^{(\alpha)}\right)^* = (\overline{\text{Range}T_\phi^{(\alpha)}})^\perp$ has finite codimension. Therefore by Proposition 2.1, $\mathcal{Z}\left(ker\left(T_\phi^{(\alpha)}\right)^*\right)$, the common zero set of $ker\left(T_\phi^{(\alpha)}\right)^*$ is a finite set. Without loss of generality, we shall assume the elements of $\mathcal{Z}\left(ker\left(T_\phi^{(\alpha)}\right)^*\right)$ are distinct. Suppose $\mathcal{Z}\left(ker\left(T_\phi^{(\alpha)}\right)^*\right) = \{a_1, a_2, ..., a_n\}$. Here $a_1, a_2, ..., a_n$ are all distinct. Then $ker\left(T_\phi^{(\alpha)}\right)^* \subset \{f \in L^2_a(dA_\alpha) : f(a_i) = 0, i = 1, 2, ..., n\}$. But $\{f \in L^2_a(dA_\alpha) : f(a_i) = 0, i = 1, 2, ..., n\} = \{f \in L^2_a(dA_\alpha) : \langle f, k_{a_i}^{1+\frac{\alpha}{2}}\rangle = 0, i = 1, 2, ..., n\} = \left\{k_{a_1}^{1+\frac{\alpha}{2}}, k_{a_2}^{1+\frac{\alpha}{2}}, ..., k_{a_n}^{1+\frac{\alpha}{2}}\right\}^\perp$. Thus $\text{sp}\left\{k_{a_1}^{1+\frac{\alpha}{2}}, k_{a_2}^{1+\frac{\alpha}{2}}, ..., k_{a_n}^{1+\frac{\alpha}{2}}\right\} \subset \left(ker\left(T_\phi^{(\alpha)}\right)^*\right)^\perp = \text{Range}T_\phi^{(\alpha)}$. In case of repeated zeros (if a is a zero of order m, say) the derivatives of the corresponding reproducing kernel up to order $m-1$, i.e., $k_a^{1+\frac{\alpha}{2}}, \frac{\partial}{\partial \bar{a}} k_a^{1+\frac{\alpha}{2}}, ...,$ $\frac{\partial^{m-1}}{\partial \bar{a}^{m-1}} k_a^{1+\frac{\alpha}{2}}$ are included in the spanned set [7]. Now since Range$T_\phi^{(\alpha)}$ has dimension n and $k_{a_1}^{1+\frac{\alpha}{2}}, k_{a_2}^{1+\frac{\alpha}{2}}, ..., k_{a_n}^{1+\frac{\alpha}{2}}$ are linearly independent therefore $\text{Range}T_\phi^{(\alpha)} = \text{sp}\left\{k_{a_1}^{1+\frac{\alpha}{2}}, k_{a_2}^{1+\frac{\alpha}{2}}, ..., k_{a_n}^{1+\frac{\alpha}{2}}\right\}$. Thus $ker\left(T_\phi^{(\alpha)}\right)^* = \left\{k_{a_1}^{1+\frac{\alpha}{2}}, k_{a_2}^{1+\frac{\alpha}{2}}, ..., k_{a_n}^{1+\frac{\alpha}{2}}\right\}^\perp = \{f \in L^2_a(dA_\alpha) : f(a_i) = 0, i = 1, 2, ..., n\}$ is an invariant subspace of the Bergman shift operator $T_z^{(\alpha)}$ defined on $L^2_a(dA_\alpha)$. Since $T_\phi^{(\alpha)}$ is finite rank implies $\left(T_\phi^{(\alpha)}\right)^*$ is finite rank therefore one can show that

$kerT_\phi^{(\alpha)}$ is also an invariant subspace of $L_a^2(dA_\alpha)$. Let $\mathcal{P}$ be the set of all polynomials in $L_a^2(dA_\alpha)$ and $M = kerT_\phi^{(\alpha)}$. Since Range$T_\phi^{(\alpha)}$ has dimension n, therefore $T_\phi^{(\alpha)}1, T_\phi^{(\alpha)}z, ..., T_\phi^{(\alpha)}z^n$ are linearly dependent. This implies there exists a non-zero polynomial p of degree at most n such that $T_\phi^{(\alpha)}p = 0$. That is, $\phi p \in (L_a^2(dA_\alpha))^\perp$. Using the facts that codimension of M is finite, and $T_z^{(\alpha)}M \subset M$, it follows that $\mathcal{P} \cap M$ is a nontrivial ideal of $\mathcal{P}$. Since $\mathcal{P}$ is a principal ideal ring, there exists $q \in \mathcal{P}$ such that $\mathcal{P} \cap M = q\mathcal{P}$, see [8]. Thus, $T_{\phi q}^{(\alpha)}g = 0$ for all polynomials $g \in \mathcal{P}$. This implies $T_{\phi q}^{(\alpha)}z^k = 0$ for all $k \geq 0$. That is, $\phi q \in (\overline{z}^k L_a^2(dA_\alpha))^\perp$ for all $k \geq 0$. Hence $\phi q \in \cap_{k\geq 0}(\overline{z}^k L_a^2(dA_\alpha))^\perp = (\cup_{k\geq 0}\overline{z}^k L_a^2(dA_\alpha))^\perp$. Therefore, it follows that $\phi q \perp \overline{z}^k z^n$ for all $k, n \geq 0$. Now, $\phi q \in L^\infty \subset L^2$ implies that $\phi q = 0$. Thus $\phi = 0$ except at the zeros of q which is a polynomial of degree at most n. Hence $\phi \equiv 0$ as $\phi \in L^\infty(\mathbb{D})$. □

For z and w in $\mathbb{D}$, let $\phi_z(w) = \frac{z-w}{1-\overline{z}w}$. These are involutive Mobius transformations on $\mathbb{D}$. In fact

1. $\phi_z \circ \phi_z(w) \equiv w$;
2. $\phi_z(0) = z, \phi_z(z) = 0$;
3. ϕ_z has a unique fixed point in $\mathbb{D}$.

Given $z \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_z f(w) = k_z^{1+\frac{\alpha}{2}}(w)f(\phi_z(w))$. Since $|k_z^{1+\frac{\alpha}{2}}|^2$ is the real Jacobian determinant of the mapping ϕ_z (see [14]), U_z is easily seen to be a unitary operator on $L^2(\mathbb{D}, dA_\alpha)$ and $L_a^2(dA_\alpha)$. It is also easy to check that $U_z^* = U_z$, thus U_z is a self-adjoint unitary operator. If $\phi \in L^\infty(\mathbb{D}, dA)$ and $z \in \mathbb{D}$ then $U_z T_\phi^{(\alpha)} = T_{\phi\circ\phi_z}^{(\alpha)}U_z$. This is because $P_\alpha U_z = U_z P_\alpha$ and for $f \in L_a^2(dA_\alpha)$, $T_{\phi\circ\phi_z}^{(\alpha)}U_z f = T_{\phi\circ\phi_z}^{(\alpha)}((f\circ\phi_z)k_z^{1+\frac{\alpha}{2}}) = P_\alpha((\phi\circ\phi_z)(f\circ\phi_z)k_z^{1+\frac{\alpha}{2}}) = P_\alpha(U_z(\phi f)) = U_z P_\alpha(\phi f) = U_z T_\phi^{(\alpha)}f$. Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$, and G_0 the isotropy subgroup at 0; i.e., $G_0 = \{\psi \in Aut(\mathbb{D}) : \psi(0) = 0\}$. Notice also that $\phi_a(z)$, as a function in a, is one-one and onto for any fixed z in $\mathbb{D}$.

Proposition 2.2. *If $\phi \in L^\infty(\mathbb{D})$ then $T_\phi^{(\alpha)}$ is finite rank if and only if $T_{\phi\circ\phi_z}^{(\alpha)}$ is finite rank. In this case Rank of $T_\phi^{(\alpha)}$ = Rank of $T_{\phi\circ\phi_z}^{(\alpha)}$.*

Proof. Notice that $f \in kerT_{\phi}^{(\alpha)}$ if and only if $U_z f \in kerT_{\phi\circ\phi_z}^{(\alpha)}$. Further since U_z is unitary, $\dim ker\left(T_{\phi}^{(\alpha)}\right)^* = \dim ker\left(T_{\phi\circ\phi_z}^{(\alpha)}\right)^*$. Thus dim Range$T_{\phi}^{(\alpha)} =$ codim $ker\left(T_{\phi}^{(\alpha)}\right)^* =$ codim $ker\left(T_{\phi\circ\phi_z}^{(\alpha)}\right)^* =$ dim Range$T_{\phi\circ\phi_z}^{(\alpha)}$. □

For $\phi \in L^\infty(\mathbb{D})$, let $\tilde{\phi}(z) = \langle T_{\phi}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\rangle$, the Berezin transform of ϕ. It is easy to check that $\left(H_{\phi}^{(\alpha)}\right)^* H_{\phi}^{(\alpha)} = T_{|\phi|^2}^{(\alpha)} - T_{\overline{\phi}}^{(\alpha)} T_{\phi}^{(\alpha)}$(see [13]). The following also holds.

Proposition 2.3. *For $\phi \in L^\infty(\mathbb{D})$, $MO(\phi)^2(z) = \widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2 \leq \left\|H_{\phi}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\|^2 + \left\|H_{\overline{\phi}}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\|^2$.*

Proof. Observe that

$$\begin{aligned}&\left\|H_{\phi}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\| = \left\|(I-P_\alpha)(\phi k_z^{1+\frac{\alpha}{2}})\right\| = \|(I-P_\alpha)U_z(\phi\circ\phi_z)\| \\ &= \|U_z(I-P_\alpha)(\phi\circ\phi_z)\| = \|(I-P_\alpha)(\phi\circ\phi_z)\| = \|\phi\circ\phi_z - P_\alpha(\phi\circ\phi_z)\|.\end{aligned}$$

Similarly, we have

$$\left\|H_{\overline{\phi}}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\| = \left\|\overline{\phi}\circ\phi_z - P_\alpha(\overline{\phi}\circ\phi_z)\right\| = \left\|\phi\circ\phi_z - \overline{P_\alpha(\overline{\phi}\circ\phi_z)}\right\|.$$

Since $\tilde{\phi}(z) = P_\alpha(\phi\circ\phi_z)(0)$ and $P_\alpha\overline{g}(z) = \overline{g}(0)$ for any $g \in L_a^2(dA_\alpha)$ and all $z \in \mathbb{D}$, we have

$$\begin{aligned}MO(\phi)^2(z) &= \widetilde{|\phi|^2}(z) - |\widetilde{\phi}(z)|^2 \\ &= \|\phi\circ\phi_z - P_\alpha(\phi\circ\phi_z)(0)\|^2 \\ &= \|\phi\circ\phi_z - P_\alpha(\phi\circ\phi_z)\|^2 + \|P_\alpha(\phi\circ\phi_z) - P_\alpha(\phi\circ\phi_z)(0)\|^2 \\ &= \left\|H_{\phi}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\|^2 + \left\|P_\alpha(\phi\circ\phi_z) - \overline{P_\alpha(\overline{\phi}\circ\phi_z)(0)}\right\|^2 \\ &= \left\|H_{\phi}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\|^2 + \left\|P_\alpha(\phi\circ\phi_z - \overline{P_\alpha(\overline{\phi}\circ\phi_z)})\right\|^2 \\ &\leq \left\|H_{\phi}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\|^2 + \left\|\phi\circ\phi_z - \overline{P_\alpha(\overline{\phi}\circ\phi_z)}\right\|^2 \\ &= \left\|H_{\phi}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\|^2 + \left\|H_{\overline{\phi}}^{(\alpha)} k_z^{1+\frac{\alpha}{2}}\right\|^2.\end{aligned}$$

□

Proposition 2.4. *If* $T \in \mathcal{L}(L_a^2(dA_\alpha))$ *and* $\left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle = 0$ *for all* $z \in \mathbb{D}$ *then* $T \equiv 0$.

Proof. Define $\rho_\alpha(T)(z) = \left\langle Tk_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle$ for all $z \in \mathbb{D}$. If $\rho_\alpha(T) = 0$ identically, then also $\langle TK_z^\alpha, K_z^\alpha\rangle = K_\alpha(z,z)\rho_\alpha(T)(z) = 0$ identically where $K_z^\alpha = K_\alpha(.,z)$ is the unnormalized reproducing kernel. Thus the function $F(x,y) = \langle TK_{\overline{x}}^\alpha, K_z^\alpha\rangle$; which is holomorphic in x and y, vanishes on the "anti-diagonal" $x = \overline{y}$. Passing to the variables u, v defined by $x = u + iv$, $y = u - iv$, we get a holomorphic function $G(u,v)$ of u, v, which vanishes when u, v are real. Thus $F(x,y) = G(u,v) \equiv 0$. Thus even $\left\langle TK_x^\alpha, K_y^\alpha\right\rangle = 0$ for any x, y. Since linear combinations of $K_x^\alpha, x \in \mathbb{D}$ are dense in $\mathcal{L}(L_a^2(dA_\alpha))$; it follows that $T \equiv 0$. □

Notice that if for all $z \in \mathbb{D}, P_\alpha(\phi \circ \phi_z) = P_\alpha(\phi \circ \phi_z)(0) = 0$ then $U_z T_\phi^{(\alpha)} k_z^{1+\frac{\alpha}{2}} = U_z T_\phi^{(\alpha)} U_z 1 = T_{\phi\circ\phi_z}^{(\alpha)} 1 = 0$. Hence $T_\phi^{(\alpha)} k_z^{1+\frac{\alpha}{2}} = 0$ for all $z \in \mathbb{D}$ and therefore $\left\langle T_\phi^{(\alpha)} k_z^{1+\frac{\alpha}{2}}, k_z^{1+\frac{\alpha}{2}}\right\rangle = 0$ for all $z \in \mathbb{D}$. By proposition 2.4, $T_\phi^{(\alpha)} \equiv 0$ and thus $\phi \equiv 0$.

3 Fredholm Toeplitz operators of index zero

In this section we prove the main results of the paper. We have shown that if $\phi \in (L_h^2(dA_\alpha))^\perp \cap L^\infty(\mathbb{D})$ and Range$T_\phi^{(\alpha)}$ is closed, then $T_\phi^{(\alpha)}$ is a Fredholm operator of index zero and $T_\phi^{(\alpha)}$ is not of finite rank. Further we show that if $\phi \in L_{M_n}^\infty(\mathbb{D})$ such that $T_\phi \in \mathcal{F}\left(L_a^{2,\mathbb{C}^n}(dA_\alpha)\right)$ and $j(T_\phi) = 0$, then there exists $\psi \in E_{n\times n}$ such that $T_{\phi+\delta\psi}$ is invertible for all sufficiently small nonzero δ.

For $\psi \in L^\infty(\mathbb{D})$, define $B_{\overline{\psi}}^{(\alpha)} : L_h^2(dA_\alpha) \to (L_h^2(dA_\alpha))^\perp$ as $B_{\overline{\psi}}^{(\alpha)} f = (I - Q_\alpha)(\overline{\psi} f)$ where Q_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto $L_h^2(dA_\alpha)$. The operator $B_{\overline{\psi}}^{(\alpha)}$ is well defined and it is easy to see that

$$Q_\alpha(\overline{z}^n z^k) = \begin{cases} \frac{n-k+1}{n+1}\overline{z}^{n-k} & \text{if } k \le n; \\ \frac{k-n+1}{k+1} z^{k-n} & \text{if } k \ge n. \end{cases}$$

Further we can verify that $B_{\overline{z}^n}^{(\alpha)}(z^k) \perp B_{\overline{z}^n}^{(\alpha)}(z^j)$ if $j \ne k$ and if $\phi = \sum_{k=s_0}^{\infty} a_k z^k$, $a_{s_0} \ne 0$ (that is,$a_0 = ... = a_{s_0-1} = 0$) then $B_\phi^{(\alpha)}(\overline{z}^n) = \sum_{k=s_0}^{\infty} a_k \cdot$

$\cdot B^{(\alpha)}_{\overline{z}^n}(z^k)$. It is shown in [12] that if Ψ and Ω are two functions in $L^2_a(dA_\alpha)$ such that $\Psi(0) = 0 = \Omega(0)$ then the operators $B^{(\alpha)}_{\Psi}$ and $B^{(\alpha)}_{\overline{\Omega}}$ are not of finite rank in $L^2_h(dA_\alpha)$. In fact the set $\left\{B^{(\alpha)}_{\Psi}(\overline{z}^n)\right\}_{n=1}^{p}$ is linearly independent for all $p > 0$ and the set $\left\{B^{(\alpha)}_{\overline{\Omega}}(z^k)\right\}_{k=1}^{p}$ is linearly independent for all $p > 0$. If $g \in L^2_h(dA_\alpha)$ and $g = \Psi + \overline{\Omega}$, where Ψ and Ω are from $L^2_a(dA_\alpha)$ then $B^{(\alpha)}_{\overline{\Omega}} = B^{(\alpha)}_g|_{L^2_a(dA_\alpha)}$ is of finite rank if and only if $\Omega \equiv 0$ and similarly $B^{(\alpha)}_{\Psi} = B^{(\alpha)}_g|_{\overline{L^2_a(dA_\alpha)}}$ is of finite rank if and only if $\Psi \equiv 0$. Notice that for $f \in L^2_a(dA_\alpha), B^{(\alpha)}_{\overline{f}} L^2_h(dA_\alpha) = B^{(\alpha)}_{\overline{f}} L^2_a(dA_\alpha)$ and we shall also write $B^{(\alpha)}_{\overline{f}}|_{L^2_a(dA_\alpha)} = B^{(\alpha)}_{\overline{f}}$. Thus $B^{(\alpha)}_{\overline{f}} : L^2_a(dA_\alpha) \to (L^2_h(dA_\alpha))^\perp$ is defined as $B^{(\alpha)}_{\overline{f}} h = (I - Q_\alpha)(\overline{f}h)$ for all $h \in L^2_a(dA_\alpha)$ and $ker B^{(\alpha)}_{\overline{f}} = \{h \in L^2_a(dA_\alpha) : \overline{f}h \in L^2_h(dA_\alpha)\}$.

From [2], it follows that if f is not constant, $ker B^{(\alpha)}_{\overline{f}} = \mathrm{sp}\{1\}$. If $f \equiv$ constant then $ker B^{(\alpha)}_{\overline{f}} = L^2_a(dA_\alpha)$, hence $B^{(\alpha)}_{\overline{f}} \equiv 0$. Now $\left(B^{(\alpha)}_{\overline{f}}\right)^*$ maps $\left(L^2_h(dA_\alpha)\right)^\perp$ into $L^2_a(dA_\alpha)$ and $\left(B^{(\alpha)}_{\overline{f}}\right)^* k = P_\alpha(fk)$ for all $k \in \left(L^2_h(dA_\alpha)\right)^\perp$. It therefore follows that for $f \in L^2_a(dA_\alpha)$,

$$\overline{\mathrm{Range}\left(B^{(\alpha)}_{\overline{f}}\right)^*} = \begin{cases} \{0\} & \text{if } f \equiv \text{constant;} \\ (\mathrm{sp}\{1\})^\perp & \text{if } f \neq \text{constant.} \end{cases}$$

Theorem 3.1. *If $\phi \in (L^2_h(dA_\alpha))^\perp \cap L^\infty(\mathbb{D})$ and $Range T^{(\alpha)}_\phi$ is closed, then $T^{(\alpha)}_\phi$ is a Fredholm operator of index zero and $T^{(\alpha)}_\phi$ is not of finite rank.*

Proof. Given that $T^{(\alpha)}_\phi 1 = \left(T^{(\alpha)}_\phi\right)^* 1 = 0$, hence $\phi \in \left(L^2_h(dA_\alpha)\right)^\perp$. If $\psi \in L^\infty(\mathbb{D})$ then it is not difficult to verify that $S^{(\alpha)}_{\overline{\psi}} \equiv 0$ if and only if $\overline{\psi} \in \left(\overline{L^2_a(dA_\alpha)}\right)^\perp$. Thus $S^{(\alpha)}_{\overline{\phi}} \equiv 0$ and $S^{(\alpha)}_{\overline{\phi}} f = 0$ for all $f \in L^2_a(dA_\alpha)$. Hence $(\overline{\phi} f) \in \left(\overline{L^2_a(dA_\alpha)}\right)^\perp$ for all $f \in L^2_a(dA_\alpha)$ and

$$\begin{aligned} ker\left(T^{(\alpha)}_\phi\right)^* &= \left\{f \in L^2_a(dA_\alpha) : (\overline{\phi} f) \in \left(L^2_a(dA_\alpha)\right)^\perp\right\} \\ &= \left\{f \in L^2_a(dA_\alpha) : (\overline{\phi} f) \in \left(L^2_h(dA_\alpha)\right)^\perp\right\}. \end{aligned}$$

Thus

$$\begin{aligned}
\ker\left(T_{\phi}^{(\alpha)}\right)^{*} &= \left\{f \in L_a^2(dA_\alpha) : (\overline{\phi} f) \in \left(L_h^2(dA_\alpha)\right)^{\perp}\right\} \\
&= \left\{f \in L_a^2(dA_\alpha) : \langle \overline{\phi} f, g\rangle = 0 \text{ for all } g \in L_h^2(dA_\alpha)\right\} \\
&= \left\{f \in L_a^2(dA_\alpha) : \langle \overline{\phi}, \overline{f} g\rangle = 0 \text{ for all } g \in L_h^2(dA_\alpha)\right\} \\
&= \left\{f \in L_a^2(dA_\alpha) : \langle \overline{\phi}, (I - Q_\alpha)(\overline{f} g)\rangle = 0 \text{ for all } g \in L_h^2(dA_\alpha)\right\} \\
&= \left\{f \in L_a^2(dA_\alpha) : \left\langle \overline{\phi}, B_{\overline{f}}^{(\alpha)} g\right\rangle = 0 \text{ for all } g \in L_h^2(dA_\alpha)\right\}.
\end{aligned}$$

Case-1: Let $f \in L_a^2(dA_\alpha)$ be a polynomial of degree $k \geq 1$, and $H(z) = z^{k+1} \in L_a^2(dA_\alpha)$. Then $\overline{f} H \in (E_p^\alpha)^\perp$ for all $p \geq 0$ and $H_{\overline{f}H}^{E_p^\alpha} \equiv 0$. Since $\overline{f} z^k \notin (E_p^\alpha)^\perp$, and $ker H_{\overline{f}}^{E_p^\alpha}$ is an invariant subspace of z, hence $ker H_{\overline{f}}^{E_p^\alpha} = z^{k+1} L_a^2(dA_\alpha)$. Therefore $H_{\overline{f}H}^{*E_p^\alpha} \equiv 0$ and $ker H_{\overline{f}H}^{*E_p^\alpha} = \overline{z}^{k+1} E_p^\alpha$ for all $p \geq 0$. Now

$$\begin{aligned}
ker\left(B_{\overline{f}}^{(\alpha)}\right)^{*} &= \left\{g \in (L_h^2(dA_\alpha))^\perp : P_\alpha(fg) = 0\right\} \\
&= \left\{g \in (L_h^2(dA_\alpha))^\perp : fg \in (L_a^2(dA_\alpha))^\perp\right\}
\end{aligned}$$

and $ker H_{\overline{f}}^{*E_p^\alpha} = \left\{g \in E_p^\alpha : fg \in (L_a^2(dA_\alpha))^\perp\right\}$. Hence $ker\left(B_{\overline{f}}^{(\alpha)}\right)^{*} \bigcap E_p^\alpha = ker H_{\overline{f}}^{*E_p^\alpha} \bigcap (L_h^2(dA_\alpha))^\perp$ and therefore

$$\begin{aligned}
ker\left(B_{\overline{f}}^{(\alpha)}\right)^{*} &= \bigcup_{p\geq 0}\left(\ker\left(B_{\overline{f}}^{(\alpha)}\right)^{*} \bigcap E_p^\alpha\right) \\
&= \bigcup_{p\geq 0}\left(\ker H_{\overline{f}}^{*E_p^\alpha} \bigcap (L_h^2(dA_\alpha))^\perp\right) \\
&= \left(\bigcup_{p\geq 0} \ker H_{\overline{f}}^{*E_p^\alpha}\right) \bigcap (L_h^2(dA_\alpha))^\perp \\
&= \left(\bigcup_{p\geq 0} \overline{z}^{k+1} E_p^\alpha\right) \bigcap (L_h^2(dA_\alpha))^\perp = \{0\}.
\end{aligned}$$

Thus if $\overline{\phi} \in (L_h^2(dA_\alpha))^\perp$ and $\overline{\phi} \neq 0$ then $\left(B_{\overline{f}}^{(\alpha)}\right)^{*} \overline{\phi} \neq 0$.

Case-2: If $f \in L^2_a(dA_\alpha)$ is a constant, then $B^{(\alpha)}_{\overline{f}} \equiv 0$ and hence $\left(B^{(\alpha)}_{\overline{f}}\right)^* \equiv 0$ and therefore $\left(B^{(\alpha)}_{\overline{f}}\right)^* \overline{\phi} = 0$ if $\overline{\phi} \neq 0$.

Case-3: If $f \in L^2_a(dA_\alpha)$ is not a polynomial then $\overline{f}g \notin (E^\alpha_p)^\perp$ for any $g \in L^2_a(dA_\alpha)$, $g \neq 0$, $p \geq 0$. Hence $kerH^{E^\alpha_p}_{\overline{f}} = \{0\}$ and therefore by Lemma 2.2, we obtain $kerH^{*E^\alpha_p}_{\overline{f}} = \{0\}$ for all $p \geq 0$. This implies $ker\left(B^{(\alpha)}_{\overline{f}}\right)^* = \bigcup_{p\geq 0}\left(\mathrm{ker}H^{*E^\alpha_p}_{\overline{f}} \bigcap (L^2_h(dA_\alpha))^\perp\right) = \{0\}$ and $\left(B^{(\alpha)}_{\overline{f}}\right)^* \overline{\phi} \neq 0$ if $\overline{\phi} \in (L^2_h(dA_\alpha))^\perp$ and $\overline{\phi} \neq 0$.

Thus from the above three cases, it follows that

$$\begin{aligned}
\ker\left(T^{(\alpha)}_{\phi}\right)^* &= \left\{f \in L^2_a(dA_\alpha) : \left\langle \overline{\phi}, B^{(\alpha)}_{\overline{f}} g\right\rangle = 0 \text{ for all } g \in L^2_h(dA_\alpha)\right\}. \\
&= \left\{f \in L^2_a(dA_\alpha) : \left(B^{(\alpha)}_{\overline{f}}\right)^* \overline{\phi} = 0\right\}. \\
&= \begin{cases} L^2_a(dA_\alpha) & \text{if } \phi \equiv 0; \\ \mathrm{sp}\{1\} & \text{if } \phi \neq 0. \end{cases}
\end{aligned}$$

Hence either $\phi \equiv 0$ or $T^{(\alpha)}_{\phi}$ is not of finite rank.

□

Notice that if $\phi \in (L^2_h(dA_\alpha))^\perp_0$ then $P_\alpha \phi \equiv (P_\alpha\phi)(0) \equiv b$, a constant.

Lemma 3.2. *Let $\mathcal{L}$ be the vector space consisting of all square matrices of order k that are singular. Then there is a sequence of elementary operations (row and column operations) which when applied to the matrices of $\mathcal{L}$ results in a set of matrices all of which have entry zero in the same position.*

Proof. We shall use mathematical induction to establish the Lemma. Assume the Lemma is valid for matrices of order $< k$. Notice that if $A \in \mathcal{L}$ and $A \neq 0$ then A can be written in the form $A = \begin{pmatrix} I_{r\times r} & 0_{s\times s} \\ 0_{r\times r} & 0_{s\times s}\end{pmatrix}$ by elementary row and column operations where $r + s = k, r > 0, s > 0$. Without loss of generality, we shall assume to begin with that $A \in \mathcal{L}$. Let $B \in \mathcal{L}$ and assume $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}$ where B_{11} and B_{21} are of order r and B_{12} and B_{22} are of orders s. Then $det(B + \lambda A) = 0$ for any $\lambda \in \mathbb{C}$. Now $det(B + \lambda A)$ is a polynomial in λ whose highest coefficient is $detB_{22}$. Hence $detB_{22} = 0$

for any $B \in \mathcal{L}$. This implies that the set of lower right hand blocks of the matrices of $\mathcal{L}$ is a vector space of singular matrices of order $< k$. Using our induction hypothesis, we obtain that there is a sequence of elementary operations which when applied to these blocks results in a set of matrices all of which have entry zero in the same position. The lemma follows. $\square$

Let X be a Banach space and X^* be its dual space. A subspace M of X^* is said to be total if for every $0 \neq x \in X$ there is an $f \in M$ such that $f(x) \neq 0$. Let E be a total subspace of $L^\infty(\mathbb{D})$ (thought as a set of linear functionals on $L^1(\mathbb{D}, dA)$). Let $E_{n\times n} = E \otimes M_n$ and $\mathcal{A} = \{T_\phi : \phi \in E_{n\times n}\}$.

Theorem 3.3. *Let $\phi \in L^\infty_{M_n}(\mathbb{D})$ such that $T_\phi \in \mathcal{F}\left(L_a^{2,\mathbb{C}^n}(dA_\alpha)\right)$ and $j(T_\phi) = 0$. Then there exists $\psi \in E_{n\times n}$ such that $T_{\phi+\delta\psi}$ is invertible for all sufficiently small nonzero δ.*

Proof. First assume that there exists no $\psi \in E_{n\times n}$ such that $T_{\phi+\delta\psi}$ is invertible for all sufficiently small nonzero δ. We shall show then that there exists $0 \neq f \in kerT_\phi$ such that $T_\psi f \in \left(kerT_\phi^*\right)^\perp$. Let $T_\phi \in \mathcal{F}\left(L_a^{2,\mathbb{C}^n}(dA_\alpha)\right)$ and $j(T_\phi) = 0$ where $\phi \in L^\infty_{M_n}(\mathbb{D})$. Let $\psi \in L^\infty_{M_n}(\mathbb{D})$. Then $T_{\phi+\delta\psi}$ also belongs to $\mathcal{F}\left(L_a^{2,\mathbb{C}^n}(dA_\alpha)\right)$ and $j(T_{\phi+\delta\psi}) = 0$ for sufficiently small δ. If $T_{\phi+\delta\psi}$ is not invertible in $\mathcal{L}\left(L_a^{2,\mathbb{C}^n}(dA_\alpha)\right)$ then $ker(T_{\phi+\delta\psi}) \neq \{0\}$. Hence for each sequence $(\delta_n)_{n=1}^\infty$ tending to 0, there exist vectors f_{δ_n} satisfying $\|f_{\delta_n}\| = 1$ and $T_{\phi+\delta_n\psi} f_{\delta_n} = 0$ for all $n \in \mathbb{N}$. Let $f_{\delta_n} = g_{\delta_n} + h_{\delta_n}, g_{\delta_n} \in kerT_\phi$ and $h_{\delta_n} \in (kerT_\phi)^\perp$. Then $T_{\phi+\delta_n\psi} f_{\delta_n} = 0$ implies $T_\phi h_{\delta_n} = -\delta_n T_\psi f_{\delta_n}$. Therefore $T_\phi h_{\delta_n} \to 0$ and by the invertibility of T_ϕ as an operator from $(kerT_\phi)^\perp$ to $RangeT_\phi$ we obtain $h_{\delta_n} \to 0$. Since $kerT_\phi$ is finite dimensional, there is a subsequence of g_{δ_n} which converges to a vector f. It is not difficult to see that $f \in kerT_\phi$ and $\|f\| = 1$. Moreover, since each $T_\psi f_{\delta_n} = -\delta_n^{-1} T_\phi f_{\delta_n} \in RangeT_\phi$ and $RangeT_\phi$ is closed, we have $T_\psi f \in RangeT_\phi = \left(kerT_\phi^*\right)^\perp$. Now assume that T_ϕ is Fredholm and $j(T_\phi) = 0$. Assume $dim\ kerT_\phi = dim\ kerT_\phi^* = n < \infty$. Let $\{f_1, f_2, ..., f_n\}$ be a basis for $kerT_\phi$ and $\{g_1, g_2, ..., g_n\}$ be a basis for $kerT_\phi^*$. Suppose the conclusion of the theorem is false. From the first part, it follows that for each $T_\psi \in \mathcal{A}$ there is a nonzero vector $f = \sum \alpha_i f_i \in kerT_\phi$ such that $\langle T_\psi f, g_j\rangle = 0,\ j = 1, ..., n$. Hence for all $T_\psi \in \mathcal{A}$ the $n\times n$ matrix $(\langle T_\psi f_i, g_j\rangle)$ is singular. Since $\mathcal{A}$ is vector space, the set of all such matrices is also linear and Lemma 3.2 can be applied in this setting. But any set of elementary row and column operations applied to these matrices has the effect of simply replacing the independent sets $\{f_i\}$ and $\{g_j\}$ by other independent sets $\{f_i'\}$ and $\{g_j'\}$. Thus from Lemma 3.2 it follows that for some such pair of independent sets and for some fixed i and j, $\left\langle T_\psi f_i', g_j' \right\rangle = 0$ for all $T_\psi \in \mathcal{A}$. But this is not possible for the following reasons. Notice that $\mathcal{A} = \{T_\psi : \psi \in E_{n\times n}\}$. Let $0 \neq f = (f_1, f_2, ..., f_n) \in L_a^{2,\mathbb{C}^n}(dA_\alpha)$ and $0 \neq g = (g_1, g_2, ..., g_n) \in L_a^{2,\mathbb{C}^n}(dA_\alpha)$. Then there is a $\psi = (\psi_{ij}) \in E_{n\times n}$ such that $\langle T_\psi f, g\rangle \neq 0$. Since $P_\alpha^{\mathbb{C}^n} g = g$, we obtain $\langle \psi f, g\rangle \neq 0$ or $\sum_{i,j=1}^{n} \int \psi_{ij} f_j \overline{g_i} \neq 0$. There are indices i_0, j_0 such that $f_{j_0} \neq 0$, $g_{i_0} \neq 0$ as f and g are nonzero vectors in $L_a^{2,\mathbb{C}^n}(dA_\alpha)$. So [3] the product $f_{j_0}\overline{g_{i_0}}$ is not almost everywhere equal to zero. Hence we can find $\psi_{i_0 j_0}$ belonging to E such that $\int \psi_{i_0 j_0} f_{j_0} \overline{g_{i_0}} \neq 0$. Set $\psi_{ij} = 0$ if $i \neq i_0$ or $j \neq j_0$. This defines a ψ such that $\langle T_\psi f, g\rangle \neq 0$. Thus we obtain there exists $\psi \in E_{n\times n}$ such that $T_{\phi+\delta\psi}$ is invertible for all sufficiently small nonzero δ. □

Corollary 3.1. *Let $\phi \in C_{M_n}(\overline{\mathbb{D}}) = C(\overline{\mathbb{D}}) \otimes M_n$. The space of invertible Toeplitz operators in $\mathcal{T} = \{T_\phi : \phi \in C_{M_n}(\overline{\mathbb{D}})\}$ is dense in the space of Fredholm Toeplitz operators of index zero in $\mathcal{T}$.*

Proof. The proof follows from the Theorem 3.3 as $C(\overline{\mathbb{D}})$ is a total subspace of $L^\infty(\mathbb{D})$. □

Corollary 3.2. *Let $\phi \in (L_h^2(dA_\alpha))^\perp \cap L^\infty(\mathbb{D})$ and suppose Range$T_\phi^{(\alpha)}$ is closed. Then either $T_\phi^{(\alpha)}$ is invertible or $T_\phi^{(\alpha)} = W^{(\alpha)} + F^{(\alpha)}$ where $W^{(\alpha)} \in \mathcal{L}(L_a^2(dA_\alpha))$ is invertible and is not a Toeplitz operator and $F^{(\alpha)}$ is a nonzero finite rank operator in $\mathcal{L}(L_a^2(dA_\alpha))$.*

Proof. From Theorem 3.1, it follows that $T_\phi^{(\alpha)}$ is Fredholm and of index zero. Now there are two possibilities. Either $T_\phi^{(\alpha)}$ is invertible or $T_\phi^{(\alpha)}$ is not invertible. Suppose $T_\phi^{(\alpha)}$ is not invertible. It follows from [11] that $T_\phi^{(\alpha)} = W^{(\alpha)} + F^{(\alpha)}$ where $W^{(\alpha)}$ is invertible and $F^{(\alpha)}$ is a finite rank operator in $\mathcal{L}(L_a^2(dA_\alpha))$. If $F^{(\alpha)} \equiv 0$ then $T_\phi^{(\alpha)} = W^{(\alpha)}$ is invertible. If $F^{(\alpha)} \not\equiv 0$ and $W^{(\alpha)} = T_\psi^{(\alpha)}$ for some $\psi \in L^\infty(\mathbb{D})$. Then $F^{(\alpha)} = T_\phi^{(\alpha)} - T_\psi^{(\alpha)} = T_{\phi-\psi}^{(\alpha)}$ is a finite rank Toeplitz operator in $\mathcal{L}(L_a^2(dA_\alpha))$. From [9] it follows that $F^{(\alpha)} \equiv 0$. But $F^{(\alpha)} \not\equiv 0$. Thus $T_\phi^{(\alpha)} = W^{(\alpha)} + F^{(\alpha)}$ where $F^{(\alpha)} \in \mathcal{L}(L_a^2(dA_\alpha))$ is a nonzero finite rank operator and $W^{(\alpha)} \in \mathcal{L}(L_a^2(dA_\alpha))$ is invertible and is not a Toeplitz operator. The corollary follows. □

Corollary 3.3. *If for $\phi \in L^\infty(\mathbb{D})$, $T_\phi^{(\alpha)}$ is invertible and $\psi \in C_0(\mathbb{D})$ (continuous functions on $\mathbb{D}$ vanishing on the boundary) is such that $T_{\phi-\psi}^{(\alpha)}$ is one-one, then $T_{\phi-\psi}^{(\alpha)}$ is invertible.*

Proof. Notice that for $\psi \in C_0(\mathbb{D})$ the Toeplitz operator $T_\psi^{(\alpha)} \in \mathcal{L}(L_a^2(dA_\alpha))$ is compact [13]. Given that $T_\phi^{(\alpha)}$ is invertible. The result follows from [10]. □

Corollary 3.4. *Let $\phi \in (L_h^2(dA_\alpha))^\perp \cap L^\infty(\mathbb{D})$. Suppose*

$$\gamma(T_\phi^{(\alpha)}) = inf\left\{\|T_\phi^{(\alpha)} f\| : f \in L_a^2(dA_\alpha), \|f\| = 1, f \in \left(kerT_\phi^{(\alpha)}\right)^\perp\right\} > 0$$

and $W^{(\alpha)} \in \mathcal{L}\left(L_a^2(dA_\alpha)\right)$ is such that $\|T_\phi^{(\alpha)} - W^{(\alpha)}\| < \gamma(T_\phi^{(\alpha)})$. If $RangeW^{(\alpha)}$ is closed then $W^{(\alpha)}$ is a Fredholm operator of index zero.

Proof. Since $\gamma(T_\phi^{(\alpha)}) > 0$, hence it follows from [4] that Range$T_\phi^{(\alpha)}$ is closed. From Theorem 3.1, it follows that $T_\phi^{(\alpha)}$ is Fredholm of index zero. Now since $\|T_\phi^{(\alpha)} - W^{(\alpha)}\| < \gamma(T_\phi^{(\alpha)})$, it again follows from [4], that $dim\ kerW^{(\alpha)} = dim\ ker\left(W^{(\alpha)}\right)^* = 0$. As Range$W^{(\alpha)}$ is closed, we obtain that $W^{(\alpha)}$ is a Fredholm operator of index zero. □

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