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POSITIVE DEFINITE SOLUTIONS OF A LINEARLY PERTURBED MATRIX EQUATION*

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Abstract

In this paper we study a special case of linearly perturbed discrete-time algebraic Riccati equation. We give some sufficient conditions for the existence of a positive definite solution of the considered equation. We propose a basic fixed point iteration and its inversion free variant for finding a positive definite solution. Moreover, by specially choosing the initial value in the basic fixed point iteration we prove that it converges to the largest solution. The theoretical results are illustrated by numerical examples.

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1 Introduction

Consider the nonlinear matrix equation

$$X - A^*XA + B^*X^{-1}B = I\,, \tag{1}$$

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where A, B are $n \times n$ complex matrices, I is the identity matrix, and A^* denotes the conjugate transpose of A. Eq. (1) is a special case of the general matrix equation

$$\begin{aligned} Y - C^*YC - \Pi_1(Y) + [L + C^*YF + \Pi_{12}(Y)] \\ \times [R + F^*YF + \Pi_2(Y)]^+ [L + C^*YF + \Pi_{12}(Y)]^* = S\,, \end{aligned} \tag{2}$$

where Z^+ is the Moore-Penrose inverse of a matrix Z. Eq. (2) is known as linearly perturbed discrete-time algebraic Riccati equation (LPDARE) and have been investigated by many authors [1, 2] and the references therein. Eq. (1) is a special case of the matrix equation

$$\begin{aligned} C^*Y + YC + S + \Pi_1(Y) - [L + YF + \Pi_{12}(Y)] \\ \times [R + \Pi_2(Y)]^+ [L + YF + \Pi_{12}(Y)]^* = 0\,, \end{aligned} \tag{3}$$

also. The last equation appears in the stochastic control and have been studied by many authors [3, 4, 5, 6]. Moreover, Eq. (1) is a combination of the well-known equations $X - A^*XA = I$ [7, 8] and $X + B^*X^{-1}B = I$ [9, 10, 11, 12, 13].

Now, we show the relationship between the equations (1) and (2) in two particular cases. Firstly, in case of $C = R = \Pi_2(Y) = \Pi_{12}(Y) = 0$, $F = I$, $\Pi_1(Y) = A_1^*YA_1$ (or $F = R = \Pi_1(Y) = \Pi_{12}(Y) = 0$, $\Pi_2(Y) = Y$), and S is a positive definite matrix, Eq. (2) is in type of Eq. (1). Secondly, consider Eq. (2) for positive semidefinite solution Y in case of $\Pi_1(Y) = A_1^*YA_1$, $L = \Pi_2(Y) = \Pi_{12}(Y) = 0$, the matrix F is nonsingular and R is a positive definite matrix, i.e, we reduce Eq. (2) to

$$Y - C^*YC - A_1^*YA_1 + C^*YF(R + F^*YF)^{-1}(C^*YF)^* = S. \tag{4}$$

Let $P = F^{-*}RF^{-1}$ and $Z = Y + P$, then from Eq. (4) it follows

$$Z - P - C^*(Z - P)C - A_1^*(Z - P)A_1 + C^*(Z - P)Z^{-1}(Z - P)C = S,$$

and

$$Z - A_1^*ZA_1 + C^*PZ^{-1}PC = S + P + C^*PC - A_1^*PA_1.$$

Now, let $Q = S + P + C^*PC - A_1^*PA_1$ be a positive definite matrix, then by multiplying both hand side of the above equation with the matrix $Q^{-\frac{1}{2}}$ we obtain Eq (1) with

$$X = Q^{-\frac{1}{2}}ZQ^{-\frac{1}{2}}, \quad A = Q^{\frac{1}{2}}A_1Q^{-\frac{1}{2}}, \quad \text{and} \quad B = Q^{-\frac{1}{2}}PCQ^{-\frac{1}{2}}.$$

Hence, Eq. (4) can be reduced to Eq. (1).

In [14] have been studied a similar equation $X - A^*XA - B^*X^{-1}B = I$. In addition, there are some contributions in the literature to the solvability and numerical solutions of the matrix equation $X + A^*X^{-1}A - B^*X^{-1}B = I$ [15, 16, 17]. Konstantinov et al. [18] have investigated for the sensitivity of the equation $A_0 + \sum_{i=1}^{k} \sigma_i A_i^* X^{p_i} A_i = 0$, which is another more general type of Eq. (1).

Motivated by the investigations in [1, 2, 14, 15], we study Eq. (1) for the existence of a positive definite solution, bounds of the solutions and iterative methods for obtaining a solution. In addition, we consider some numerical examples to illustrate the theoretical results.

Throughout this paper, $\mathcal{C}^{n\times n}$ denotes the set of $n \times n$ complex matrices, and $\mathcal{H}^n$ the set of $n\times n$ Hermitian matrices. $A > 0$ ($A \geq 0$) means that A is a Hermitian positive definite (semidefinite) matrix. If $A-B > 0$ (or $A-B \geq 0$) we write $A > B$ (or $A \geq B$). For $N \geq M > 0$ we use $[M, \infty)$, (M, ∞), and $[M, N]$ to denote the sets of matrices $\{X : X \geq M\}$, $\{X : X > M\}$, and $\{X : M \leq X \leq N\}$, respectively. We use $\lambda_1(C)$, $\lambda_n(C)$, $\sigma_1(A)$, $\sigma_n(A)$, $\rho(A)$, and $\|A\|$ to denote the largest and the smallest eigenvalues of an $n \times n$ Hermitian matrix C, the largest and the smallest singular values ($\sigma_i(A) = \sqrt{\lambda_i(A^*A)}$), the spectral radius ($\rho(A) = \max|\lambda_i(A)|$), and the spectral norm ($\|A\| = \sigma_1(A)$) of a $n \times n$ matrix A, respectively. The Hermitian solutions X_S and X_L of a matrix equation are called the *smallest* solution and the *largest* solution, respectively if $X_S \leq X \leq X_L$ for any Hermitian solution X of the equation.

2 Preliminaries

Firstly, we will present some results for the Stain's equation

$$X - A^*XA = Q, \tag{5}$$

where Q is a positive definite matrix.

Lemma 1. *[8] Let A, Q be square matrices.*

(a) If $\rho(A) < 1$, then Eq. (5) has a unique solution P_Q, and $P_Q \geq 0$ ($P_Q > 0$), when $Q \geq 0$ ($Q > 0$).

(b) If there is some $P > 0$ such that $P - A^*PA$ is positive definite (semidefinite), then $\rho(A) < 1$ ($\rho(A) \leq 1$).

Remark 1. *[17] From Lemma 1 it follows that, if $\rho(A) < 1$, $Q_1 \leq Q_2$ ($Q_1 < Q_2$), and P_i, $i = 1, 2$, are the unique solutions of the equations $P - A^*PA = Q_i$, $i = 1, 2$, respectively, then $P_1 \leq P_2$ ($P_1 < P_2$).*

Remark 2. *In case of* $\rho(A) < 1$ *the unique solution* P_Q *of Eq. (5) has representation*

$$P_Q = \sum_{k=0}^{\infty} (A^*)^k Q A^k.$$

Now, we will present some results for the equation

$$X + B^* X^{-1} B = Q, \tag{6}$$

where Q is a positive definite matrix.

In [10], the solvability of Eq. (6) has been studied in terms of properties of the corresponding rational matrix-valued function $\psi(\lambda) = Q + \lambda B + \lambda^{-1} B^*$. The function ψ is called *regular* if $\det(\psi(\lambda))$ is not identically zero, i.e., if there exists at least one value $\lambda \in \mathbb{C}$ where $\det(\psi(\lambda)) \neq 0$. Engwerda et al. [10, Theorem 2.1] proved that Eq. (6) has a positive definite solution if and only if ψ is regular and $\psi(\lambda) \geq 0$ for all λ on the unit circle. In particular, Eq. (6) has a positive definite solution if $\psi(\lambda) > 0$ for all λ on the unit circle [12]. Moreover:

Lemma 2. *[10, Theorem 3.4] Suppose* $Q > 0$ *and assume Eq. (6) has a positive definite solution. Then this equation has a largest and a smallest solution* M *and* N*, respectively. Moreover* M *is the unique solution for which* $X + \lambda B$ *is invertible for* $|\lambda| < 1$*, while* N *is the unique solution for which* $X + \lambda B^*$ *is invertible for* $|\lambda| > 1$*.*

We have also $\rho(M^{-1}B) \leq 1$ [10, Theorem 2.2].

Let us denote by $\omega(A)$ the numerical radius of a matrix A, i.e.,

$$\omega(A) = \max_{\|x\|=1} |x^* A x|.$$

Lemma 3. *[10, Theorem 5.2] Suppose* B *is nonsingular. Then Eq. (6) has a positive definite solution* X *if and only if* $\omega(Q^{-\frac{1}{2}} B Q^{-\frac{1}{2}}) \leq \frac{1}{2}$*.*

The proof of [10, Theorem 5.2] also contains the following result.

Lemma 4. *[12, Lemma 6.3]* $\psi(\lambda) > 0$ *for all* λ *on the unit circle if and only if* $\omega(Q^{-\frac{1}{2}} B Q^{-\frac{1}{2}}) < \frac{1}{2}$*.*

It is well-known [9, 10] that the largest positive definite solution M of Eq. (6) can be found by the iterative method

$$X_{k+1} = Q - B^* X_k^{-1} B, \quad X_0 = Q.$$

Moreover, we have $M \leq X_{k+1} \leq X_k, \quad k = 0, 1, \ldots$.

Meini [13] has proposed a more effective algorithm (Cyclic reduction) for computing the largest solution M of Eq. (6).

Lemma 5. *[17] If there is some $P > 0$ such that $P + B^*P^{-1}B \leq Q$, then Eq. (6) has a positive definite solution, as well as the largest positive definite solution $X_L \geq P$.*

3 Conditions for the existence of a positive definite solution

We consider the equations (5) and (6) with right-hand side $Q = I$:

$$X - A^*XA = I, \tag{7}$$

$$X + B^*X^{-1}B = I. \tag{8}$$

Note that (see Lemma 1 and Remark 1) Eq. (7) has a unique positive definite solution P_I if and only if $\rho(A) < 1$.

Theorem 1. *Let P_I be a unique positive definite solution of Eq. (7) and Eq. (1) has a positive definite solution X_+. Then $X_+ \leq P_I$.*

Proof. Let P_I be a unique positive definite solution of Eq. (7) and let X_+ be a positive definite solution of Eq. (1), i.e.,

$$\begin{array}{rcl} P_I - A^*P_IA &=& I \\ X_+ - A^*X_+A &=& I - B^*X_+^{-1}B \end{array}$$

By subtraction of the above equations, we have

$$P_I - X_+ - A^*(P_I - X_+)A = B^*X_+^{-1}B. \tag{9}$$

Since $B^*X_+^{-1}B \geq 0$ and $\rho(A) < 1$, by Lemma 1 *(i)* and (9), we have $P_I - X+ \geq 0$. □

Theorem 2. *Suppose Eq. (8) has a positive definite solution and let M be the largest solution. Then Eq. (1) has a positive definite solution $X_+ \geq M$ if and only if $\rho(A) < 1$. Let P_I be a unique positive definite solution of Eq. (7), then $X_+ \in [M, P_I]$. Moreover,*

(i) if $M < X_+$, then $\rho(X_+^{-1}B) \leq 1$,

(ii) if $M < X_+$ and B is nonsingular, then $\rho(X_+^{-1}B) < 1$,

(iii) if A is nonsingular, then $M < X_+$ and $\rho(X_+^{-1}B) < 1$.

Proof. Let Eq. (8) have positive definite solutions and let M be the largest solution.

Suppose Eq. (1) has a positive definite solution $X_+ \geq M$. Then

$$X_+ - A^*X_+A = I - B^*X_+^{-1}B \geq I - B^*M^{-1}B = M > 0$$

and by Lemma 1 *(b)*, it follows $\rho(A) < 1$. Thus, Eq. (7) has a unique positive definite solution P_I and by Theorem 1 it follows $X_+ \leq P_I$.

Hence, $X_+ \in [M, P_I]$.

Now, suppose $\rho(A) < 1$, i.e. Eq. (7) has a unique positive definite solution P_I.

By subtraction of the equations

$$P_I - A^*P_IA = I \quad \text{and} \quad M + B^*M^{-1}B = I,$$

we obtain

$$P_I - M = A^*P_IA + B^*M^{-1}B \geq 0.$$

Consider a map F, defined by

$$F(X) = I + A^*XA - B^*X^{-1}B, \quad X \in (0, \infty). \tag{10}$$

We will show that $F([M, P_I]) \subset [M, P_I]$. Let $X \in [M, P_I]$. Then

$$\begin{aligned} F(X) &= I + A^*XA - B^*X^{-1}B \\ &\leq I + A^*P_IA = P_I \end{aligned}$$

and

$$\begin{aligned} F(X) &\geq I + A^*MA - B^*M^{-1}B \\ &\geq I - B^*M^{-1}B = M. \end{aligned}$$

Therefore, for all $X \in [M, P_I]$, $F(X) \in [M, P_I]$, i.e. $F([M, P_I]) \subset [M, P_I]$. Since $[M, P_I]$ is a convex, closed and bounded set and the map F is continuous on $[M, P_I]$, by Brouwer's fixed point theorem [19, p.17] it follows that there exists a solution $X_+ \in [M, P_I]$ of Eq. (1).

Now, by substraction of the equations $M + B^*M^{-1}B = I$ and

$$X_+ - A^*X_+A + B^*X_+^{-1}B = I,$$

we have

$$X_+ - M - B^*X_+^{-1}(X_+ - M)X_+^{-1}B = R, \tag{11}$$

where $R = A^*X_+A + B^*X_+^{-1}(X_+ - M)M^{-1}(X_+ - M)X_+^{-1}B$.

Note that $R \geq 0$. In addition, $R > 0$, if A is nonsingular or if $X_+ > M$ and B is nonsingular. Thus, from (11) and Lemma 1 we obtain *(i)*, *(ii)* and *(iii)*. □

Theorem 3. *If there are numbers $\beta \geq \alpha > 0$ satisfying the inequalities*

$$\beta^2 A^*A + \beta(1-\beta)I \leq B^*B \leq \alpha^2 A^*A + \alpha(1-\alpha)I, \tag{12}$$

then Eq. (1) has a positive definite solution X_* in $[\alpha I, \beta I]$.

Proof. The proof is similar to proof of Theorem 2. Consider the map F defined in (10). Let $X \in [\alpha I, \beta I]$, then by inequalities in (12) we have

$$\begin{aligned} F(X) &= I + A^*XA - B^*X^{-1}B \\ &\leq I + \beta A^*A - \beta^{-1}B^*B \leq \beta I, \\ F(X) &\geq I + \alpha A^*A - \alpha^{-1}B^*B \geq \alpha I. \end{aligned}$$

Therefore, for all $X \in [\alpha I, \beta I]$, $F(X) \in [\alpha I, \beta I]$, i.e. $F([\alpha I, \beta I]) \subset [\alpha I, \beta I]$. By Brouwer's fixed point theorem it follows that there exists a solution of Eq. (1) in $[\alpha I, \beta I]$. □

Corollary 1. *If $\sigma_1(A) < 1$ and $\sigma_1^2(B)(1-\sigma_n^2(A)) < 1/4$ are satisfied, then there are numbers $\beta \geq \alpha > 0$ satisfying the inequalities (12).*

Proof. Let $\sigma_1(A) < 1$ and $\sigma_1^2(B)(1-\sigma_n^2(A)) < 1/4$, then we have also $\sigma_n(A) < 1$ and $\sigma_n^2(B)(1-\sigma_1^2(A)) < 1/4$. Thus, the equations

$$(1-\sigma_n^2(A))\alpha^2 - \alpha + \sigma_1^2(B) = 0,$$

$$(1-\sigma_1^2(A))\beta^2 - \beta + \sigma_n^2(B) = 0$$

have solutions $\alpha_i, \beta_i, i = 1, 2$, respectively, where

$$\begin{aligned} \alpha_1 &= \frac{1-\sqrt{1-4\sigma_1^2(B)(1-\sigma_n^2(A))}}{2(1-\sigma_n^2(A))}, \\ \alpha_2 &= \frac{1+\sqrt{1-4\sigma_1^2(B)(1-\sigma_n^2(A))}}{2(1-\sigma_n^2(A))}, \\ \beta_1 &= \frac{1-\sqrt{1-4\sigma_n^2(B)(1-\sigma_1^2(A))}}{2(1-\sigma_1^2(A))}, \\ \beta_2 &= \frac{1+\sqrt{1-4\sigma_n^2(B)(1-\sigma_1^2(A))}}{2(1-\sigma_1^2(A))}, \end{aligned}$$

and

$$0 \leq \beta_1 \leq \alpha_1 < \alpha_2 \leq \beta_2.$$

Thus,

$$\sigma_1^2(B) \le \alpha(1-\alpha) + \alpha^2\sigma_n^2(A) \iff \alpha \in [\alpha_1, \alpha_2], \tag{13}$$
$$\beta(1-\beta) + \beta^2\sigma_1^2(A) \le \sigma_n^2(B) \iff \beta \in (-\infty, \beta_1] \cup [\beta_2, \infty). \tag{14}$$

Therefore, for all $\alpha \in (\alpha_1, \alpha_2]$ and $\beta \in [\beta_2, \infty)$ the inequalities in (12) are satisfied. □

Corollary 2. *Let A, B be the matrix coefficients in Eq. (1). Then*

(a) the identity matrix I is a solution of Eq. (1) if and only if $B^*B = A^*A$;

(b) if $B^*B \le A^*A$ and $\rho(A) < 1$, then Eq. (1) has a solution $X' \in [I, P_I]$, where P_I is the unique solution of Eq. (7).

(c) if $A^*A \le B^*B \le \alpha^2 A^*A + \alpha(1-\alpha)I$ for some $\alpha \in [\frac{1}{2}, 1)$, then Eq. (1) has a solution $X'' \in [\alpha I, I]$.

Proof. The proof of *(a)* is pretty straightforward.

(b) Let $B^*B \le A^*A$ and $\rho(A) < 1$. Then from $\rho(A) < 1$ it follows that Eq. (7) has a unique positive definite solution $P_I \ge I$.

Consider the map F defined in (10). Let $X \in [I, P_I]$, then

$$F(X) \le I + A^*P_IA = P_I,$$
$$F(X) \ge I + A^*A - B^*B \ge I.$$

Therefore, for all $X \in [I, P_I]$, $F(X) \in [I, P_I]$. By Brouwer's fixed point theorem it follows that there exists a solution $X' \in [I, P_I]$ of Eq. (1).

The statement *(c)* is in case of $\beta = 1$ in Theorem 3. □

4 Iterative algorithms

We propose two iterative algorithms for obtaining a positive definite solution of Eq. (1).

Algorithm 1 (Basic fixed point iteration)**.** *For a matrix X_0, compute*

$$X_{i+1} = I + A^*X_iA - B^*X_i^{-1}B, \quad i = 0, 1, \ldots.$$

Theorem 4. *Let $\rho(A) < 1$, $\omega(B) \le \frac{1}{2}$, B be a nonsingular matrix, and let P_I and M be a unique positive definite solution of Eq. (7) and the largest solution of Eq. (8), respectively. Then Eq. (1) has the largest positive definite solution $X_L \in [M, P_I]$ and Algorithm 1 with $X_0 = P_I$ and $X_0' = M$ generates two matrix sequences $\{X_i\}$ and $\{X_i'\}$ with the following properties* *(a)* $X_i \geq X_{i+1} \geq X_L$, $i = 0, 1, \ldots$, *and* $\lim\limits_{i\to\infty} X_i = X_L$,

(b) $X'_i \leq X'_{i+1} \leq X_L$, $i = 0, 1, \ldots$, *and* $\lim\limits_{i\to\infty} X'_i = X' \leq X_L$, *where* X' *a solution of Eq. (1).*

Proof. Since $\rho(A) < 1$, by Lemma 1 Eq. (7) has a unique positive definite solution P_I. Since $\omega(B) \leq \frac{1}{2}$ and B is a nonsingular matrix, by Lemma 2 and Lemma 3, Eq. (8) has a largest positive definite solution M. Thus, by Theorem 2 it follows that Eq. (1) has a positive definite solution $X_+ \in [M, P_I]$.

Now, we consider Algorithm 1 with $X_0 = P_I$. Then

$$X_1 = I + A^* P_I A - B^* P_I^{-1} B \leq I + A^* P_I A = X_0$$

and

$$X_1 \geq I + A^* X_+ A - B^* X_+^{-1} B = X_+.$$

Assume that $X_+ \leq X_k \leq X_{k-1}$. Then

$$X_{k+1} = I + A^* X_k A - B^* X_k^{-1} B \leq I + A^* X_{k-1} A - B^* X_{k-1}^{-1} B = X_k$$

and

$$X_{k+1} \geq I + A^* X_+ A - B^* X_+^{-1} B = X_+.$$

Hence, by induction $X_i \geq X_{i+1} \geq X_+$ for $i = 0, 1, \ldots$. Thus, the sequence $\{X_i\}$ converge to a positive definite solution $X_L \geq X_+$. Therefore, X_L is the largest solution of Eq. (1). The statement *(a)* is proven.

The statement *(b)* can be proven by analogy. □

Remark 3. *Under conditions of the Theorem 4, if Eq. (1) has more than one solution in* $[M, P_I]$*, then* X' *is the smallest solution in* $[M, P_I]$*.*

Theorem 5. *If there are numbers* $\beta \geq \alpha > 0$ *satisfying the inequalities (12), then Algorithm 1 with* $X_0 = \beta I$ *and* $X'_0 = \alpha I$ *generates two matrix sequences* $\{X_i\}$ *and* $\{X'_i\}$ *for which* $X'_i \leq X'_{i+1} \leq X_{i+1} \leq X_i$, $i = 0, 1, \ldots$, *and* $\lim\limits_{i\to\infty} X_i = X_\beta$, $\lim_{i\to\infty} X'_i = X_\alpha \leq X_\beta$*, where* X_α *and* X_β *are solutions of Eq. (1).*

Proof. We have by Theorem 3 that Eq. (1) has a positive definite solution $X_* \in [\alpha I, \beta I]$. Now, we consider Algorithm 1 with $X_0 = \beta I$. Then, by the left inequality in (12), we have

$$X_1 = I + \beta A^* A - \frac{1}{\beta} B^* B \leq \beta I = X_0$$

and

$$X_1 \geq I + A^* X_* A - B^* X_*^{-1} B = X_*.$$

Assume that $X_* \leq X_k \leq X_{k-1}$. Then

$$X_{k+1} = I + A^* X_k A - B^* X_k^{-1} B \leq I + A^* X_{k-1} A - B^* X_{k-1}^{-1} B = X_k$$

and

$$X_{k+1} \geq I + A^* X_* A - B^* X_*^{-1} B = X_*.$$

Hence, by induction $X_i \geq X_{i+1} \geq X_*$ for $i = 0, 1, \ldots$. Thus, the sequence $\{X_i\}$ converges to a positive definite solution $X_\beta \geq X_*$.

Now, we consider Algorithm 1 with $X_0' = \alpha I$. Then, by the right inequality in (12), we have

$$X_1' = I + \alpha A^* A - \frac{1}{\alpha} B^* B \geq \alpha I = X_0'$$

and

$$X_1' \leq I + A^* X_* A - B^* X_*^{-1} B = X_*.$$

By induction we have $X_i' \leq X_{i+1}' \leq X_*$ for $i = 0, 1, \ldots$. Thus, the sequence $\{X_i'\}$ converges to a positive definite solution $X_\alpha \leq X_*$.

Therefore, $X_i' \leq X_{i+1}' \leq X_\alpha \leq X_* \leq X_\beta \leq X_{i+1} \leq X_i,\ \ i = 0, 1, \ldots$. □

Now, we motivated by the investigations of Zhan [11], and Guo and Lancaster [12] for Eq. (6), consider an inversion free variant of Algorithm 1.

Algorithm 2 (An inversion free variant of the basic fixed point iteration)**.** *For the matrices X_0 and $0 < Y_0 \leq X_0^{-1}$, compute*

$$\begin{cases} Y_{i+1} = Y_i(2I - X_i Y_i), \\ X_{i+1} = I + A^* X_i A - B^* Y_{i+1} B, \end{cases} \qquad i = 0, 1, \ldots.$$

Lemma 6. *[11, Lemma 3.2] Let C and P be Hermitian matrices of the same order and let $P > 0$. Then $CPC + P^{-1} \geq 2C$.*

Theorem 6. *Let P_I be a unique positive definite solution of Eq. (7) and Eq. (1) has a positive definite solution. Then the matrix sequence $\{X_i\}$ generated by Algorithm 2 with $X_0 = P_I$ and $Y_0 = I/\|X_0\|_\infty$ is monotone decreasing and converges to the maximal solution X_L.*

Proof. We prove the theorem by induction. Let X_+ be a positive definite solution of Eq. (1).

By Theorem 1 we have $X_0 = P_I \geq X_+$. Thus

$$Y_0 = \frac{1}{\|P_I\|_\infty} I \leq P_I^{-1} \leq X_+^{-1}.$$

We compute

$$\begin{aligned} Y_1 &= \frac{1}{\|P_I\|_\infty}(2\|P_I\|_\infty I - P_I)\frac{1}{\|P_I\|_\infty} \geq \frac{1}{\|P_I\|_\infty} I = Y_0, \\ X_1 &= I + A^* P_I A - B^* Y_1 B \leq I + A^* P_I A = P_I = X_0. \end{aligned}$$

We have by Lemma 6 that

$$Y_1 = 2Y_0 - Y_0 X_0 Y_0 \leq X_0^{-1} \leq X_+^{-1}.$$

Thus

$$X_1 = I + A^* P_I A - B^* Y_1 B \geq I + A^* X_+ A - B^* X_+^{-1} B = X_+.$$

Therefore, $Y_0 \leq Y_1 \leq X_+^{-1}$, $X_0 \geq X_1 \geq X_+$.

Assume that $Y_{k-1} \leq Y_k \leq X_+^{-1}$ and $X_{k-1} \geq X_k \geq X_+$. Once again, by Lemma 6 we have

$$Y_{k+1} = 2Y_k - Y_k X_k Y_k \leq X_k^{-1} \leq X_+^{-1}.$$

Hence,

$$X_{k+1} = I + A^* X_k A - B^* Y_{k+1} B \geq I + A^* X_+ A - B^* X_+^{-1} B = X_+.$$

Since $Y_k \leq X_{k-1}^{-1} \leq X_k^{-1}$, we have

$$Y_{k+1} - Y_k = Y_k(Y_k^{-1} - X_k)Y_k \geq 0,$$

and

$$X_{k+1} - X_k = -A^*(X_{k-1} - X_k)A - B^*(Y_{k+1} - Y_k)B \leq 0.$$

Therefore, $X_i \geq X_{i+1} \geq X_+$, $Y_i \leq Y_{i+1} \leq X_+^{-1}$, for $i = 1, 2, \ldots$, and the limits $\lim_{i\to\infty} X_i$, $\lim_{i\to\infty} Y_i$ exist. Let $\lim_{i\to\infty} X_i = X$ and $\lim_{i\to\infty} Y_i = Y$. Then $X \geq X_+$ for every positive definite solution X_+ of Eq. (1). Taking limits in Algorithm 2 yields

$$\begin{aligned} Y &= YXY, \\ X &= I + A^* X A - B^* Y B. \end{aligned}$$

Thus, $Y = X^{-1}$ and $X = I + A^* X A - B^* X^{-1} B$.

Hence, $X = X_L$ the largest positive definite solution of Eq. (1). $\square$

5 Numerical examples

In this section we carry out numerical experiments for computing the positive definite solutions of Eq. (1) by Algorithms 1 and 2. We use notations $\{X_i\}$ and $\{X'_i\}$ for sequences generated by Algorithm 1 with $X_0 = P_I$ (or $X_0 = \beta I$) and $X'_0 = M$ (or $X'_0 = \alpha I$), respectively (see theorems 4 and 5), and $\{X''_i\}$ generated by Algorithm 2 with $X''_0 = P_I$ and $Y''_0 = I/\|P_I\|_\infty$ (see Theorem 6).

Let us $res(X) = \|X - A^*XA + B^*X^{-1}B - I\|_\infty$. As practical stopping criterions we use $\|X_k - X_{k-1}\|_\infty \leq 10^{-10}$, where k is the number of iterations.

We use the Matlab function *dlyap* for computing the unique positive definite solution P_I of Eq. (7), and Cyclic reduction algorithm [13, Algorithm 3.1.] for computing the largest solution M of Eq. (8) if there exist.

Example 1. We consider Eq. (1) with matrix coefficients

$$A = \begin{pmatrix} 0.7 & 0.15 & 0.1 \\ 0.01 & 0.8 & 0.06 \\ 0.02 & 0.03 & 0.83 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.15 & 0.15 \\ 0.1 & 0.15 & 0.25 \end{pmatrix}.$$

By using *dlyap* and *Cyclic reduction* algorithm, we have

$$M \approx \begin{pmatrix} 0.8265 & -0.1684 & -0.1582 \\ -0.1684 & 0.8316 & -0.1633 \\ -0.1582 & -0.1633 & 0.8214 \end{pmatrix}, \quad P_I \approx \begin{pmatrix} 2.0161 & 0.6338 & 0.6009 \\ 0.6338 & 3.5247 & 1.2988 \\ 0.6009 & 1.2988 & 4.0809 \end{pmatrix}.$$

In Table 1 we report the results of experiments for Example 1. We obtain $X_L \approx X_{89}$ and $X' \approx X'_{101}$ by Algorithm 1 with $X_0 = P_I$ and $X'_0 = M$, respectively, and $X_L \approx X''_{89}$ by Algorithm 2. Moreover, we have $\|X_{89} - X'_{101}\|_\infty = 5.0950e - 10$ and $\|X''_{89} - X_{89}\|_\infty = 2.0438e - 11$. Hence, $X_L \equiv X'$.

Table 1: Numerical results of Example 1.

Algorithm	X_0	k	$\|X_k - X_{k-1}\|_\infty$	$res(X_k)$
1. (BFPI)	M	101	$8.7228e - 11$	$6.8676e - 11$
1. (BFPI)	P_I	89	$8.5141e - 11$	$6.7034e - 11$
2. (IFV-BFPI)	P_I	89	$9.2056e - 11$	$7.2477e - 11$

Example 2. We consider Eq. (1) with matrix coefficients

$$A = \begin{pmatrix} 0.7 & 0.2 & 0.3 \\ 0 & 0.8 & 0.6 \\ 0 & 0 & 0.8 \end{pmatrix}, \qquad B = \frac{1}{8}\begin{pmatrix} 2 & 0 & 0 \\ 2 & 1.5 & 0 \\ 1 & 1.5 & 2.5 \end{pmatrix}.$$

By using *dlyap* and *Cyclic reduction* algorithm, we have

$$M \approx \begin{pmatrix} 0.8025 & -0.0976 & -0.0601 \\ -0.0976 & 0.9135 & -0.0727 \\ -0.0601 & -0.0727 & 0.8887 \end{pmatrix}, \ P_I \approx \begin{pmatrix} 1.9608 & 0.6239 & 1.5314 \\ 0.6239 & 3.5502 & 6.3649 \\ 1.5314 & 6.3649 & 26.4569 \end{pmatrix}.$$

In Table 2 we report the results of experiments for Example 2. We obtain $X_L \approx X_{76}$ and $X' \approx X'_{84}$ by Algorithm 1 with $X_0 = P_I$ and $X'_0 = M$, respectively, and $X_L \approx X''_{76}$ by Algorithm 2. Moreover, $\|X_{76} - X'_{84}\|_\infty = 4.1598e - 10$ and $\|X''_{76} - X_{76}\|_\infty = 4.9603e - 12$. Hence, for Example 2 $X_L \equiv X'$, also.

Table 2: Numerical results of Example 2.

Algorithm	X_0	k	$\|X_k - X_{k-1}\|_\infty$	$res(X_k)$
1. (BFPI)	M	84	$9.5680e - 11$	$6.9057e - 11$
1. (BFPI)	P_I	76	$7.6954e - 11$	$5.5539e - 11$
2. (IFV-BFPI)	P_I	76	$7.9021e - 11$	$5.7024e - 11$

Example 3. We consider Eq. (1) with matrix coefficients

$$A = \frac{1}{50}\begin{pmatrix} 40 & 0 & 0 & 0 & 0 \\ 25 & 42 & 0 & 0 & 0 \\ 23 & 27 & 48 & 0 & 0 \\ 35 & 45 & 16 & 42 & 0 \\ 66 & 21 & 24 & 65 & 46 \end{pmatrix}, \quad B = \frac{1}{300}\begin{pmatrix} 11 & 21 & 23 & 25 & 32 \\ 21 & 31 & 60 & 42 & 33 \\ 23 & 60 & 34 & 18 & 26 \\ 25 & 42 & 18 & 44 & 30 \\ 32 & 33 & 26 & 30 & 50 \end{pmatrix}.$$

By using *dlyap* we compute the unique positive definite solution P_I of Eq. (7), since $\rho(A) = 0.96 < 1$. For Example 3, Theorem 4 can not be used, since $\omega(B) = \rho(B) = 0.5396 > 0.5$. But, $A^*A \geq B^*B$. Thus, by Corollary 2 *(b)* we have that Eq. (1) has a solution $X' \in [I, P_I]$. We use Algorithm 1 with $X_0 = P_I$ and $X'_0 = I$ (with $\alpha = 1$).

In Table 3 we report the results of experiments for Example 3. We obtain $\|X_{412} - X'_{464}\|_\infty = 1.0984e - 08$ and $\|X''_{434} - X_{412}\|_\infty = 4.7026e - 10$.

Table 3: Numerical results of Example 3.

Algorithm	X_0	k	$\|X_k - X_{k-1}\|_\infty$	$res(X_k)$
1. (BFPI)	I	464	$7.2760e-12$	$2.3283e-10$
1. (BFPI)	P_I	412	$3.6380e-11$	$4.0939e-10$
2. (IFV-BFPI)	P_I	434	$7.1054e-15$	$9.6634e-13$

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